SLICES OF PARAMETER SPACES OF GENERALIZED NEVANLINNA FUNCTIONS

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Abstract. In the early 1980's, computers made it possible to observe that in complex dynamics, one often sees dynamical behavior reflected in parameter space and vice versa. This duality was first exploited by Douady, Hubbard and their students in early work on rational maps.

Here, we extend these ideas to transcendental functions.

In [16], it was shown that for the tangent family, $\{\lambda \tan z\}$, the hyperbolic components meet at a parameter λ^* such that $f_{\lambda*}^n(\lambda^* i) = \infty$ for some n. The behavior there reflects the dynamic behavior of $\lambda^* \tan z$ at infinity. In Part 1. we show that this duality extends to a more general class of transcendental meromorphic functions $\{f_\lambda\}$ for which infinity is not an asymptotic value. In particular, we show that in "dynamically natural" one-dimensional slices of parameter space, there are "hyperbolic-like" components Ω with a unique distinguished boundary point such that for $\lambda \in \Omega$, the dynamics of f_λ reflect the behavior of f_λ at infinity. Our main result is that *every* parameter point λ in such a slice for which the iterate of the asymptotic value of f_λ is a pole is such a distinguished boundary point.

In the second part of the paper, we apply this result to the families $\lambda \tan^p z^q$, $p, q \in \mathbb{Z}^+$, to prove that all hyperbolic components of period greater than 1 are bounded.

Introduction. In the early 20th century, Fatou and Julia studied dynamical systems generated by holomorphic functions and, more particularly, rational functions. With the growth of computing power in the 1980's, it became possible to draw pictures of these systems. These pictures, even in the simplest case of the family $\{z^2 + c\}$ generated tremendous interest. Studying pictures of Julia sets of these polynomials and of the Mandelbrot set, Douady, Hubbard and their students, using techniques developed from the theory of quasiconformal mappings, discovered,

among other things, that the points c of the Mandelbrot set M for which an iterate of c under $z^2 + c$ belongs to a periodic cycle play a key role in understanding the structure of M. See *e.g.* [10, 19, 25]. If the cycle is attracting, c is a center of a component of the Mandelbrot set in which the structures of the Julia sets are all quasiconformally conjugate. The centers give a combinatorial description of the Mandelbrot set. If the cycle is repelling, there is a natural map from the Julia set of $z^2 + c$ to a neighborhood of c in the plane which shows that the dynamical behavior is reflected in parameter space and vice versa.

Families of entire functions, like the exponential and sine families, whose parameter spaces are quite different, have also been studied to the same end. In the exponential family $\exp z + c$ where the essential singularity is also an asymptotic value, for example, all the hyperbolic components are unbounded. See [8, 22, 24].

The simultaneous presence of poles and essential singularities make transcendental meromorphic functions more difficult to analyze. The first such results were for the tangent family, $\lambda \tan z$ in [16]. Such functions have two asymptotic values and no critical values. In the quadratic family $z^2 + c$, the orbit of the critical value c determines the dynamical behavior. For the tangent family, the parameter λ also determines the dynamical behavior, because that behavior is determined by the orbits of the asymptotic values $\pm\lambda i$. The hyperbolic components of the parameter space are enumerated in terms of a distinguished boundary point on the component, a "virtual center", so called because the multiplier of the attracting cycle corresponding to the hyperbolic component goes to zero at that boundary point. It was also proved that the virtual centers are in one to one correspondence with those parameters for which some iterate of the asymptotic value λi is a pole. The key idea here is that there is a duality between these particular parameters and the dynamical plane near infinity.

In this paper we define a general class of transcendental meromorphic functions. The main result is that families of this class exhibit the same duality as the tangent family: that is, in a properly chosen one dimensional slice of the parameter space, at every point where an iterate of the asymptotic value is a pole, hyperbolic-like components meet in a way that reflects the dynamical behavior of the functions at infinity.

To describe the classes of functions we concern ourselves with and to state our theorems, we need some background. The dynamical plane of a meromorphic map is divided into two sets: the *Fatou or stable set* on which the iterates are well defined and form a normal family, and the *Julia set*, its complement. The Julia set can be characterized as the closure of the set of repelling periodic points, or equivalently, the closure of the set of pre-poles, points that map to infinity after finite iteration. A good introduction to meromorphic dynamics can be found in [5].

We begin with a transcendental meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ with an essential singularity at infinity. The points over which a meromorphic function f is not a regular covering map are called singular values. Singular values may be: *critical values* (images of zeroes of f', the *critical points*), *asymptotic values* (points $v = \lim_{t\to\infty} f(\gamma(t))$ where $\gamma(t) \to \infty$ as $t \to \infty$) and accumulation points of the previous types. If an asymptotic value is isolated, the local inverse there is holomorphically equivalent to the logarithm. The class of meromorphic functions with finitely many singular values, called finite type functions, is particularly tractable: all the asymptotic values of these functions are isolated; the Fatou domains of these functions have a simple classification because there are no wandering domains (see

Section 1 for definitions); the parameter spaces of such functions have natural embeddings into Teichmüller spaces of dimension n, where n is a simple function of the number of singular values. See *e.g.* [14]. The focus in this paper is on the subclass $\mathcal{M}_\infty$ of functions of finite type for which infinity is the only essential singularity and is not an asymptotic value. These necessarily have infinitely many poles and this behavior at infinity has consequences for the dynamical and parameter spaces (see *e.g.* [7, 1, 2, 3, 4]). In particular, it is very different from that for entire functions or transcendental functions with finitely many poles. See *e.g.* [12, 22, 23, 24].

A general principle in dynamics is that each stable dynamical phenomenon is "controlled" by a singular value. For example, each attracting or parabolic periodic cycle always attracts a singular value; the boundary of each rotation domain contains the accumulation set of the orbit of a singular value. This principle is an important tool in studying parameter spaces. *Dynamically natural* one-dimensional slices of parameter spaces families of functions in $\mathcal{M}_\infty$ were defined in [13] using this principle. See Definition 2.9 for the precise definition. The idea is that the slice is compatible with the dynamics in the sense that all but one of the singular values controls a stable phenomenon that persists throughout the slice. The remaining singular value, called the "free singular value $v = v(\lambda)$", can exhibit varying behaviors at points λ in the slice. For example, at some point λ_0, $v(\lambda_0)$ may be attracted by an attracting periodic cycle; if so, this will also be true for all other λ's in an open "hyperbolic-like" region of the slice where the functions are all quasiconformally conjugate on their Julia sets. If, however, in every neighborhood of λ_0 the orbits of $v(\lambda)$ exhibit many different types of dynamical behavior then λ_0 belongs to the "bifurcation locus". In this paper we study how the hyperbolic-like regions fit together around the bifurcation locus in a dynamically natural slice.

The bifurcation locus of dynamically natural slices of families in $\mathcal{M}_\infty$ contains parameters distinguished by functional relations. In analogy with rational maps, there are *Misiurewicz points* where v lands on a repelling periodic cycle. Another type of distinguished parameter, not seen for rational maps, is a *virtual cycle parameter*, where an iterate of v is a pole. See [13]. As we will see, these are somewhat analogous to the *centers* of hyperbolic components of rational maps where a critical value lands on a super-attracting cycle.

Dynamically natural slices contain two different kinds of hyperbolic-like domains: *capture components*, where the free asymptotic value is attracted to one of the fixed dynamical phenomena such as an attracting cycle of fixed multiplier, and *shell components*, where the free asymptotic value is the only singular value attracted to an attracting cycle with a variable multiplier. In a shell component the period of the attracting cycle is constant and the multiplier map is a well-defined universal covering map onto the punctured disk. Properties of shell components for general families in $\mathcal{M}_\infty$ were studied in detail in [13]. In particular, it was proved that the boundary of every shell component contains a special point, the *virtual center* where the limit of the multiplier map is zero. One of the main results proved there is

Theorem [FK]. *For families in $\mathcal{M}_\infty$, a virtual center on the boundary of a shell component in a dynamically natural slice is a virtual cycle parameter and any virtual cycle parameter on the boundary of a shell component is a virtual center.*

In this paper, we concern ourselves with a special subclass in $\mathcal{M}_\infty$. We start with *Nevanlinna functions*, these are functions of finite type with one essential singularity

at infinity, finitely many asymptotic values and no critical values. We let $\mathcal{N}_r$ be the subclass of Nevanlinna functions with r asymptotic values, none of which is infinity. We next form the family which we call *Generalized Nevanlinna functions* by pre- and post-composing Nevanlinna functions with polynomials. If P and Q are polynomials of respective degrees p and q and $g \in \mathcal{N}_r$, we define the subfamily $\mathcal{M}_{p,q,r} = \{P \circ g \circ Q\}$. These are generalized Nevanlinna functions all of whose asymptotic values are finite; note that infinity, however, is allowed to be a critical value. The following proposition is proved in [13].

Proposition [FK]. *If h is a meromorphic function topologically conjugate to a function $f = P \circ g \circ Q$ in $\mathcal{M}_{p,q,r}$ then h also belongs to $\mathcal{M}_{p,q,r}$; that is there is a function $\tilde{g} \in \mathcal{N}_r$ and polynomials $\tilde{P}, \tilde{Q}$ of degrees p, q respectively such that $h = \tilde{P} \circ \tilde{g} \circ \tilde{Q}$.*

In this paper we obtain the corollary,

Corollary [FK]. *There is a natural embedding of $\mathcal{M}_{p,q,r}$ into $\mathbb{C}^{p+q+r+3}$.*

Theorem [FK] does not preclude the existence of virtual cycle parameters that are not on the boundary of a shell component. For example, such a parameter might be buried in the bifurcation locus. Our first new theorem says this cannot happen for slices of $\mathcal{M}_{p,q,r}$.

Theorem A. *In a dynamically natural slice in $\mathcal{M}_{p,q,r}$, every virtual cycle parameter lies on the boundary of a shell component.*

Corollary A, which follows directly from Theorems [FK] and A, says that in $\mathcal{M}_{p,q,r}$ the notions of virtual center and virtual cycle parameter are equivalent.

Corollary A. *In a dynamically natural slice of $\mathcal{M}_{p,q,r}$, every virtual cycle parameter is a virtual center and vice versa.*

In [13] it was proved that for families in $\mathcal{M}_\infty$, shell components of period 1 in dynamically natural slices where the parameter is an affine function of the free asymptotic value are always unbounded and it was conjectured that those of period greater than 1 are always bounded. This is in contrast to general families of entire functions, where because infinity is asymptotic value, there are unbounded shell components of arbitrary period. The conjecture was proved true in the tangent family in [16]. In this paper we prove it for the generalization of the tangent family, $\mathcal{F} = \{\lambda \tan^p(z^q)\}$. This is a subfamily of $\mathcal{M}_{p,q,r}$.

Theorem B. *Every shell component of period strictly greater than 1 in the λ plane of the family $\mathcal{F}$ is bounded.*

The paper is organized as follows. Part 1. is a discussion of the general family $\mathcal{M}_{p,q,r}$ and dynamically natural slices of its parameter space. In Section 1 we give a brief overview of the basic theory, set our notation and discuss the theorem of Nevanlinna, Theorem 1.3, which we use to define the class of Nevanlinna functions. Next, in Section 2, we define the class of generalized Nevanlinna functions, state Proposition [FK] and prove Corollary [FK]. In Section 2.1 we first recall the classification of Fatou components in our context. We then define dynamically natural slices of parameter spaces and their shell components. In Section 3, we define virtual cycle parameters and virtual centers, state Theorem A and prove Theorem B.

Part [2] is a discussion of the special symmetric subfamily $\mathcal{F} \subset \mathcal{M}_{p,q,2}$. In Section [4] we classify the shell components of $\mathcal{F}$ by period and discuss the special properties of the components of periods 1 and 2. Finally, in Section [5] we prove Theorem B.

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Part 1. The family $\mathcal{M}_{p,q,r}$

1. Basics and tools.

1.1. **Meromorphic functions.** In this paper, unless we specifically say otherwise, we always assume that an entire or meromorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is transcendental and so has infinite degree. If f has essential singularities in addition to one at infinity, it is not defined at them. If we mean a map of finite degree we call it polynomial or rational. We need the following definitions:

Definition 1.1. A point $v \in \widehat{\mathbb{C}}$ is called a *singular value*[1] of f if, for some small neighborhood of v, some branch of f^{-1} is not well defined. If c is a zero of f', it is a critical point and its image $v = f(c)$ is a *critical value*. The branch with $f^{-1}(v) = c$ is not well defined so the critical value v is singular. A point is also singular if there is a path $\gamma(t)$ such that $\lim_{t \to \infty} \gamma(t) = s$ where s is an essential singularity of f and $\lim_{t \to \infty} f(\gamma(t)) = v$. The limit v is called an *asymptotic value* of f. Singular values may be critical, asymptotic or accumulations of such points. We denote the set of singular values of f by $\mathcal{S}_f$.

If an asymptotic value v is isolated, and γ is an asymptotic path for v, we can find a nested sequence $\{D_{r,v}\}$, of disks of radius r centered at v, with $r \to 0$, and a particular branch g of f^{-1} such that $\{V_r = g(D_{r,v})\}$ is a nested sequence of neighborhoods containing γ and $\cap g(D_{r,v}) = \emptyset$. Then r can be chosen small enough so that the map $f : V_r \to D_{r,v} \setminus \{v\}$ is a holomorphic universal covering map. In this case V_r is called an *asymptotic tract* for the asymptotic value v and v is called a *logarithmic singular value*. The number of distinct asymptotic tracts of a given asymptotic value is called its *multiplicity*[2].

Definition 1.2. (See *e.g.* [17]) A ray $\beta = te^{i\theta}$, $t \in [t_0, \infty)$, is called a *Julia ray* and θ is called a *Julia direction* for the meromorphic function f if for large t_0, f assumes all (but at most two) values infinitely often in any sector containing β.

An example to keep in mind is e^z with asymptotic values at 0 and ∞. The asymptotic tracts are the left and right half planes and the Julia directions are $\theta = \pm \pi/2$. Another example is $\tan z$ with asymptotic values $\pm i$, asymptotic tracts the upper and lower half planes and Julia directions 0 and π.

In this paper, whenever we talk about the number of critical points and/or asymptotic values, we tacitly assume that we count them with multiplicity.

Infinity is always both an asymptotic value and an essential singularity of an entire function f.

We will only be interested in meromorphic functions with $\#\mathcal{S}_f < \infty$ and a single essential singularity at infinity. These are called functions of *finite type*. Note that because all the asymptotic values of a finite type function are isolated, they are logarithmic. Such functions may have infinity as a singular value or a critical value.

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[1]In the literature these are sometimes called singularites of f^{-1}, see e.g. [6] for a complete discussion.

[2]Note this notion is not the same as multiplicity of a critical point.

Their iterates, however, are not of finite type and have essential singularities at pre-images of infinity.

1.1.1. *Nevanlinna functions.* Nevanlinna, in [20] and [21], characterized families of meromorphic functions with finitely many asymptotic values, finitely many critical points and a single essential singularity at infinity. (See [9, 16, 11] for further discussion.)

Recall that the Schwarzian derivative of a function g is defined by

$$S(g) = (g''/g')' - \frac{1}{2}(g''/g')^2. \tag{1}$$

Because Schwarzian derivatives satisfy the cocycle relation

$$S(f \circ g)(z) = S(f)(g'(z))^2 + S(g)(z)$$

and the Schwarzian derivative of a Möbius transformation is zero, solutions to equation (1) are determined only up to post-composition by a Möbius transformation.

Nevanlinna's theorem says

Theorem 1.3 (Nevanlinna)**.** *([21], Chap XI) Every meromorphic function g with $p < \infty$ asymptotic values and $q < \infty$ critical points has the property that its Schwarzian derivative is a rational function of degree $p + q - 2$. If $q = 0$, the Schwarzian derivative is a polynomial $P(z)$. In the opposite direction, for every polynomial function $P(z)$ of degree $p - 2$, the solution to the Schwarzian differential equation $S(g) = P(z)$ is a meromorphic function with exactly p asymptotic values and no critical points. The only essential singularity is at infinity.*

Remark 1.1. The proof of the first part of this theorem involves the construction of the function as a limit of holomorphic functions whose Schwarzians are rational of bounded degree. By construction, the only essential singularity of the limit function is at infinity. The proof of the second part of the theorem involves understanding the asymptotic properties of solutions to the equation $S(g) = P(z)$. In particular, there are exactly p "truncated solutions" $g_0, \ldots, g_{p-1}$ that, for any $\epsilon > 0$, have asymptotic developments of the form

$$\log g_k(z) \sim (-1)^{k+1} z^{p/2}$$

defined in the sector $|\arg z - 2\pi k/p| < 3\pi/p - \epsilon$. Each g_k is entire and tends to zero as z tends to infinity along each ray of the sector $|\arg z - 2\pi k/p| < \pi/p$ and tends to infinity in the adjacent sectors. The rays separating the sectors are the Julia rays for $g(z)$. It follows that the asymptotic values corresponding to adjacent sectors cannot be equal. See [9, 15, 20] for details.

Definition 1.4. We denote the family of meromorphic functions with $p < \infty$ asymptotic values and no critical values by $\widetilde{\mathcal{N}}_p$ and call the functions *Nevanlinna functions.* Note that for functions in this family infinity may be an asymptotic value.

One immediate corollary of Theorem 1.3 is that Nevanlinna functions cannot have exactly one asymptotic value. Moreover, since polynomials of degree $p - 2$ depend on their $p - 1$ coefficients and the solutions to the Schwarzian equation are determined up to post-compostion by a Möbius transformations which depends on four coefficents satisfying a relation, the most general solution depends on $p + 2$ parameters.

Since infinity is an essential singularity, if it is also an asymptotic value, the dynamics generated by the function are affected. Below we concern ourselves with systems generated by functions none of whose asymptotic values is infinity. This condition imposes a second relation on the coefficients of the Möbius transformation. Therefore the space of Nevanlinna functions is a subspace of $\widetilde{\mathcal{N}}_p$ of dimension $p+1$; we denote it by $\mathcal{N}_p$.

We state this as a corollary of Nevanlinna's theorem.

Corollary 1.5. *The family $\mathcal{N}_p$ with $p < \infty$ asymptotic values, none of which is at infinity, and no critical values has a natural embedding into $\mathbb{C}^{p+1}$.*

A third immediate corollary of Nevanlinna's theorem is that the family $\mathcal{N}_p$ is topologically closed. That is,

Corollary 1.6. *If f is topologically conjugate to a meromorphic function g in $\mathcal{N}_p$, and if f is meromorphic, then it is also in $\mathcal{N}_p$.*

2. **The family $\mathcal{M}_{p,q,r}$.** A family that is more general than $\mathcal{N}_p$ is the family of functions

$$\mathcal{M}_{p,q,r} = \{f = P \circ g \circ Q \,|\, g \in \mathcal{N}_r, \, P, Q \text{ polynomials of degrees } p, q\}.$$

Because P and Q are required to be polynomial, the functions in the family have an essential singularity at infinity. Moreover, they have only finitely many singular values. Infinity is an asymptotic value of f if it is an asymptotic value of g. This family is topologically closed. The precise statement of this, proved in [13], is,

Proposition A ([13],Theorem 8.3). *If h is topologically conjugate to a meromorphic function $f = P \circ g \circ Q$ in $\mathcal{M}_{p,q,r}$, and if h is meromorphic, then it is also in $\mathcal{M}_{p,q,r}$; that is there is a function $\tilde{g} \in \mathcal{N}_r$ and polynomials $\tilde{P}, \tilde{Q}$ of degrees p, q respectively such that $h = \tilde{P} \circ \tilde{g} \circ \tilde{Q}$.*

The above proposition says that a function with the same dynamics as one in $\mathcal{M}_{p,q,r}$ also belongs to this family. A corollary to the proposition is

Corollary A. *The space of functions $\mathcal{M}_{p,q,r}$ has a natural embedding into $\mathbb{C}^{p+q+r+3}$.*

Proof. The polynomials P and Q of degrees p and q are determined by their $p + q + 2$ coefficients. By the discussion before Corollary 1.5 g is determined by $r + 1$ parameters. Therefore $\mathcal{M}_{p,q,r}$ has a natural embedding into $\mathbb{C}^{p+q+r+3}$. □

2.1. **Dynamics of meromorphic functions.** Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function with essential singularity at infinity and let f^n denote the nth iterate of f, that is $f^n(z) = f(f^{n-1}(z))$ for $n \geq 1$. Then f^n is well-defined, except at the poles of $f, f^2, \cdots, f^{n-1}$, which form a countable set. These points have finite orbits that end at infinity.

The basic objects studied are the *Fatou set* and *Julia set* of the function f. The Fatou set $F(f)$ of f is defined by

$$F(f) = \{z \in \mathbb{C} \mid f^n \text{ is defined and normal in a neighborhood of } z\},$$

and the Julia set by

$$J(f) = \widehat{\mathbb{C}} \setminus F(f).$$

Note that the point at infinity is always in the Julia set. If f is a meromorphic function with more than one pole, then the set of prepoles, $\mathcal{P} = \cup_{n \geq 1} f^{-n}(\infty)$ is

infinite. By Montel's theorem, f^n is normal on $\widehat{\mathbb{C}} \setminus \overline{\mathcal{P}}$. Since it is not normal on $\overline{\mathcal{P}}$, $J(f) = \overline{\mathcal{P}}$, (see also [1]).

A point z is called a periodic point of $p \geq 1$, if $f^p(z) = z$ and $f^k(z) \neq z$ for any $k < p$. The multiplier of the cycle is defined to be $\mu = (f^p)'(z)$. The periodic point is *attracting* if $0 < |\mu| < 1$, *super-attracting* if $\mu = 0$, *parabolic* if $\mu = e^{2\pi i \theta}$, where θ is rational number, and *neutral* if θ is not rational. It is *repelling* if $|\mu| > 1$.

If D is a component of the Fatou set, then $f(D)$ is either a component of the Fatou set or a component missing one point. For the orbit of D under f, there are only two cases:

- there exist integers $m \neq n \geq 0$ such that $f^m(D) \subset f^n(D)$, and D is called *eventually periodic*;
- for all $m \neq n$, $f^n(D) \cap f^m(D) = \emptyset$, and D is called a *wandering domain*.

Suppose that $\{D_0, \cdots, D_{p-1}\}$ is a periodic cycle of Fatou components, then either:

(a) The cycle is (super)attracting: each D_i contains a point of a periodic cycle with multiplier $|\mu| < 1$ and all points in each domain D_i are attracted to this cycle. If $\mu = 0$, the critical point itself belongs to the periodic cycle and the domain is called super-attracting.

(b) The cycle is parabolic: the boundary of each D_i contains a point of a periodic cycle with multiplier $\mu = e^{2\pi i q/m}$, $(q, m) = 1$, m a divisor of p, and all points in each domain D_i are attracted to this cycle.

(c) The components of the cycle are Siegel disks: that is, each D_i contains a point of a periodic cycle with multiplier $\mu = e^{2\pi i \theta}$, where θ is irrational and there is a holomorphic homeomorphism mapping each D_i to the unit disk Δ, and conjugating the first return map f^p on D_i to an irrational rotation of Δ. The preimages under this conjugacy of the circles $|\xi| = r, r < 1$, foliate the disks D_i with f^p forward invariant leaves on which f^p is injective.

(d) The components of the cycle are Herman rings: each D_i is holomorphically homeomorphic to a standard annulus and the first return map is conjugate to an irrational rotation of the annulus by a holomorphic homeomorphism. The preimages under this conjugacy of the circles $|\xi| = r, 1 < r < R$, foliate the disks with f^p forward invariant leaves on which f^p is injective.

(e) D_i is an essentially parabolic (Baker) domain: that is, the boundary of each D_i contains a point z_i (possibly ∞) and $f^{np}(z) \to z_i$ for all $z \in D_i$, but f^p is not holomorphic at z_i. If $p = 1$, then $z_0 = \infty$.

Definition 2.1. Define the *post-singular set* of f as the closure of the orbits of the singular values; that is,

$$PS_f = \overline{\cup_{n=0}^{\infty} f^n(\mathcal{S}_f)}.$$

For notational simplicity, if a pre-pole s is a singular value, $\cup_{n=0}^{\infty} f^n(s)$ is a finite set and includes infinity.

Each non-repelling periodic cycle is associated to some singular point. In particular we have, see e.g. [19], chap 8-11 or [5], Sect.4.3,

Theorem 2.2. *If $\{D_0, \cdots, D_{p-1}\}$ is an attracting, superattracting, parabolic or Baker periodic cycle of Fatou components, then for some $i = 0, 1, \ldots, p - 1$, D_i contains a singular value. If $\{D_0, \cdots, D_{p-1}\}$ is a cycle of rotation domains (Siegel disks or Herman rings) the boundary of each D_i contains the ω-limit set of some singular value.*

For a discussion of hyperbolicity for general meromorphic functions, see [26] and the references therein. In this paper, we use the following definition of a hyperbolic function in $\mathcal{M}_{p,q,r}$. This definition is equivalent to what is called *hyperbolic in the spherical metric* in [26].

Definition 2.3. An $f \in \mathcal{M}_{p,q,r}$ is called *hyperbolic* if
$$P\mathcal{S}_f \cap J(f) = \emptyset.$$

Note that if all the singular values of f are attracted to attracting or super-attracting cycles then f is hyperbolic.

Singularly finite maps may have Baker domains but

Theorem 2.4. [4] *If $\#\mathcal{S}_f$ is finite, then there are no wandering domains in the Fatou set.*

2.2. **Holomorphic families.** In this paper, we are interested in the family $\mathcal{M}_{p,q,r}$ of generalized Nevanlinna functions for which infinity is not an asymptotic value. For each choice of triples (p, q, r) this is an example of a holomorphic family. Below we state the general definitions and results we need.

Definition 2.5 (Holomorphic family). A *holomorphic family* of meromorphic maps over a complex manifold X is a map $\mathcal{F} : X \times \mathbb{C} \to \widehat{\mathbb{C}}$, such that $\mathcal{F}(x, z) =: f_x(z)$ is meromorphic for all $x \in X$ and $x \mapsto f_x(z)$ is holomorphic for all $z \in \mathbb{C}$.

Definition 2.6 (Holomorphic motion). A *holomorphic motion* of a set $V \subset \widehat{\mathbb{C}}$ over a connected complex manifold with basepoint (X, x_0) is a map $\phi : X \times V \to \widehat{\mathbb{C}}$ given by $(x, v) \mapsto \phi_x(v)$ such that

(a) for each $v \in V$, $\phi_x(v)$ is holomorphic in x,
(b) for each $x \in X$, $\phi_x(v)$ is an injective function of $v \in V$, and,
(c) at x_0, $\phi_{x_0} = \mathrm{Id}$.

A holomorphic motion of a set V respects the dynamics of the holomorphic family $\mathcal{F}$ if $\phi_x(f_{x_0}(v)) = f_x(\phi_x(v))$ whenever both v and $f_{x_0}(v)$ belong to V .

The following equivalencies are proved for rational maps in [18] and extended to the transcendental setting in [16].

Theorem 2.7. *Let $\mathcal{F}$ be a holomorphic family of meromorphic maps with finitely many singularities, over a complex manifold X, with base point x_0. Then the following are equivalent.*

(a) *The number of attracting cycles of f_x is locally constant in a neighborhood of x_0.*
(b) *There is a holomorphic motion of the Julia set of f_{x_0} over a neighborhood of x_0 which respects the dynamics of $\mathcal{F}$.*
(c) *If in addition, for $i = 1, \ldots, N$, $s_i(x)$ is are holomorphic maps parameterizing the singular values of f_x, then the functions $x \mapsto f_x^n(s_i(x))$ form a normal family on a neighborhood of x_0.*

Definition 2.8 (J−stability). A parameter $x_0 \in X$ is a J-stable parameter for the family $\mathcal{F}$ if it satisfies any of the above conditions.

The set of J-stable points is thus the set where the dynamics do not change or bifurcate. The set of non J-stable parameters, on the other hand, is precisely the set where bifurcations do occur, and it is often called the *bifurcation locus* of the

family $\mathcal{F}$, and denoted by $\mathcal{B}_X$. In families of maps with more than one singular value, however, it makes sense to consider subsets of the bifurcation locus where only some of the singular values are bifurcating, in the sense that the families $\{g_n^i(x) := f_x^n(s_i(x))\}$ are normal in a neighborhood of x_0 for some values of i, but not for all. We define

$$\mathcal{B}_X(s_i) = \{x_0 \in X \mid \{g_n^i(x)\} \text{ is not normal in any neighborhood of } x_0.\}$$

In this paper we investigate *dynamically natural* one dimensional slices of the holomorphic family $\mathcal{M}_{p,q,r}$. In these slices, roughly speaking, all the dynamic phenomena but one are fixed and the last is determined by a "free asymptotic value". We will study the components of the complement of the bifurcation locus in these slices. The parameters in them are J-stable.

Precisely, (see [13])

Definition 2.9. A one dimensional subset $\Lambda \subset X$ is a *dynamically natural slice* with respect to $\mathcal{F}$ if the following conditions are satisfied.

(a) Λ is biholomorphic to the complex plane punctured at points where the function is not in the family; for example, points where the number of singular values is reduced. The removed points are called *parameter singularities*. By abuse of notation we denote the image of Λ in $\mathbb{C}$ by Λ again, and denote the variable in Λ by λ.

(b) The singular values are given by distinct holomorphic functions $s_i(\lambda)$, $i = 1, \ldots, N-1$, and an asymptotic value v_λ. For convenience we require that v_λ is an affine function of λ.[3] We call v_λ the *free asymptotic value*; we require that $\mathcal{B}_\Lambda(v_\lambda) \neq \emptyset$.

(c) The poles (if any) are given by distinct holomorphic functions $p_i(\lambda)$, $\lambda \in \Lambda$, $i \in \mathbb{Z}$.

(d) The critical values and some of the asymptotic values are attracted to an attracting or parabolic cycle[4] whose period and multiplier are constant for all $\lambda \in \Lambda$. We call these *dynamically fixed singular values*. The remaining singular values include v_λ; if there is only one remaining value, it may have arbitrary behavior. If there are several, they satisfy a functional relation that persists throughout Λ so that the remaining dynamical behavior is controlled by the behavior of v_λ.

(e) Suppose v_{λ_0} is attracted to $\mathcal{A}_{\lambda_0}$, the basin of attraction of an attracting cycle that does not attract any of the dynamically fixed singular values. Then the slice Λ contains, up to affine conjugacy, all meromorphic maps $g : \mathbb{C} \to \widehat{\mathbb{C}}$ that are quasiconformally conjugate to f_{λ_0} in $\mathbb{C}$ and conformally conjugate to f_{λ_0} on $\mathbb{C} \setminus \mathcal{A}_{\lambda_0}$.

(f) Λ is maximal in the sense that if $\Lambda' = \Lambda \cup \{\lambda_0\}$ where λ_0 is a parameter singularity, then Λ' does not satisfy at least one of the conditions above.

Examples of case (d) are discussed in Section 4.

In the components of the complement of the bifurcation locus in these slices v_λ is attracted to an attracting cycle $\bar{\mathbf{a}}$ of fixed period; this is *the period of the component*. We distinguish two cases:

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[3] This can always be arranged by a holomorphic change of coordinates in X.

[4] If the attracting cycle is parabolic, the functions in the slice are not hyperbolic but the results hold. See [13] for further discussion.

(i) $\bar{\mathbf{a}}$ does not attract one of the dynamically fixed singular values. We call
these *Shell components* and denote the individual components by Ω and the
collection by $\mathcal{S}$.
(ii) $\bar{\mathbf{a}}$ attracts one of the dynamically fixed singular values in addition to attracting
v_λ. We call these *Capture components*.

The properties of capture components are very different from those of the shell
components and we leave a discussion of them to future work.

3. **Shell components.** The properties of shell components are described in detail
in [13]. We summarize them here. We assume $\mathcal{F}_\lambda$ is the restriction of a holomorphic
family in $\mathcal{M}_{p,q,r}$ to a dynamically natural slice Λ and denote a function in $\mathcal{F}_\lambda$ by
f_λ. We need the following definitions

Definition 3.1. Suppose that for some λ, v_λ is a prepole of order $k-1$. Then λ is
called a *virtual cycle parameter of order k*. Set $a_1 = v_\lambda$, and $a_{i+1} = f_\lambda(a_i)$ where
i is taken   mod k so that $a_0 = \infty$. We call the set $\{a_1, a_2, \ldots, a_{k-1}, a_0\}$ a *virtual
cycle*.

This definition is justified by the following. Let $\gamma(t)$ be an asymptotic path for
$v_\lambda = a_1$ and let h be the branch of the inverse of f^k taking ∞ to the prepole a_1,
Then
$$\lim_{t \to \infty} f^k(h(\gamma(t)) = v_\lambda = a_1$$
so in this limiting sense, the points form a cycle.

Definition 3.2. Let Ω be a shell component in Λ and let
$$\bar{\mathbf{a}}_\lambda = \{a_0, a_1, \ldots, a_{k-2}, a_{k-1}\}$$
be the attracting cycle of period k that attracts v_λ. Suppose that as $n \to \infty$,
$\lambda_n \to \lambda^* \in \partial\Omega$ and the multiplier $\mu_{\lambda_n} = \mu(\bar{\mathbf{a}}_{\lambda_n}) = \Pi_{i=0}^{k-1} f'(a_i^n) \to 0$. Then λ^* is
called a *virtual center* of Ω.

Since the attracting basin of the cycle $\bar{\mathbf{a}}_\lambda$ must contain v_λ, we will assume
throughout that the points in the cycle are labeled so that v_λ and a_1 are in the
same component of the immediate basin.

The next theorem collects the main results in [13] about shell components in a
dynamically natural slice Λ for a holomorphic family of transcendental functions
with finite singular set, none of whose asymptotic values is at infinity. Note that
Theorem A of the introduction is part (c) of the theorem. Let $\mathbb{D}^* = \{z : 0 < |z| < 1\}$. We have

Theorem 3.3. *Let Ω be a shell component in Λ. Then*

(a) *The map $\mu_\lambda : \Omega \to \mathbb{D}^*$ is a universal covering map. It extends continuously
to $\partial\Omega$ and $\partial\Omega$ is piecewise analytic; either Ω is simply connected and μ_λ is
infinite to one or Ω is isomorphic a punctured disk and the puncture is a
parameter singularity.*
(b) *There is a unique virtual center on $\partial\Omega$. If the period of the component is 1,
the virtual center is at infinity.*
(c) *Every (finite) virtual center of a shell component is a virtual cycle parameter
and any virtual cycle parameter on the boundary of a shell component is a
virtual center.*

Here we prove a stronger theorem for slices of the family $\mathcal{M}_{p,q,r}$. Note that because the functions are of the form $f(z) = Q \circ g \circ P(z)$, and all the asymptotic values are finite, if for some z, $v = P(z)$ is an asymptotic value of g, then $Q(v)$ is an asymptotic value of f. There are p distinct asymptotic tracts corresponding to each asymptotic tract of v so that there are pr distinct asymptotic tracts at infinity separated by the Julia directions. The asymptotic values, tracts and Julia directions depend holomorphically on the parameters. At each pole of f of order k there are kqr pre-asymptotic tracts and the pull-backs of the rays in the Julia directions separate them.

Theorem B. *Let Λ be a dynamically natural slice for the meromorphic family $\mathcal{M}_{p,q,r}$ consisting of meromorphic functions of the form $f_\lambda = Q \circ g \circ P$ all of whose asymptotic values are finite, and let λ^* be a virtual center parameter of order k. Then λ^* is on the boundary of a shell component of order k in Λ. That is, in any neighborhood of λ^* there exists $\lambda \in \Lambda$ such that f_λ has an attracting cycle of period k.*

An immediate corollary to Theorem 3.3 and Theorem B is

Corollary B. *In a dynamically natural slice of $\mathcal{M}_{p,q,r}$, every virtual cycle parameter is a virtual center and vice versa.*

Remark 3.1. The essence of the theorem is that the dynamic picture at the poles is reflected in the parameter picture at the virtual center parameters.[5] If infinity is an asymptotic value, both the dynamics and parameter pictures are different.

Proof of Theorem B. Since λ^* is a virtual center parameter of order k, f_{λ^*} has a virtual cycle $\bar{\mathbf{a}}^* = \{a_1^* = v_{\lambda^*}, \ldots, a_{k-1}^*, a_0^* = \infty\}$. For each $j = 2, \ldots, k-1$, the cycle uniquely determines a branch of the inverse of f_{λ^*}, $f_{\lambda^*,j}^{-1}$ such that $f_{\lambda^*,j}^{-1}(a_j^*) = a_{j-1}^*$. By abuse of notation, for readability, we drop the j and denote all of these branches by $f_{\lambda^*}^{-1}$. Because we are in a dynamically natural slice of a holomorphic family, the analytic continuations of the $f_{\lambda^*}^{-1}$, denoted by f_λ^{-1} are well defined.

Note however, that in a neighborhood of the asymptotic value a_1^*, the inverse branch of f_{λ^*} is not uniquely determined. Since v_{λ^*} is part of a virtual cycle, a punctured neighborhood U of v_{λ^*} has at least one pre-image that is in an asymptotic tract. If the multiplicity of v_{λ^*} is one, then by definition there is a unique asymptotic tract that is determined by the virtual cycle of f_{λ^*}; we denote it by A_{λ^*} and take as $f_{\lambda^*}^{-1}$ the branch that maps U to A_{λ^*}. Then taking the analytic continuation of this $f_{\lambda^*}^{-1}$ we obtain A_λ as the analytic continuation of A_{λ^*}. If the multiplicity of v_{λ^*} is greater than one there will be a choice among the tracts corresponding to v_{λ^*} (and hence the inverse branch $f_{\lambda^*}^{-1}$). Similarly, if there is more than one asymptotic value that varies with λ (and satisfies a functional relation with λ), we choose the tract (or one of them if there is more than one) corresponding to the free asymptotic value[6]. In the argument below, we take A_{λ^*}, or choose one of the tracts as A_{λ^*}, along with the corresponding branch f_{λ^*}. Since, in general, the asymptotic value is not omitted, there may also be infinitely many inverse branches of the neighborhood that are bounded. We don't need to concern ourselves with them here.

Because we are in a dynamically natural slice we have the following holomorphic functions:

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[5] This is analogous to the situation for Misiurewicz points in the parameter plane for quadratic polynomials.

[6] See the examples in Part 2.

1. $v(\lambda) = v_\lambda$ is the free asymptotic value of f_λ.
2. If $p^* = a^*_{k-1}$, then $p(\lambda) = p_\lambda$ is the holomorphic function defining the pole of f_λ such that $p(\lambda^*) = p^*$.
3. Note that for λ in a neighborhood $V \subset \Lambda$ of λ^*, the affine function $\lambda \mapsto v_\lambda$ determines a corresponding punctured neighborhood U of v^*_λ in the dynamic plane of f_{λ^*} and vice versa. Define the map $h : V \to \mathbb{C}$ by $h(\lambda) = f^{k-2}_\lambda(v_\lambda) = h_\lambda$. Then if V is small enough, both p_λ and h_λ are in $\widehat{U} \subset U$, a small neighborhood of p^* in the dynamic plane of f_{λ^*}.
4. In the dynamic plane of f_λ, set $u(\lambda) = f^{-(k-2)}_\lambda(p_\lambda) = u_\lambda$. Then u_λ is the preimage of the pole p_λ in a neighborhood of the asymptotic value v_λ.
5. Each u_λ has infinitely many inverses in the asymptotic tract A_λ; denote these by $w_{\lambda,j}$, $j \in \mathbb{Z}$.

The main ideas for the proof are first to use the relation between λ and the free asymptotic value v_λ to carefully choose and fix a λ close to λ^* and then to construct a domain $\mathcal{T} \subset A_\lambda$ such that $f^k_\lambda(\mathcal{T}) \subset \mathcal{T}$. It will then follow from Schwarz's lemma that f^k_λ has an attracting fixed point in $\mathcal{T}$.
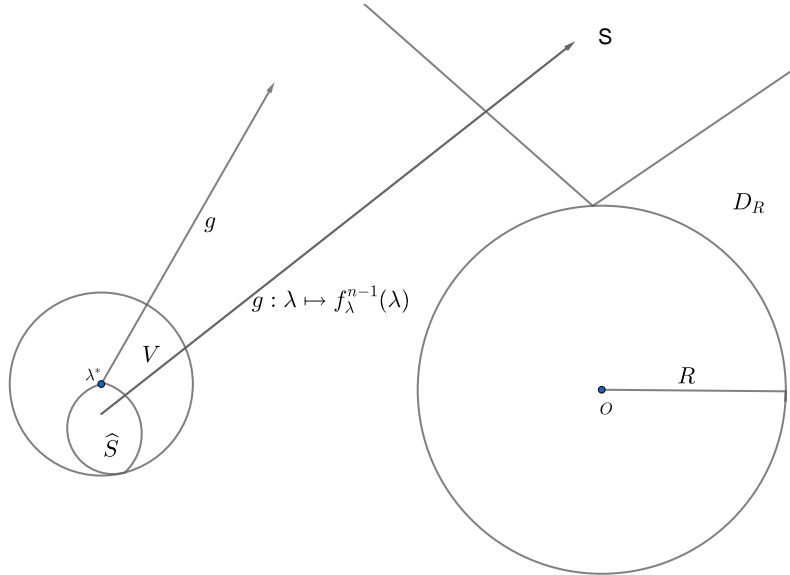

FIGURE 1. The map g_λ on parameter space. S is a sector inside all the asymptotic tracts A_λ, $\lambda \in V$. Note that $g(\lambda^*) = \infty$.

Choosing λ: (See Figure 1.) The asymptotic tract A_{λ^*} lies between two Julia rays r^*_1, r^*_2 and these span an angle $\theta_{pr} = 2\pi/pr$. We choose a small enough neighborhood V of λ^* such that for each $f_\lambda, \lambda \in V$, the Julia rays $r_1(\lambda), r_2(\lambda)$ of f_λ, lie within $\delta = \delta(V)$ of r^*_1, r^*_2 respectively. Each f_λ has an asymptotic tract A_λ between these rays and we can find a domain $S \subset \cap_{\lambda \in V} A_\lambda$ which lies inside the sector in $\widehat{\mathbb{C}}$ bounded by the rays r^*_1 and r^*_2 such that S is a sector in $\widehat{\mathbb{C}}$ with vertex at infinity and angle $\theta_{pr} - 2\delta$.

As above, let U be a neighborhood of v_{λ^*} and let V be the corresponding neighborhood in parameter space. For each $\lambda \in V$, $v_\lambda \in U$ and $f^{k-1}_\lambda(v_\lambda)$ is defined.

Define $g_\lambda : V \to \mathbb{C}$ by $g(\lambda) = f_\lambda^{k-1}(v_\lambda)$. Since V contains λ^* and g_λ is holomorphic, we can find $R = R_V$ such that $g(\lambda) = |f_\lambda^{k-1}(v_\lambda)| > R$. Set $D_R = \{z \in \mathbb{C} \,|\, |z| > R\}$ and let $\widehat{S}_V = g_\lambda^{-1}(S \cap D_R)$ be the "triangular" subset of V with vertex at λ^*. Note that the number of such triangular sets is equal to the order of the pole. If it is one, the set is unique. If it is greater than one, we choose one arbitrarily.
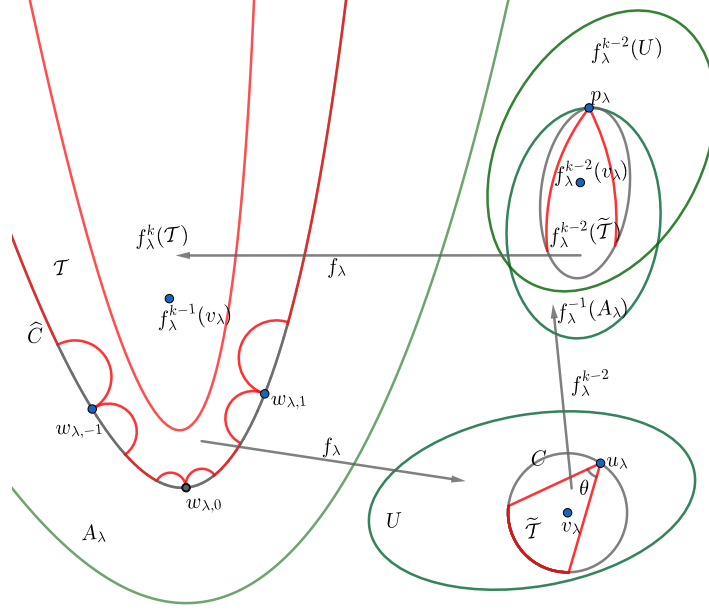

FIGURE 2. The dynamic plane for f_λ. The region $f_\lambda^n(\mathcal{T})$ is contained inside $\mathcal{T}$.

Constructing $\mathcal{T}$: (See Figure 2.) Now we work with a fixed $\lambda \in \widehat{S}_V$ and we set $U = f_\lambda(A_\lambda)$. Since $S \subset A_\lambda$ we have $f_\lambda(S) \subseteq U$. In U we have v_λ and $u_\lambda = f_\lambda^{-(k-2)}(p_\lambda)$. Let C be a circle with center v_λ and radius $|v_\lambda - u_\lambda| = \eta_\lambda$. Taking λ closer to λ^* if necessary, we may assume C lies in a compact subset of U; that is the disk $\{z \,|\, |z - v_\lambda| \leq \eta_\lambda\} \subset U$. Now $\widehat{C} = f_\lambda^{-1}(C) \subset A_\lambda$ and $f_\lambda : \widehat{C} \to C$ is an infinite to one cover so $\widehat{C}$ contains preimages $w_{\lambda,j}$, $j \in \mathbb{Z}$, of u_λ.

We want to approximate the distance from a point on $\widehat{C}$ to ∂A_λ. Let $\phi : A_\lambda \to \mathbb{H}_l$ be a conformal homeomorphism from A_λ to the left half plane $\mathbb{H}_l$ and let $\psi : U \setminus \{v_\lambda\} \to \mathbb{D}^*$ be a conformal homeomorphism from $U \setminus \{v_\lambda\}$ to the punctured unit disk $\mathbb{D}^*$, respectively chosen such that $f_\lambda = \psi^{-1} \circ \exp \circ \phi$.

Because ψ is a homeomorphism, $|\psi'(z)|$ is bounded above and below for z in a closed disk containing C. Applying Koebe's distortion theorem, there are positive constants K_1, K_1' such that for $z \in C$,

$$\frac{K_1 \eta_\lambda}{(1 + \eta_\lambda)^2} < |\psi^{-1}(z)| < \frac{K_1' \eta_\lambda}{(1 - \eta_\lambda)^2}.$$

We are assuming η_λ is small so we may assume it is less than $1/2$. Thus we can find closed annuli in $\mathbb{D}^*$ and U containing $\psi(C)$ and C respectively.

We then have, for appropriate positive constants, K_2, K_2',

$$K_2 + |\log \eta_\lambda| < |\operatorname{Re} \log \psi^{-1}(z)| < K_2' + |\log \eta_\lambda|. \tag{2}$$

This says that the lift $\log \psi^{-1}(C)$ of the circle C lies in a vertical strip W of bounded horizontal width in $\mathbb{H}_l$. Let $\widehat{W}$ denote $\phi^{-1}(W) \subset A_\lambda$; it is the lift of the annulus in U to A_λ.

The covering group $\mathbb{Z}$ for the exponential map acting on $\mathbb{H}_l$ pulls back to an infinite cyclic group Γ generated by a map $\gamma : z \mapsto \gamma(z)$ that is the covering group acting on A_λ under the map f_λ so that

$$\phi(\gamma(z)) = w + 2\pi i.$$

Thus

$$\phi'(z)dz = dw \text{ and } \phi'(\gamma(z))\gamma'(z)dz = dw.$$

This says that

$$\frac{dw}{dz} = \phi'(z) = \phi'(\gamma(z))\gamma'(z).$$

Because $|\phi'(z)|$ is bounded in the closure of a fundamental domain for Γ intersected with $\widehat{W}$, it is bounded for all w in $\widehat{W}$. Now because ϕ is a conformal homeomorphism, it preserves the hyperbolic density so that

$$\rho_{A_\lambda}(\gamma(z))|d\gamma(z)| = \frac{|dw|}{|\operatorname{Re} w|}$$

$$\rho_{A_\lambda}(\gamma(z))|\gamma'(z)dz| = \frac{|dw + 2\pi i|}{|\operatorname{Re}(w + 2\pi i)|} = \frac{|dw|}{|\operatorname{Re} w|}.$$

Therefore the hyperbolic density in A_λ is invariant under the action of γ. Equation (2), says that $|\operatorname{Re} w|$ is comparable to $|\log \eta_\lambda|$ which gives us the estimate we want; that is, there are positive constants K_3, K_3' such that

$$\min_{\zeta \in \partial A_\lambda, y \in \widehat{C}} |y - \zeta| \sim K_3 + K_3'|\log \eta_\lambda|.$$

Note that because we have a group action on A_λ, there will be a $\zeta \in \partial A_\lambda$ and a $y \in \widehat{C}$ in each fundamental region. Now let $\widetilde{\mathcal{T}}$ be a triangular region in U with one vertex at u_λ, two sides spanning an angle of $\theta < \theta_{pr}/m$, where m is the order of the pole p_λ joining u_λ to C, and third side an arc of the circle C; the sides are chosen so that $\widetilde{\mathcal{T}}$ contains v_λ in its interior. Next set $\mathcal{T} = f_\lambda^{-1}(\widetilde{\mathcal{T}})$. First note that $\mathcal{T} \subset A_\lambda$ and $f_\lambda^{-1}(\partial\widetilde{\mathcal{T}})$ is a doubly infinite curve in A_λ that stays a bounded distance from $\widehat{C}$, where the bound depends on η_λ and θ. The inverse images of sides of the triangle form scallops "above" $\widehat{C}$ (further inside A_λ).

Now look at $f_\lambda^{k-2}(\widetilde{\mathcal{T}})$. This is a triangular shaped region contained in $f^{k-2}(U)$ with a boundary point p_λ; it contains $f_\lambda^{k-2}(v_\lambda)$. Choose $y \in \widehat{C}$ realizing the minimum above so that $f_\lambda^{-1}(y)$ is contained in $f^{k-2}(U)$. Then, as p_λ is a pole of order m, we have

$$|f_\lambda^{-1}(y) - p_\lambda| \sim \frac{K_3}{|\log \eta_\lambda|^{\frac{1}{m}}}. \tag{3}$$

Since the derivative of f_{λ^*} along the orbit of v_{λ^*} from v_{λ^*} to p_{λ^*} is bounded and varies holomorphically with λ, the derivative of f_λ along the orbits of v_λ from v_λ to $f_\lambda^{k-2}(v_\lambda)$ and u_λ to p_λ are also bounded for $\lambda \in \widehat{S}_V$. Because $v_\lambda \in \widetilde{\mathcal{T}}$ and η_λ is small, when we map by f_λ^{k-2} the images of v_λ and points on $C \cap \widetilde{\mathcal{T}}$ are comparably close to the pole p_λ. Specifically, for some positive constant K_4, we have

$$|f_\lambda^{k-2}(v_\lambda) - p_\lambda| \sim K_4\eta_\lambda << \frac{K_3}{|\log \eta_\lambda|^{\frac{1}{m}}}.$$

Comparing this with the estimate (3) it follows that

$$f_\lambda^{k-1}(C \cap \widetilde{\mathcal{T}}) \subset \mathcal{T} \text{ and } f_\lambda^{k-1}(v_\lambda) \in f_\lambda^k(\mathcal{T}).$$

Therefore because v_λ is inside $\widetilde{\mathcal{T}}$,

$$f_\lambda^k(\mathcal{T}) \subset \mathcal{T}$$

so that f_λ^k has a fixed point in $\mathcal{T}$ by the Schwarz lemma. $\square$

Part 2. The Extended Tangent Family $\lambda \tan^p z^q$.

In this part of the paper we use the results above to prove that for family $\mathcal{F} = \{\lambda \tan^p z^q\}$, every shell component of period $n > 1$ is bounded. A corollary is that every capture component is bounded as well. The proof will follow from by studying the period 1 and period 2 components.

4. **The shell components of $\mathcal{F}$.** The family $\mathcal{F} = \{f_\lambda = \lambda \tan^p z^q\}$ is a subfamily of $\mathcal{M}_{p,q,2} = \{P \circ g \circ Q\}$ with $P(z) = z^p$ and $Q(z) = z^q$ and g in the one dimensional slice of $\mathcal{N}_2$ consisting of functions that fix 0 and have symmetric asymptotic values. The functions in $\mathcal{F}$ have one fixed critical point at 0 and have either one asymptotic value with mulitplicity $2q$ or two asymptotic values with multiplicity q that are opposite in sign. Specifically, the map $\tan z$ has two distinct asymptotic tracts and two asymptotic values, $\pm i$. Each of these tracts has q pre-images under Q^{-1}. If p is even, all $2q$ of the asymptotic tracts map onto punctured neighborhoods of the single point $i^p \lambda = (-1)^{p/2} \lambda$ which is the free asymptotic value v_λ. If p is odd, q of the asymptotic tracts map onto punctured neighborhoods of $i^p \lambda$ and the other q tracts map onto punctured neighborhoods of $(-i)^p \lambda$. In this case we choose $v_\lambda = i^p \lambda$ as the free asymptotic value. The other asymptotic value satisfies the relation $v_\lambda' = -v_\lambda$.

The punctured plane $\lambda \neq 0$ is thus a *dynamically natural slice* in $\mathcal{M}_{p,q,2}$ and the dynamics are determined by the forward orbit of v_λ.

The full set of shell components in this slice is denoted by $\mathcal{S} = \{\Omega\}$ and we divide it into subsets depending on the period of the cycle as follows:

Definition 4.1. If pq is even

$$\mathcal{S}_n = \{\Omega_n |\ f_\lambda \text{ has an attracting cycle of period } n\},$$

otherwise

$$\mathcal{S}_n = \{\Omega_n \mid f_\lambda \text{ has one attracting cycle of period } 2n \text{ or}$$

$$\Omega_n' \mid f_\lambda \text{ has two attracting cycles of period } n\}.$$

For readability we only include the subscript on Ω if the period is not obvious from the context.

Figure 3 shows the parameter plane for the family $f_\lambda(z) = \lambda \tan^2 z^3$. The period 1 shell components are yellow, the period 2 shell components are cyan blue. The capture components are green. The virtual centers are the black dots. They are enlarged because of roundoff error.
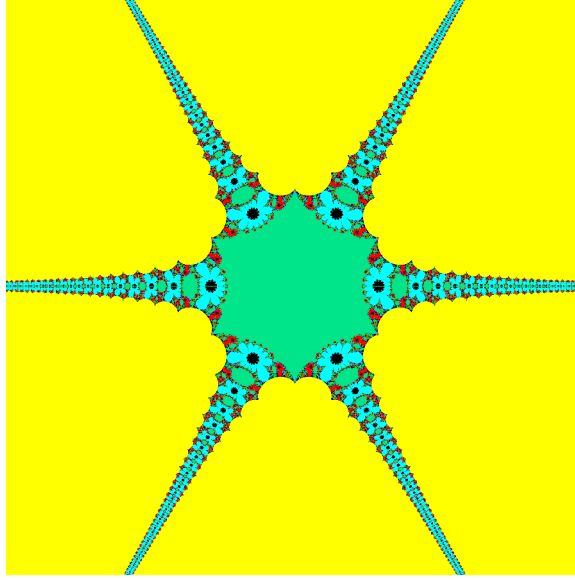
FIGURE 3. The parameter plane for $\lambda \tan^2 z^3$.

4.1. **Symmetries.** Note that for any λ, $f_{\bar{\lambda}}(\bar{z}) = \overline{f_\lambda(z)}$. In addition, if $\omega_k, k = 0, \ldots, q - 1$, are the qth roots of unity,

$$f_{\omega_k \lambda}(\omega_k z) = \omega_k \lambda \tan^p (\omega_k z)^q = \omega_k f_\lambda(z).$$

It follows that if $\Omega \in \mathcal{S}_n$ then both $\overline{\Omega}$ and $\omega_k \Omega$ are also.

Suppose pq is even so that f_λ has a single attracting cycle $\{z_1, \ldots, z_n\}$ of period n with multiplier $\mu(\lambda)$. Then $\{-z_1, \ldots, -z_n\}$ is a cycle for $f_{-\lambda}$ and $\mu(-\lambda) = \mu(\lambda)$.

If pq is odd then

$$f_\lambda(-z) = -f_\lambda(z) = f_{-\lambda}(z) \text{ and } f_{\bar{\lambda}}(z) = \overline{f_\lambda(\bar{z})}.$$

Assume that f_λ has two cycles of period n. They must be symmetric: that is they are $\{z_1, \ldots, z_n\}$ and $\{-z_1, \ldots, -z_n\}$ and they have the same multiplier.

Now $f_{-\lambda}(z_1) = -z_2$, $f_{-\lambda}(-z_2) = z_3$, $\ldots$, $f_{-\lambda}^m(z_1) = (-1)^m z_{m+1}$, so that if n is even, $f_{-\lambda}^n(z_1) = z_1$ and $f_{-\lambda}$ also has two cycles of period n. The set of periodic points of $f_{-\lambda}$ is the same as that for f_λ but they divide into different cycles for λ and $-\lambda$; again, $\mu(-\lambda) = \mu(\lambda)$. If, however, n is odd, $f_{-\lambda}^n(z_1) = -z_1$ and $f_{-\lambda}$ has a single cycle of period $2n$; it has multiplier $\mu^2(\lambda)$. We summarize this discussion as follows.

Proposition 4.2. *If Ω is a hyperbolic component of the λ plane, then $-\Omega, \bar{\Omega}$ and $\omega_k \Omega$, $k = 0, q - 1$ are all hyperbolic components. That is, the parameter plane is symmetric with respect to reflection in the real and imaginary axes, rotation by π and rotation by q^{th} roots of unity.*

If pq is even, $\mu(\lambda) = \mu(-\lambda)$ for $\lambda \in \Omega$, while if pq is odd there are two cases depending on the parity of n: either n is even and there are two cycles for $\lambda \in \Omega$, each has multiplier $\mu(\lambda)$ and there is a single cycle of double the period for $-\lambda \in -\Omega$ such that for this cycle, $\mu^2(\lambda) = \mu^2(-\lambda)$ or n is odd and the statement holds with the roles of λ and $-\lambda$ reversed.

4.2. **Components of $\mathcal{S}_1$.** In this section we will show that $\mathcal{S}_1$ consists of exactly $2q$ unbounded components arranged symmetrically around the origin. Let $\eta_k^+, k = 0, \ldots, q-1$, be the roots of $(\eta_k^+)^q = i$ and let $\eta_k^-, k = 0, \ldots, q-1$, be the roots of $(\eta_k^-)^q = -i$. If q is odd these are labeled so that $\eta_k^- = -\eta_k^+$ whereas if q is even they are labeled so that $\eta_k^- = \overline{\eta_k^+}$. We denote the set of rays in the directions of these vectors by $\tilde{\mathcal{R}} = \{\ell_k^\pm = si^{-p}\eta_k^\pm, s > 0, k = 0, \ldots, q-1\}$.

Theorem 4.3. *The set $\mathcal{S}_1$ consists of $2q$ unbounded components, $\Omega_k^\pm$, $k = 0, \ldots, q-1$, such that each is symmetric about the ray $\ell_k^\pm$. Note that if $\lambda \in \ell_k^\pm$, then $v_\lambda = s\eta_k^\pm$, $s > 0$.*

Moreover, for $\lambda \in \ell_k^\pm \cap \Omega^\pm$:

(i) *When both p and q are even f_λ has a single attracting fixed point z_λ and $\arg z_\lambda = \arg v_\lambda$;*

(ii) *When p is odd and q is even, again f_λ has a single attracting fixed point z_λ and either $\arg z_\lambda = \arg v_\lambda$ or $\arg(-v_\lambda)$;*

(iii) *When pq is odd f_λ either has two attracting fixed points, z_λ and $-z_\lambda$, or a single attracting period two cycle $\{z_\lambda, -z_\lambda\}$. Moreover, if $\lambda \in \ell_k^+$, f_λ has two attracting fixed points, z_λ on ℓ_k^+ and $-z_\lambda$ on ℓ_k^-, which attract v_λ and $-v_\lambda$ respectively; if $\lambda \in \ell_k^-$, then f_λ has one attracting cycle of period two, $\{z_\lambda, -z_\lambda\}$ which attracts both v_λ and $-v_\lambda$ and we can label the points so that $\arg z_\lambda = \arg v_\lambda$.*

Proof. If $f_\lambda(z) \in \mathcal{S}_1$ and has an attracting fixed point $z = z_\lambda$, then using the relation

$$\lambda = \frac{z}{\tan^p z^q}, \tag{4}$$

we compute that the multiplier $\mu(z)$ is

$$\mu(z) = \lambda pq z^{q-1} \tan^{p-1} z^q \sec^2 z^q = 2pq \frac{z^q}{\sin 2z^q}.$$

Set $u = 2z^q$, so that $\mu(z) = h(u) = pqu/\sin u$. The locus $|h(u)| = 1$ in the $u = x + iy$ plane consists of two branches, one in the upper half plane and one in the lower half plane. Call the unbounded regions defined by these curves $U^\pm$ respectively. Each is symmetric with respect to both the real and imaginary axes. Moreover the regions intersect the imaginary axis in the intervals $u = \pm ir^q$, $r > r_0$, respectively, where $h(\pm ir_0^q) = 1$. Inside these domains $|h(u)| < 1$. As $|y| \to \infty$, the branches are asymptotic to the curves

$$\frac{\pm e^{|y|}}{2pq} + iy.$$

For each u in $U^\pm$ there are q corresponding $z = z_\lambda$'s and these form $2q$ regions $V_k^\pm$, $k = 0, \ldots, q-1$ in $\mathbb{C}$. The rays in the directions $\eta_k^\pm$ are axes of symmetry for the $V_k^\pm$. We give the argument for $\eta_k = \eta_k^+$; the argument for η_k^- follows similarly. That is, we want to show that if $z_1 \in V_k$, then $z_2 = \eta_k^2 \bar{z}_1$ is also in V_k. Now if $z_1 \in V_k$, then $z_1^q \in U^+$ and so $z_2^q = -\bar{z}_1^q$ is also in U^+. Taking appropriate q^{th}-roots, the symmetry follows.

From equation (4) the images $\Omega_k^\pm$ of the $V_k^\pm$ are in $\mathcal{S}$. First, the symmetry of each of the $V_k^\pm$'s translates into a symmetry of the corresponding $\Omega_k^\pm$. Suppose

$z_1, z_2 = (\eta_k^+)^2 \bar{z}_1 \in V_k^+$. If p is even we have

$$\lambda(z_2) = \frac{\eta_k^{+2} \bar{z}_1}{\tan^p(-\bar{z}_1^q)} = \eta_k^{+2} \overline{\lambda(z_1)}$$

which implies that the image of V_k^+ is symmetric about the lines in direction η_k^+. A similar statement holds for V_k^- about the lines in direction η_k^-. If, however, p is odd,

$$\lambda(z_2) = \frac{\eta_k^{\pm 2} \bar{z}_1}{\tan^p(-\bar{z}_1^q)} = -\eta_k^{\pm 2} \overline{\lambda(z_1)}$$

so that the images of $V_k^{\pm}$ are symmetric about the rays in directions $i\eta_k^{\pm}$.

To see that there are $2q$ distinct domains $\Omega_k^{\pm}$, assume that z is on a line of symmetry for $V_k^{\pm}$; that is, $z = s\eta_k^{\pm}$, $s > r_0$. By equation (4) we have

$$\lambda(i\eta_k^{\pm}) = \frac{s\eta_k^{\pm}}{\tan^p(\pm is^q)} = \frac{s\eta_k^{\pm}}{(\pm i)^p \tanh^p(s^q)}.$$

If pq is even, on the rays where $z = s\eta_k^+$ and $z = s\eta_k^-$ the arguments of the corresponding λ's under the relation (4) are different, so that the images of V_k^+ and V_k^- are different. Therefore there are $2q$ components in $\mathcal{S}_1$ corresponding to $V_k^{\pm}$; these are denoted by $\Omega_k^{\pm}$. Moreover, if $z = s\eta_k^+$ and p is even, then $\arg v_\lambda = \arg z$. If however p is odd and q is even, then if $z = s\eta_k^+$, $\arg v_\lambda = \arg z$ but if $z = s\eta_k^-$, then $\arg v_\lambda = \arg z + \pi$. In either case, both z and v_λ are perpendicular to λ.

If pq is odd, then $\lambda(z)$ is an even function,

$$\frac{z}{\tan^p z^q} = \frac{-z}{\tan^p(-z)^q},$$

so that both $V_k^{\pm}$ have the same image Ω_k^+. This gives us q components in $\mathcal{S}$. For λ in Ω_k^+, f_λ has two fixed points z_1 and z_2 and they both have the same multiplier, $\mu(\lambda)$. Moreover on the symmetry lines, $z = s\eta_k^{\pm}$, we have $\arg v_\lambda = \arg s\eta_k^{\pm}$. Consider the q components $\Omega_k^- = \{\lambda \mid -\lambda \in \Omega_k^+\}$. As we saw in the discussion of Proposition 4.2, when $\lambda \in \Omega_k^-$, f_λ has one attracting cycle with period 2 and multiplier $\mu^2(\lambda)$ so these components are also in $\mathcal{S}_1$ and there are $2q$ components in $\mathcal{S}_1$ in this case as well. $\square$

The proof of Theorem B and the statement of Theorem 4.3 tell us something about the structure of the parameter plane for the family $\mathcal{F}_\lambda$ in a neighborhood of a virtual center. We have

Corollary 4.4. *In the parameter plane of the family $\mathcal{F}_\lambda$, there are $2pq$ shell components of period k that meet at every virtual center parameter of order $k > 1$. That is, the virtual center parameter is a common boundary point of these $2pq$ components. Infinity is a common boundary component of $2q$ components.*

Proof. In the proof of Theorem B we used that fact that the virtual center parameter λ^* corresponds to a particular prepole p_{λ^*} of f_{λ^*} and used one of the asymptotic tracts at p_{λ^*} in our construction to find the shell component at λ^*. There are $2pq$ tracts at each prepole; these correspond to the asymptotic values tied to v_λ by the functional relation. We could have used any one of these in the argument above to obtain a shell component at the virtual center. $\square$

4.3. **Separating lines.** In this section we discuss the rays in the λ plane spanned by the roots of $v_\lambda^q = 1$ and their negatives.

Let $\omega_k^\pm$, $k = 0, \ldots, q - 1$ be the roots of $\omega_k^q = 1$ and $(\omega_k^-)^q = -1$ respectively. Since $v_\lambda = i^p \lambda$ is the free asymptotic value, the rays in the λ plane we are interested in are those spanned by $i^{-p}\omega_k^\pm$. Denote them by $\tilde{\ell}_k = i^{-p} r \omega_k^\pm$, $r > 0$.

Lemma 4.5. *For the family $\mathcal{F}_\lambda$,*

(i) *If pq is even, none of the rays in the set*

$$\tilde{\mathcal{R}}' = \{i^{-p} r \omega_k^\pm, r > 0\}_{k=1}^q$$

intersects any component of $\mathcal{S}$; that is $\mathcal{S} \cap \tilde{\mathcal{R}}' = \emptyset$ for all k.

(ii) *if pq is odd, at each virtual cycle parameter λ on the ray $\tilde{\ell}_k$, such that v_λ is a pre-pole of f_λ of order n, n is odd and there are two components of $\mathcal{S}_{n+1}$ intersecting the ray. In one of these components there are two periodic cycles of order $n+1$ and in the other there is a single cycle of order $2(n+1)$ attracting both asymptotic values.*

Proof. Set $\lambda = r i^{-p} \omega_k^\pm$ for some real r so that v_λ^q is real. Notice that this implies $f_\lambda(v_\lambda)$ lies in the same line as λ. We have to look at the parities of p and q.

To prove (i), suppose first that p is even. Then λ and v_λ are in the same line. Moreover, since λ^q is real, it follows that λ, v_λ, and all its images under f_λ, lie on the same line, although perhaps on opposite sides of the origin. Therefore this line is invariant under f_λ and the restriction of f_λ to this line is conjugate to a real-valued function $f_r(t) = r \tan^p t^q$.

Next suppose p is odd and q is even. Then λ and v_λ are in perpendicular lines. Since q is even, λ^q is real so $(f_\lambda(v_\lambda))^q$ is real, and $f_\lambda^2(v_\lambda)$ is also on the line through λ, as is the rest of its orbit. That is, f_λ is invariant on the line through λ and the orbit of v_λ eventually lands on the line. Again the restriction of f_λ to this line is conjugate to a real-valued function $f_r(t) = r \tan^p t^q$.

We claim that in these cases, f_λ cannot have an attracting cycle other than the super-attracting fixed point 0. We consider the family of real valued functions $f_r(t) = r \tan^p t^q$. On the interval $(0, \sqrt[q]{\pi/2})$, f_r, f_r' and f_r'' are all positive and f_r goes from 0 to infinity. Therefore there is one fixed point z_0 inside this interval. Because 0 is a superattracting fixed point, its basin contains the interval $(0, z_0)$. By symmetry, about the imaginary axis if p is even and about the origin if p is odd, the basin of 0 contains the interval $I_0 = (-z_0, z_0)$. Since $f_r''(t) > 0$, $f_r'(t) > 1$ for t in $(z_0, \sqrt[q]{\pi/2})$, and $\tan t$ has period π, $|f_r'(t)| < 1$ only in $\mathcal{I}$, the union of the translates of the interval I_0.

We claim there are no other attracting periodic cycles in $\tilde{\mathcal{R}}'$. If there were such a cycle, $\{z_1, \ldots, z_n\}$, its points would have to be outside $\mathcal{I}$ and its multiplier would have to satisfy $\Pi_{i=1}^n |f_r(z_i)| < 1$. This cannot happen since none of these factors can be less than 1.

To prove (ii), we have both p and q odd. Then λ^q and $(f_\lambda(v_\lambda))^q$ are pure imaginary. Therefore

$$f_\lambda^2(v_\lambda) = \lambda \tan^p (f_\lambda(v_\lambda))^q = \pm i \tanh^p (f_\lambda(v_\lambda)^q$$

is in the line containing v_λ. The orbit of v_λ thus alternates between two perpendicular lines. So if it approaches an attracting cycle, that cycle must have even period and hence λ does not belong to $\mathcal{S}_1$.

Suppose λ is on the ray spanned by $ri^{-p}\omega_k$, and is inside one of the components Ω_k intersecting it. The multiplier of the cycle is real and monotonic in Ω_k. The virtual center λ^* of Ω_k is therefore on the ray. By Corollary 4.4, there are $2pq$ components at λ^* and since pq is odd, as we move around λ^*, they alternate between those with two cycles of period $n+1$ and one cycle of period $2(n+1)$ attracting both asymptotic values. Again since pq is odd, there is one of each type intersecting the ray. $\qquad\square$

4.4. **Components of $\mathcal{S}_2$.** By Theorem 3.3, each component in $\mathcal{S}$ is simply connected and the multiplier map is a universal covering. Moreover, each component has a virtual center on its boundary where the limit of the multiplier is zero, and if this is finite, it is a virtual center parameter so that the asymptotic value is a pre-pole. For completeness, we include a proof of the last statement for components of $\mathcal{S}_n$ so that we can use it to characterize the virtual centers of $\mathcal{S}_2$.

Lemma 4.6. *If the shell component $\Omega \in \mathcal{S}_n$, is bounded, then the virtual center λ^* satisfies $f_{\lambda^*}^{n-1}(v_{\lambda^*}) = \infty$. That is, the asymptotic value is a pre-pole of order n and the virtual center is a virtual cycle parameter.*

Proof. Without loss of generality, we assume f_λ has an attracting periodic cycle of period n; the proof is similar if f_λ has one attracting cycle of period $2n$. Let $\{z_0, z_1, \cdots, z_{n-1}\}$ be the attracting cycle of f_λ, and suppose z_0 lies in an asymptotic tract. The multiplier of the cycle is

$$\prod_{i=0}^{n-1} f_\lambda'(z_i) = \prod_{i=0}^{n-1} \frac{\lambda pq z_i^{q-1} \tan^{p-1} z_i^q}{\cos^2(z_i^q)} = (2pq)^n \prod_{i=0}^{n-1} \frac{z_i^q}{\sin(2z_i^q)}$$

If the multiplier tends to 0 as $\lambda \to \lambda^*$, then at least one of the factors $z^q/\sin(2z^q)$ tends to 0, which in turn implies that the imaginary part z_0^q is unbounded. Since z_0 is in the asymptotic tract, it follows that as $\lambda \to \lambda^*$, $|\operatorname{Im} z_0^q| \to \infty$ and $z_1 = \lambda \tan^p z_0^q \to v_{\lambda^*}$. Because $z_0 = \lambda \tan^p z_{n-1}^q$ and λ is bounded, it follows that $z_{n-1}^q \to k\pi + \pi/2$ as $\lambda \to \lambda^*$ and $f_{\lambda^*}^{n-1}(v_{\lambda^*}) = \infty$. $\qquad\square$

Lemma 4.7. *If Ω is a component of $\mathcal{S}_2$, then its virtual center is a solution of $v_\lambda^q = k\pi + \pi/2$ for some integer k.*

Proof. By Lemma 4.6, we only need to show that each component of $\mathcal{S}_2$ is bounded.

Suppose there is an unbounded component $\Omega \in \mathcal{S}_2$ and f_λ has a periodic cycle $\{z_1, z_0\}$. We will show that if Ω is unbounded then both points lie in asymptotic tracts, and therefore that p and q are both odd. This in turn implies that f_λ has only one attracting cycle of period 2 so that $\Omega \in \mathcal{S}_1$ and not in $\Omega \in \mathcal{S}_2$.

Let $\lambda(t) \to \lambda^*$, $t \to \infty$ be a path in Ω such that the multiplier of the periodic cycle $\mu(\lambda(t)) \to 0$. If Ω is unbounded, the multiplier of the periodic cycle has absolute value 1 for any finite point on the boundary. Therefore, the "virtual center" λ^* of Ω is at infinity. (We use quotes because infinity is a parameter singularity.) From the proof of Lemma 4.6, the point $z_0(\lambda(t))$ of the cycle must be in the asymptotic tract and $|\operatorname{Im} z_0(\lambda(t))^q| \to \infty$ and thus $z_1(\lambda(t)) = \lambda(t)\tan^p z_0(\lambda(t))^q \asymp i^p\lambda(t)$ also goes to infinity.

Suppose $z_1(\lambda(t))^q = x_1(t) + iy_1(t) \asymp i^{pq}\lambda(t)^q$. We claim that if $y_1(t)$ goes to infinity, then $z_1(\lambda(t))$ is also in an asymptotic tract. If not, then $x_1(t) \to \infty$ but $y_1(t)$ stays bounded. Take a sequence $z_1(t_k)$ such that $x_1(t_k) = k\pi$. Then

$$\tan(x_1(t_k) + iy_1(t_k)) = \tan(iy_1(t_k)) = i\tanh(y_1(t_k)) = B_k i,$$

where B_k is real and bounded.

Since $z_0(t_k) = \lambda \tan^p(z_1(t_k))^q = \lambda(B_k i)^p$, $z_0(t_k)^q/z_1(t_k)^q = B_k^{pq} \in \mathbb{R}$; that is, $z_0(t_k)$ is asymptotically parallel to $z_1(t_k)$ with a bounded ratio so that the imaginary part of $z_0^q(t)$ remains bounded. This contradicts our assumption that $z_0^q(t_k)$ is unbounded, so it follows that Ω is bounded as claimed. $\square$

5. **Theorem C.** We now have all the ingredients to prove Theorem C.

Theorem C. *All the components of $\mathcal{S}_n$, $n > 1$ are bounded.*

Proof. We proved in Theorem 4.3 and Lemma 4.5 that for the family $\mathcal{F}_\lambda$, $\mathcal{S}_0$, consists of $2q$ unbounded components, $\{\pm\Omega_1, \cdots, \pm\Omega_q\}$. These are symmetric about the rays $\tilde{\mathcal{R}} = \{\pm r\eta_k, r > r_0\}_{k=1}^q$ and separated by the rays $\tilde{\mathcal{R}}' = \{\pm r\omega_k, r > 0\}_{k=1}^q$.

The boundary of each Ω_k is an analytic curve defined by the relation $|f_\lambda'(z_0)| = 1$ where z_0 is the fixed point or, if pq is odd and there is a single period 2 cycle, $\{z_0, z_1\}$, by the relation $|f_\lambda'(z_0)f_\lambda'(z_1)| = 1$. We assume now that there are two fixed points. The discussion for the period 2 cycle case is essentially the same. Along $\partial\Omega_k$ there is a sequence of points where $f_\lambda'(z_0) = e^{2\pi i p/q}$ with $p/q \in \mathbb{Q}$ and the fixed points have become parabolic. At these points there is a standard bifurcation creating a new "bud component" attached to $\partial\Omega_k$, and tangent to it, in which the fixed point is now repelling and a new attracting cycle of period q appears. At the sequence $\nu_{k,m}$ where $f_\lambda'(z_0) = -1$ the new cycle is of period two. We label the bud component $\Omega_{2,k,m}$; it belongs to $\mathcal{S}_2$. By Lemma 4.7, $\Omega_{2,k,m}$ is bounded and has a virtual center, say $s_{m,k}$ on the ray in $\tilde{\mathcal{R}}$ separating Ω_k from the next one in order around the origin; for argument's sake assume it is $-\Omega_{k+1}$.

If we now look at the points on the boundary of $-\Omega_{k+1}$, there is a sequence $\nu_{k+1,m'}$ where the multiplier of the cycle is -1 and there is a bud component $\Omega_{2,k+1,m'}$ with virtual center $s_{m',k+1}$ on the same ray of $\tilde{\mathcal{R}}$ separating Ω_k and $-\Omega_{k+1}$. Choose $\nu_{k+1,m'}$ so that $s_{m',k+1} = s_{m,k}$. We can do this since the boundaries of Ω_k and $-\Omega_{k+1}$ are both asymptotic to the ray containing the centers.

At $s_{m,k}$ there are $2pq$ shell components and $\Omega_{2,k,m}$ and $W_{2,k+1,m'}$ are two of them. Now we draw a curve in $\Omega_{2,k,m}$ from $\nu_{k,m}$ to $s_{m,k}$ and another from $s_{m,k}$ to $\nu_{k+1,m'}$ in $\Omega_{2,k+1,m'}$. We then choose another parabolic point $\nu_{k+1,m''}$ on the boundary of $-\Omega_{k+1}$ whose bud component $\Omega_{2,k+1,m''}$ has a center $s_{m'',k+1}$ on the next ray in order around the origin. We draw a simple curve in $-\Omega_{k+1}$ from $\nu_{k+1,m'}$ to $\nu_{k+1,m''}$. Continuing around, in the next component Ω_{k+2}, we find the bud component that shares $s_{m'',k+1}$ as its center and draw a simple curve through the bud components joining $-\Omega_{k+1}$ and Ω_{k+2}. We continue in this manner until we get back around to the original Ω_k. We join all the curves in the $\pm\Omega_k$ and the bud components between them. The result, γ_m is a simple closed around the origin.

In this way, choosing the $\nu_{k,m}$ carefully and systematically, we can create a nested sequence of curves γ_m around the origin. Any other component of $\mathcal{S}$ is disjoint from the components $\pm\Omega_k$ and the bud components $\Omega_{2,k,m}$ and so must lie inside one of the γ_m and is therefore bounded. $\square$

An immediate corollary of the above proof is

Corollary 5.1. *All the capture components in the dynamically natural slice are bounded.*

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