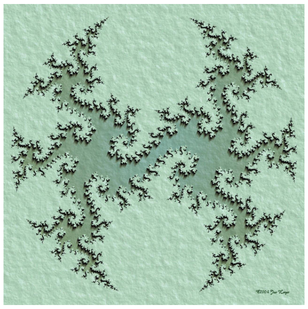
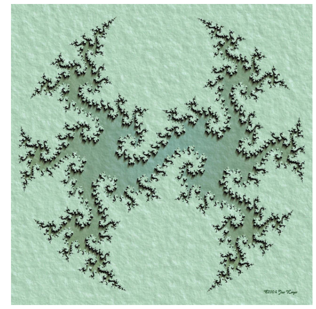


Linda Keen

Lehman College

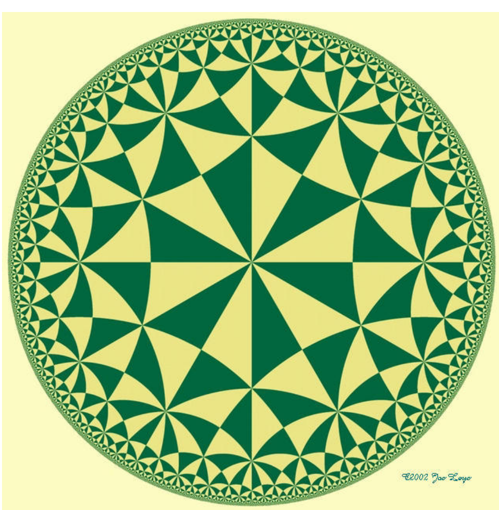


Professor Keen's research spans the three classical areas of mathematics: Geometry, Algebra and Analysis. In Geometry her work has focussed on Riemann Surfaces and Hyperbolic Geometry. In Algebra, she has specialized in Kleinian and Fuchsian Groups. In Analysis she has had major advances in both Complex Analysis and Hyperbolic Dynamics. She has employed the techniques of each subfield to explore the others achieving new insights into these three areas of mathematics.

We now present a brief description of these fields and her contribution to each. Although occasionally the description can become quite technical we believe scientists should be able to follow the main idea.

Geometry

Hyperbolic Geometry is the geometry which satisfies all the axioms of Euclidean geometry except the parallel postulate. In fact, for every given line and point, there are infinitely many lines (or geodesics) parallel to the given line passing through the point! The Poincare model of Hyperbolic geometry is the collection of points in a disk with geodesics that curve like semicircles until they hit the boundary.



Jos Leys, an admirer of Linda Keen and a well known digital artist, has given us this picture of the Poincare Model. Here he has tiled the entire Hyperbolic plane with triangles of the same size. As you approach the edge of the disk you must travel over more and more triangles so that you can never truely get there. Notice how the geodesic sides seem to bend away from each other like semicircles.

Other hyperbolic geometries share this property of diverging geodesics, but the geodesics can wrap around and back to themselves. A common flat geometry which has this strange wrapping effect is the torus, the space depicted in the computer game, Pacman, where one who leaves the screen out of one side appears on the other. The same effect can be achieved on a hyperbolic space by cutting the disk along the boundaries of some triangles and imagining that leaving through one edge enters through another. If a geodesic leaves through one edge and comes back and loops onto itself it is called a geodesic loop.

Keen's Collar Lemma, one of the most influencial tools used to study hyperbolic surfaces, concerns these geodesic loops. It states that regardless of the length of the closed geodesic there is a collar around it which has an area of at least 8/5. In other words, if the geodesic is long, this collar may be quite thin, but if the geodesic is short, then it has a very fat collar around it. Recently Keen's lemma was applied by Vourinen and Sugawa in obtaining specific estimates on the hyperbolic metric of any plane domain. It has also been fundamental in studying the thick and thin parts of Riemann surfaces, the univalence radius, uniformly perfect domains, Teichmuller theory, and complex dynamics.

Riemann Surfaces are special hyperbolic surfaces created by solving equations like $w=z^2$ where z is any complex number. Ordinarily, one would say that inverting this process gives two solutions, yet mathematicians prefer to study functions which have one output for each input. Thus the Riemann Surface is created to allow for unique solutions to these equations. Teichmuller theory is the study of spaces of such Riemann Surfaces. Linda Keen is considered the world expert on the Teichmuller space of a punctured torus: a torus with holes punched out of it.

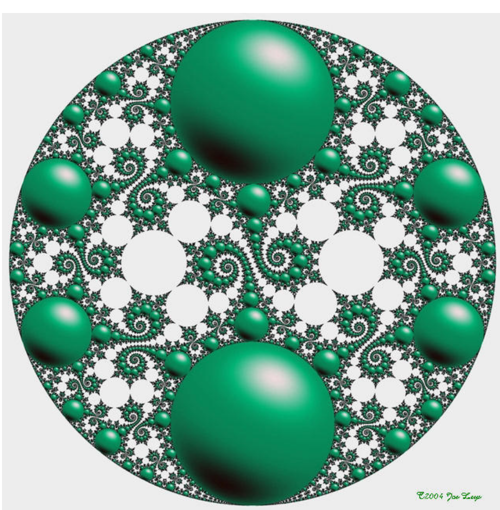
In 2005, Professor Keen has published an article in the prestigious journal, *Topology*, with her long time coauthor Caroline Series, on new "pleating invariants" for this interesting space. These invariants are special constants associated with the space much like the number pi is associated with a circle, but rather than coming from measuring something like a circumference they are found by foliating the space with sheets much like one can fill in a solid block of space with a pile of paper. Of course, this being hyperbolic geometry, the sheets don't line up so neatly. As John Parker writes in his review of the article on MathSciNet:

"there is much subtlety and the whole picture is strikingly elegant."

Linda Keen is also known for her work on Fricke, intrinsic and geometric moduli of Riemann Surfaces and is the keystone of the Teichmuller theory group at the CUNY Graduate Center. Her research on spaces like the Teichmuller space of the punctured torus, has lead to important contributions to the study of Kleinian and Fuchsian groups.

Algebra

The field of algebra is an extension of the concepts from arithmetic. Symbols are used to represent both the operations (like multiplication and arithmetic) and the individual elements (for example, numbers). When one only considers elements with a single operator, this is called group theory. Often the elements of a group are not numbers but functions or actions. A commonly understood group is the group of isometries on a lattice. Such groups have been used to understand the symmetries of crystals.



More complicated groups like Kleinian Groups can be understood as symmetries of hyperbolic spaces. The diagram on the left was created by Jos Leys using a computer program similar to "Klein", designed by David Wright. Wright's program credits the results of Keen and Series, whose precise geometric interpretations of such groups helped bring such images to life. By defining new structures and giving geometric interpretations to Kleinian and Fuchsian groups, Professor Keen has lead to the production of a practical and beautiful means of studying these groups.

Linda Keen has also studied groups of Moebius transformations on the Riemann sphere. In 2005, in joint work with Jane Gilman, she extended concepts used to study groups on surfaces in the past to higher dimensional spaces called three manifolds. This is a strong generalization of her earlier work on Fuchsian groups.

While Linda Keen has not worked in the field of number theory, the deeper understanding of Kleinian and Fuchsian groups has had an influence on that fourth classical field of mathematics, as has her work on geodesic loops.

Analysis

Professor Keen is a pioneer in the study of complex dynamics of entire functions. While the complex dynamics of rational maps had been a popular object of study for many years and many results had been established in that area, relatively little was known about iterations of entire functions until Linda Keen started working on the subject.

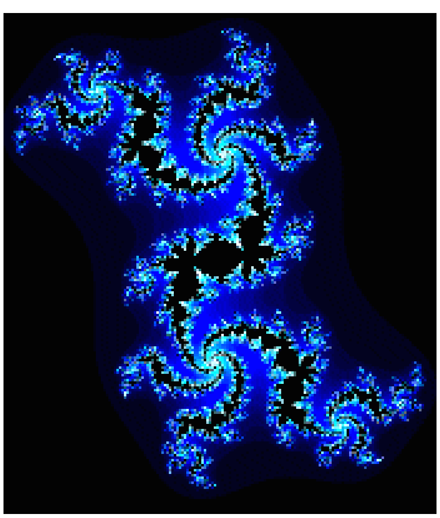
In particular, Professor Keen has studied the well known tangent function from trigonometry. Given a number, *x*, she studied iterates of tangent, creating what is called the "orbit of x":

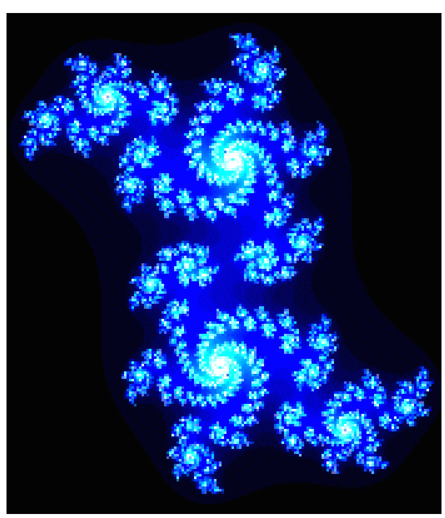
$$O(x)=\{x, \tan(x), \tan(\tan(x)), \tan(\tan(\tan(x))), \tan(\tan(\tan(\tan(x)))),...\}$$

If a small change in the input data, *x*, changes the orbit only slightly, we say the system is stable, or predictable. If a small change in z, on the other hand, causes a drastic change in the orbit, we say the system is chaotic. Weather is well known to be such a chaotic system, where the flutter of wings of a butterfly in India can affect the formation of a strom system over Florida.

Keen has focussed on stable dynamical systems similar to the well known predator prey model in biology. The system created using the iterations of the tangent function above is stable. For most stable systems the orbits tend to a periodic orbit.

More generaly Linda Keen and Janina Kotus have studied families of functions like $f(c,z) = c \tan(z)$ so that one can study how the orbits of a fixed z vary when one varies the parameter c. If, for a stable system, a small change in the parameter c changes the limiting periodic orbit slightly, the system is called hyperbolic. It would seem that hyperbolic behavior is the norm, but this is often difficult to prove. Professor Keen has proved that hyperbolic behavior occurs for a collection of open subsets of the parameter plane and she has given a full description of them.





Keen and Kotus have related the dynamics of $f(c,z) = c \tan(z)$ to an analysis of the geometry of their orbits. These spaces of orbits have special sets in them called Julia sets. Here are two pictures of Julia sets produced with different values of the parameter c by Jim Loy.

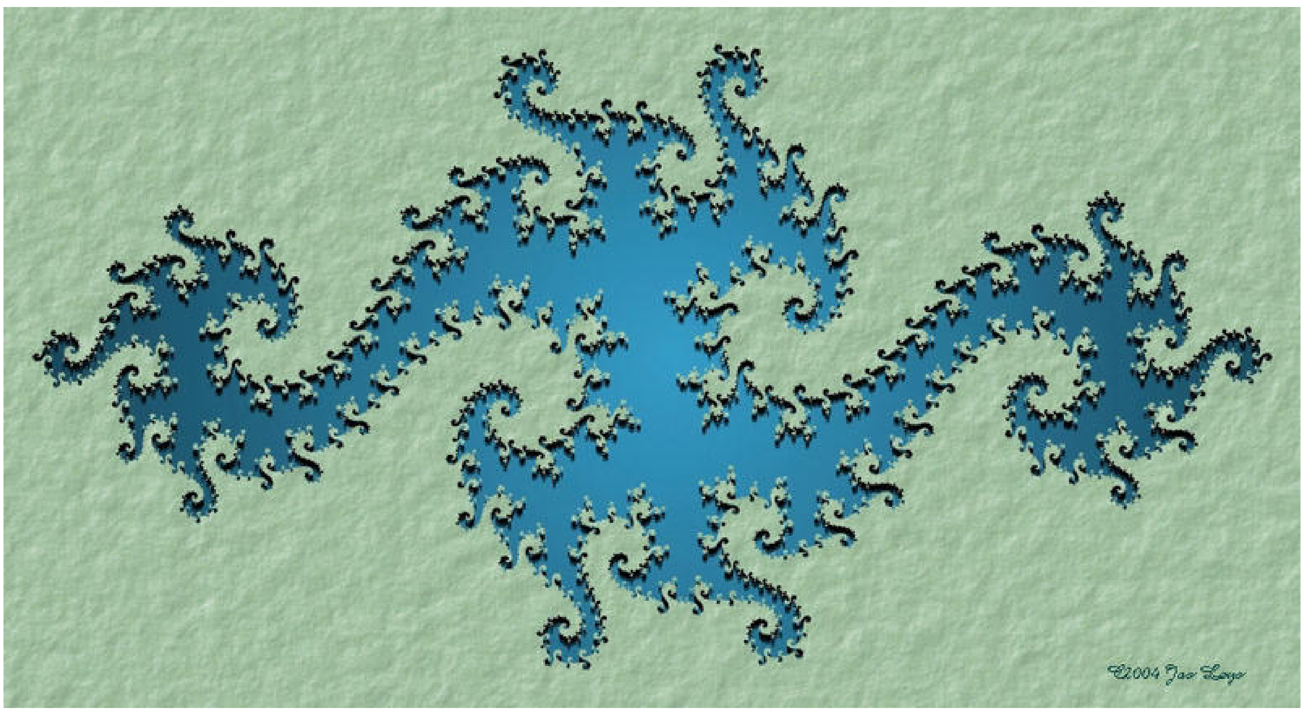
It is easy to see that the Julia sets are fractal sets closely related to the Fuchsian groups described above. Keen and Kotus employ this relationship in their proofs.

In a series of recent papers, Keen and Lakic have developed a new approach to looking at the limits of iterated function systems. The iterated function systems are similar to the iterated tangent functions described above except that now the functions are allowed to change with each iteration:

$$\{z, f_1(z), f_2(f_1(z)), f_3(f_2(f_1(z))), \ldots\}.$$

To keep things under control one requires all the functions f_i to map a common set U of points z to a fixed subset X. The key is to find conditions on the sets U and X so that accumulations of orbits must be constant maps. In particular Keen and Lakic show this is the case for a special class of functions when U is the disk but not when U is a more general hyperbolic domain.

Even more technically difficult to describe but of equal if not more importance are Linda Keen's results on the "period doubling phenomenon" and the possibility of "wandering domains" for entire functions, not to mention the "Sharkowski-like ordering" of the periods in hyperbolic components. Nevertheless these results can easily be explained to a graduate student in mathematics.



In fact Professor Keen's research is an analysis of concepts which are so fundamental to mathematics that the majority of doctoral students who have passed their qualifying exams can understand the importance of her results. The proofs, on the other hand, require a deeper understand of all three of the classical fields she works in.

It is Linda Keen's intuitive understanding of this interconnection between Analysis, Geometry and Algebra that has lead to the great success of her research.

Written by Professors Nikola Lakic, Christina Sormani, Katherine StJohn,
colleagues of Linda Keen at Lehman College and the CUNY Graduate Center.
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