Holomorphic Motions and Lipman Bers
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Frederick P. Gardiner and Linda Keen [1]

Preface

We remember Lipman Bers' with affection. Especially memorable were his expository lectures and his research efforts with students. Whether to a newcomer or a veteran, he offered enthusiasm, encouragement, and a wealth of ideas.

Towards the end of his life, the proliferation of work generated by his research made it impossible for any one person to keep up with all the new developments. In the seventies, Thurston presented a program to classify three-dimensional manifolds that incorporated a new approach to Teichmüller theory. By using Teichmüller theory to glue together in a geometric way the surface boundaries of certain open three dimensional manifolds, he was able to show that a wide class of compact three dimensional manifolds can be realized as three dimensional hyperbolic space factored by certain discrete subgroups of the group of Möbius transformations. Teichmüller theory, which had appeared to be a subject concerning geometry in only one or two dimensions, became important for understanding three-dimensional space. In the eighties conformal dynamics, the study of which had been started by Julia at the beginning of the century, was revived. Impetus for research here came from the recently acquired ability to obtain computer-drawn pictures of geometrical shapes forced by the simple algorithms implicit in the subject. Also in the eighties, a school of physicists took an interest in Teichmüller theory as a preliminary foundation for part of string theory. The intensive

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activity in low-dimensional topology, conformal dynamics and string theory forced researchers to start from scratch and rethink all of Teichmüller theory.

Bers was thrilled by the influence of Thurston, Sullivan and others who entered the field to use it in other contexts. It spurred him to further efforts and enhanced the extent and quality of his research. His joy in these developments is an inspiring example of a youthfulness of mind lasting to the end of a long life.

In the more than twenty years since Bers' death, there has been substantial progress: Thurston's Geometrization Program giving a full characterization of all possible geometric structures for three manifolds was completed last year and many of the open questions in conformal dynamical systems have been answered. A major tool in all of this work is the idea of a *holomorphic motion*. The term was first coined by Màñé, Sad and Sullivan in their paper [38]. Although Bers first learned about it in a lecture by Sullivan in the mid-eighties, the underlying idea goes back to his own work. It plays a big role in *Simultaneous Uniformization* and the representation of Teichmüller space as a complex analytic Banach space of holomorphic quadratic differentials, the *Bers embedding*. (See the article by Wolpert in this volume for an historical discussion and more details.) The ensuing theorems on the extendibility of holomorphic motions by Sullivan and Thurston [46], of Bers and Royden [8] and, finally, of Slodkowski [45] and of Astala [5], clarified the whole subject of quasiconformal mapping and deformation theory for dynamical systems and Kleinian groups.

In the 1998 Proceedings of the First Ibero-American Congress on Geometry we contributed an expository article dedicated to Lipman Bers [28]. It was organized around the topic of holomorphic motions with applications to holomorphic dynamics and quasifuchsian groups. Because we wrote that description more than fifteen years ago, this topic has ballooned into many areas of research. On the occasion of the 100th anniversary of Lipa's birth and in light of the importance holomorphic motions have played since then we have decided to reprint that exposition here with minor editorial corrections and to add an appendix that summarizes some of the subsequent developments. Of course while we cannot vouch for what Lipa's mathematical tastes might now be, we have tried to choose topics that we believe would most appeal to him.

Introduction

This article is an expository account of why mappings in a holomorphic motion $f_t(z)$ of the $z-$plane parameterized by a complex variable t with $|t| < 1$ are necessarily quasiconformal maps and an account of some geometric applications of this fact. Our approach is different from the one ordinarily given. In the first section we demonstrate how the generalized Schwarz lemma implies f_t is quasiconformal with dilatation less than or equal to $\frac{1+|t|}{1-|t|}$ and Hölder continuous with Hölder exponent $\frac{1-|t|}{1+|t|}$. Extremal length methods are not used to obtain the precise Hölder exponent. In the second section, we describe Thurston's construction of a quasifuchsian manifold with prescribed geometry as a holomorphic motion.

A holomorphic motion of a closed set J in the extended complex plane $\bar{\mathbb{C}}$ is a curve $f_t(z)$, defined for every z in J and for every t in the unit disk, such that:

i) $f_0(z) = z$ for all z in J,
ii) $z \mapsto f_t(z)$ is injective as a function from J into $\overline{\mathbb{C}}$ and
iii) $t \mapsto f_t(z)$ is holomorphic for $|t| < 1$ and for each fixed z in J.

The variable t is the complex time-parameter and as t changes the set $J_t = f_t(J)$ moves in $\overline{\mathbb{C}}$. Although J may start out to be as smooth as a circle and although the points of J are moving holomorphically, for every $t \neq 0$, J_t can be an interesting fractal set with fractional Hausdorff dimension.

Any holomorphic motion of a closed subset of $\overline{\mathbb{C}}$ defined for $|t| < 1$ automatically extends to a holomorphic motion of $\overline{\mathbb{C}}$. Moreover, for each fixed t, the mapping $z \mapsto f_t(z)$ is quasiconformal with dilatation K less than or equal to $\frac{1+|t|}{1-|t|}$.

Thus injectivity and holomorphic dependence automatically lead to quasiconformality. Viewed from first principles, this is surprising because quasiconformality is a geometric idea about distortion. By definition, an arbitrary orientation preserving homeomorphism is quasiconformal if it distorts standard shapes by a bounded amount. It may distort sizes by a large amount; no assumption is made on this point. A beautiful first result of the theory is that quasiconformal mappings are necessarily Hölder continuous with Hölder exponent $1/K$. (This result is due to Lavrientiev or to Ahlfors, although it is difficult to know who deserves credit because each attributed it to the other [7].) Thus, quasiconformal control of distortion of shape implies some control on distortion of size. Moreover, a normalized family of quasiconformal mappings with a common bound on the dilatation of every member is equicontinuous.

The realization that any holomorphic motion defined for $|t| < 1$ and for z in some closed set could be extended to all z in $\overline{\mathbb{C}}$ and for all $|t| < 1$ was the combined result of several research papers. The first one by Màñé, Sad and Sullivan presented the idea of a holomorphic motion and showed that mappings in the motion necessarily have quasiconformal extensions. Then the papers by Sullivan and Thurston [46] and by Bers and Royden [8] showed the local holomorphic extendibility to a domain $|t| < \epsilon$ where ϵ is universal, not depending on J or the particular holomorphic motion. (Bers and Royden showed that $\epsilon \geq 1/3$.) Finally, Slodkowski [45] showed that any such motion could be extended to a holomorphic motion of the whole complex plane defined on the full unit disk, $|t| < 1$.

Slodkowski's result complements nicely the 1961 paper by Ahlfors and Bers [4] in which singular operators are used to solve the global Beltrami equation

$$f_{\overline{z}}(z) = t\mu(z) f_z,$$

where μ is an arbitrary L_∞ complex-valued function defined on $\overline{\mathbb{C}}$ with $||\mu||_\infty = 1$ and $|t| < 1$. The use of singular integral operators to solve the Beltrami equation had been carried through earlier by Bojarski in [10]. The solution $f^{t\mu}$ is a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ uniquely determined up to post-composition by a Möbius transformation. A major part of the emphasis of the paper by Ahlfors and Bers was that $f^{t\mu}(z)$ depends holomorphically on t for $|t| < 1$. When $t = 0$, $f^{t\mu}$ is the identity and, for each fixed $|t| < 1$, $f^{t\mu}(z)$ is an injective function of z. In other words, their paper showed the existence of many holomorphic motions of $\overline{\mathbb{C}}$. The new theory of holomorphic motions showed their emphasis on the holomorphic dependence had further significance: holomorphic dependence combined with injectivity of $z \mapsto f^{t\mu}(z)$ implies $z \mapsto f^{t\mu}(z)$ is quasiconformal; the holomorphic dependence of the quasfuchsian groups defined by $G_0 \mapsto f^{t\mu}(z) \circ G \circ (f^{t\mu}(z))^{-1}$ for fixed quasifuchsian $G_0 \subset PSL(2, \mathbb{C})$ and quasiconformal $f^{t\mu}(z)$ satisfying $\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z)$ for all $g \in G_0$ has been used to develop the complex structure on Teichmüller space.

In Section 1 of this paper we give an analytic application of holomorphic motions. We show why a holomorphic motion of injective mappings of the plane starting at the identity necessarily consists of mappings which are Hölder continuous and which are quasiconformal. The method of proof of Hölder continuity shows how the $(|\epsilon \log \epsilon|)^{-1}$ growth of the infinitesimal form of the Poincaré metric near a puncture is related, after integration of an ordinary differential equation, to the Hölder continuity of the mappings.

In Section 2 we give a geometric application. In his Princeton lecture notes [**?**], Thurston described a method to construct hyperbolic 3-manifolds. The main step is to construct a quasifuchsian manifold. Such a manifold is the quotient of hyperbolic 3-space by a quasifuchsian group. The construction, which is a generalization of topological methods of Dehn, is geometric. It involves "bending" the surface along geodesics; the resulting manifold not only is quasifuchsian, but it has prescribed geometric properties. We describe Thurston's construction and indicate how to prove, as a consequence of the main theorems on holomorphic motions, that it yields quasifuchsian manifolds with the prescribed geometry.

1 Quasiconformal mapping

The goal of this section is to explain why a holomorphic motion of $\mathbb{C}$ necessarily moves through quasiconformal mappings. Our first step is to show that f_t is Hölder continuous with Hölder exponent $\frac{1-|t|}{1+|t|}$.

We show this exponent by analyzing the tangent vectors to the curve f_t. Let $V_t(z)\frac{\partial}{\partial z}$ be the vector field defined by the equation

$$f_{s+t} \circ (f_t)^{-1}(z) = z + sV_t(z) + o(t).$$

Since $f_t(0) = 0, f_t(1) = 1$ and $f_t(\infty) = \infty$, for each $|t| < 1$, the vector field $V_t(z)\frac{\partial}{\partial z}$ vanishes at $0, 1$ and ∞. Note that a continuous vector field $V(z)\frac{\partial}{\partial z}$ vanishes at ∞ if, and only if, $V(z)/z^2$ approaches 0 as $z \to \infty$.

A cross-ratio Q of four distinct points cannot equal $0, 1$ or ∞. Since f_t is injective, the same cross-ratio Q_t of f_t applied to these four distinct points is also not equal to $0, 1$ or ∞. Therefore, by applying Schwarz's lemma to the map $t \mapsto Q_t$, from $|t| < 1$ into $\Sigma = \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$, we find that the speed of Q_t at time t measured with the Poincaré metric on Σ does not exceed $\frac{2}{1-|t|^2}$.

Let ρ be the infinitesimal form of the Poincaré metric on Σ normalized to have constant negative curvature equal to -4. Assume z_1, z_2, z_3 and z_4 are four distinct points and let $A = z_2 - z_1$, $B = z_3 - z_2$, $C = z_4 - z_3$, and $D = z_1 - z_4$. We call z_1 and z_2 the initial and terminal points of A, and use the notation

$$V'(A) = \frac{V(\textit{terminal point}) - V(\textit{initial point})}{\textit{terminal point} \ - \ \textit{initial point}}.$$

If we choose Q to be the cross-ratio $\frac{AC}{BD}$, a calculation shows that the speed of Q_t at $t = 0$ is

$$|\rho(Q)Q\{V'(A) - V'(B) + V'(C) - V'(D)\}|. \tag{1}$$

Since $V(z)/z^2$ approaches 0 as $z \to \infty$, it follows that if $z_4 = \infty$ then the term $V'(C) - V'(D)$ in (1) vanishes. By letting A be the interval from z to 0 and B be the interval from 0 to 1, we find $Q = z$, $V'(A) = -V(z)/z$ and $V'(B) = 0$. Thus,

$$|\rho(z)V_t(z)| \leq \frac{2}{1-|t|^2}.$$

Consequently, by the well-known estimate for the Poincaré metric in a neighborhood of a puncture [2], for every $\epsilon > 0$, there is a $\delta > 0$, such that if $|z| < \delta$ then

$$|V_t(z)| \leq (\frac{1+\epsilon}{1-\epsilon})|z|(\log\frac{1}{|z|})\frac{2}{1-|t|^2}. \tag{2}$$

In fact, if $\epsilon = \exp(-d_n)$ where $d_n = \log\log 2^n$, one can choose $\delta = \frac{1}{2^n}$.

Lemma 1 *Suppose f_t is a holomorphic motion of $\mathbb{C}$ such that $f_t(0) = 0, f_t(1) = 1$ and $f_t(\infty) = \infty$, for each $|t| < 1$. Then there exists a constant C depending only on R such that if $|z| < R$ and $|w| < R$ and if $|z - w| < \frac{1}{2}$, then*

$$|f_t(z) - f_t(w)| \leq C|z-w|^{\frac{1-|t|}{1+|t|}}.$$

PROOF. Let $\Delta(t) = |f_t(z) - f_t(w)|$.

Step 1. We first consider the case $w = 0$ and $|z| < 1/2$ so $\Delta(t) = |f_t(z)|$. Since one of the cross ratios of the quadruple $f_t(z), 0, 1, \infty$ is $f_t(z)$, by the bound on the velocity given in (2), we find, for $|z| \leq \frac{1}{2^n}$,

$$\frac{d}{dt}\Delta(t) \leq B\frac{2}{1-|t|^2}\Delta(t)\log\frac{1}{\Delta(t)},$$

where $B = \frac{1+\epsilon_n}{1-\epsilon_n}$ and $\epsilon_n = 1/\log 2^n$. The solution to this differential inequality with the inequality replaced by equality is

$$\Delta(t) = \Delta(0)^{(\frac{1-|t|}{1+|t|})^B}.$$

By the inequality principle for ordinary differential equations, we find

$$|f_t(z)| \leq |z|^{(\frac{1-|t|}{1+|t|})^B}.$$

Step 2. The last term in the previous inequality can be replaced by $C|z|^{\frac{1-|t|}{1+|t|}}$. To prove this assume that $\frac{1}{2^{n+1}} \leq |z| \leq \frac{1}{2^n}$ and let $\alpha = \frac{1-|t|}{1+|t|}$. We wish to find a constant C such that

$$|z|^{\alpha^B} \leq C|z|^{\alpha}.$$

It is sufficient to find C such that

$$|1/2^n|^{\alpha^B} \leq C|1/(2^{n+1})|^{\alpha}.$$

This is equivalent to finding an upper bound for

$$(\alpha - \alpha^B)(n \log 2) = \alpha(1 - \alpha^{(B-1)})n \log 2.$$

Since $0 < \alpha < 1$, this is equivalent to finding an upper bound for

$$(B-1)n(\log 2)\alpha(\log(\frac{1}{\alpha})). \tag{3}$$

But by (2), $B - 1 = \frac{2\epsilon_n}{1-\epsilon_n}$ and $\epsilon_n = \frac{1}{n \log 2}$ and thus the quantity displayed in (3) is bounded independently of n.

Step 3. Now suppose g_t is a holomorphic motion fixing three points $0, z$ and ∞ and assume $|z - w| \leq \min\{1/2, |z|/2\}$. Then the same arguments as above yield the similar inequality:

$$|g_t(w) - z| \leq C|w - z|^{\frac{1-|t|}{1+|t|}}.$$

Step 4. Apply Step 3 to $g_t(w) = \frac{f_t(w)}{f_t(z)}$, a motion which fixes $0, z$ and ∞. Since $f_t(z)$ is bounded for $|t| \leq |t_0| < 1$, we obtain

$$|f_t(w) - f_t(z)| \leq C|w - z|^{\frac{1-|t|}{1+|t|}},$$

where C is a new constant equal to the product of the constant in Step 3 and the bound on $f_t(z)$. $\square$

The following result is given on page 60 of [1].

Lemma 2 *If $|t| < 2-\sqrt{3}$, equivalently, if the hyperbolic distance from 0 to t is less than $\frac{1}{2}\log 3$, and if $f_t(\infty) = \infty$, then the image of oriented vertices of any equilateral triangle under $z \mapsto f_t(z)$ are mapped to vertices of a triangle with the same orientation.*

PROOF. Let z_1, z_2, z_3 be the vertices of such a triangle labeled in counter-clockwise order and let A be the segment from z_1 to z_2, B be the segment from z_2 to ∞, C be the segment from ∞ to z_3 and D the segment from z_3 to z_1. Then

$$Q = \frac{AC}{BD} = \frac{A}{-D} = \frac{z_2 - z_1}{z_3 - z_1},$$

which for an equilateral triangle is equal to

$$\frac{1}{2} + i\frac{1}{2}\sqrt{3}.$$

The mapping $z \mapsto f_t(z)$ maps the vertices z_1, z_2, z_3 to vertices w_1, w_2, w_3. Let Q_t be a cross ratio of w_1, w_2, w_3, ∞. As the points move, the orientation of the triangle remains counterclockwise until the points become collinear. This happens precisely when Q_t is real and cannot happen if $d_\Sigma(Q, Q_t) < d_\Sigma(Q, \mathbb{R})$. By Schwarz's lemma

$$d_\Sigma(Q, Q_t) \leq \log\frac{1+|t|}{1-|t|}.$$

In order to complete the proof of the lemma it suffices to show the Poincaré distance from $\omega = \frac{1}{2} + \frac{1}{2}\sqrt{3}$ to the real axis is $\log\sqrt{3}$. Consider $S_1(z) = 1 - z, S_2(z) = 1/z$ and $B(z) = \overline{z}$. Then $B \circ S_1$ and $B \circ S_2$ are anticonformal automorphisms of Σ. Moreover, the intersection of the locus of fixed points of $B \circ S_1$ and $B \circ S_2$ consists of two points, $\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$, each of which is equidistant from the real axis. In the standard universal covering of Σ by the upper half plane, the upper half of Σ is covered by part of a fundamental domain lying to the right of the y-axis, to the left of the vertical line with x-coordinate equal to 1, and above the semicircle with diameter coinciding with the unit interval $[0, 1]$. The point ω in Σ is covered by the point $\frac{1}{2} + i\frac{1}{2}\sqrt{3}$. Using the infinitesimal form $|d\zeta|/\mathrm{Im}(\zeta)$ for the Poincaré metric in the universal covering, we see that the Poincaré distance from ω to the real axis is equal to

$$\int_{1/2}^{\sqrt{3}/2} \frac{dt}{t} = \log\sqrt{3}.$$

□

Lemma 3 *Let $f_t(z)$ be a holomorphic motion of the whole plane fixing* $0, 1$ *and* ∞*, defined for* $|t| < 1$*. If* $|t| < 2 - \sqrt{3}$*, then the mapping* $z \mapsto f_t(z)$ *is quasiconformal.*

PROOF. To show $z \mapsto f_t(z)$ is quasiconformal we construct a normalized sequence f_n of quasiconformal mappings such that $K(f_n) \leq K_0$ which converges pointwise to $f_t(z)$. Let Γ_n be a grid of equilateral triangles which fill the plane. Assume each triangle of Γ_n has side-length equal to $\frac{1}{2^n}$ and one of the triangles of Γ_n, which we call the initial triangle, lies in the first quadrant, has a vertex at the origin and one side on the positive real axis. The mapping f_n is defined by declaring that it coincides with f_t on the vertices of Γ_n and is affine in each triangle of Γ_n. To calculate the complex dilatation of f_n on a triangle Δ of Γ_n, note that that this dilatation is the same as the dilatation of $A \circ f_n \circ B$ where A is affine and $B(z) = cz + d$, where c is real. Select B so that it maps the triangle with vertices at $0, 1$ and $\tau_0 = \frac{1}{2} + i\frac{1}{2}\sqrt{3}$ to a triangle Δ of Γ_n and select A so that $A \circ f_n \circ B$ fixes 0 and 1. Let $\tau = A \circ f_n \circ B(\tau_0)$. Then the complex dilatation μ_n of f_n in the triangle Δ satisfies:

$$\mu_n = -\frac{\tau - \tau_0}{\tau - \overline{\tau_0}}.$$

Assuming $|t| \leq 2 - \sqrt{3} - \epsilon_0$, the argument of the previous lemma implies $|\frac{\tau - \tau_0}{\tau - \overline{\tau_0}}| \leq k_0$, where $k_0 < 1$ and k_0 depends only on $\epsilon_0 > 0$.

We conclude that the normalized family of mappings $\{f_n\}$ satisfies $K(f_n) \leq K_0 = \frac{1+k_0}{1-k_0}$. From the previous lemma, this implies that, restricted to any bounded set in the plane, the family $\{f_n\}$ is equicontinuous. Therefore f_n converges pointwise to f_t.

At this point we adopt the geometric definition of quasiconformality, which says a mapping is quasiconformal if it distorts standard shapes by a bounded amount. What we take as a model for a standard shape is somewhat arbitrary but in this case, by a standard shape we mean a square with horizontal and vertical sides. The image $f_n(S)$ of the square S is a topological rectangle and there is a conformal map g_n from $f_n(S)$ to a rectangle R_n such that $g_n \circ f_n$ preserves horizontal and vertical sides. If a_n is the length of the rectangle and b_n is its height, then by Grötzsch's argument ([1], page 7), $1/K_0 \leq a_n/b_n \leq K_0$. This is true for each n and, since f_n converges locally uniformly to f_t, the same inequality is also true for the side lengths of the rectangle $R = g \circ f_t(S)$. Therefore, $K(f_t) \leq \frac{1+|t|}{1-|t|}$. $\square$

Lemma 4 *Let $f_t(z)$ be a holomorphic motion of the whole plane fixing $0, 1$ and ∞, defined for $|t| < 1$. Then $z \mapsto f_t(z)$ is quasiconformal for every $|t| < 1$.*

Proof. Assume $|t| < 1$. On the hyperbolic geodesic joining 0 to t choose points $0 = t_0, t_1, ..., t_n = t$ so that the hyperbolic distance from t_{k-1} to t_k is equal to $1/n$ times the hyperbolic distance from 0 to t. We choose n large enough so that this distance is less than $\frac{1}{2}\log 3$. Let $g_k = f_{t_k} \circ (f_{t_{k-1}})^{-1}$ so that

$$f_t = g_n \circ g_{n-1} \circ ... \circ g_1.$$

Also, let $F_s = f_t \circ (f_{t_{k-1}})^{-1}$ where $t = \frac{s+t_{k-1}}{1+\overline{t_{k-1}}t}$. Then the dependence of F_s on s for $|s| < 1$ is holomorphic, when $s = 0$, F_s is the identity, and when $t = t_k$, $F_s = g_k$. Since for this value of t, $|s| = |\frac{t_k - t_{k-1}}{1-\overline{t_{k-1}}t_k}| < 2 - \sqrt{3}$, it follows from the previous lemma g_k is quasiconformal. Therefore, f_t is quasiconformal because it is a composition of quasiconformal mappings. $\square$

It is known that if a mapping f satisfies the geometric definition of quasiconformality, then it has first partial derivatives in the sense of distributions which are locally L_2 and, moreover, the Beltrami coefficient $\mu_f = f_{\overline{z}}/f_z$ is an $L_\infty-$function with $||\mu||_\infty < 1$.

Lemma 5 *The map*

$$t \mapsto \mu_t = \textit{the Beltrami coefficient of } f_t,$$

is a holomorphic mapping from $|t| < 1$ *into the open unit ball of the Banach space* $L_\infty(\mathbb{C})$.

PROOF. We must show that $t \mapsto \mu_t$ is differentiable in the complex sense. We may assume f_t is normalized to fix $0, 1$ and ∞ because, if it does not, we may select Möbius transformations A_t so that $A_t \circ f_t$ is normalized this way. But the Beltrami coefficient of $A_t \circ f_t$ is is equal to the Beltrami coefficient of f_t. Let

$$g_s(z) = f_{t+s} \circ (f_t)^{-1}(z) = z + sV_t(z) + o(s).$$

Let z_1, z_2 and z_3 be the vertices of an equilateral triangle and consider the motion of the quadruple z_1, z_2, z_3, ∞ induced by g_s, which depends holomorphically on s. Let $\omega_1 = z_2 - z_1$, $\omega_2 = z_3 - z_2$ and $\omega_3 = z_1 - z_3$ and let W be an affine function whose values at z_1, z_2 and z_3 coincide with the values v_1, v_2 and v_3 of V_t at these points. On the one hand,

$$\overline{\partial} W = \frac{\omega_2 v_1 + \omega_3 v_2 + \omega_1 v_3}{\overline{\omega}_2 \omega_1 - \overline{\omega}_1 \omega_2}.$$

On the other hand, we know from (1) that

$$\overline{\partial} W = \frac{\omega_2 v_1 + \omega_3 v_2 + \omega_1 v_3}{\omega_1^2}$$

is bounded. Approximating V_t by a sequence affine functions V_n which coincide with V_t at the vertices of a hexagonal grid whose equilateral triangles have side length $1/2^n$, we find that $\frac{\partial}{\partial \bar{z}} V_t(z) = \nu(z)$, exists as a generalized partial derivative and $||\nu||_\infty < \infty$. Then

$$[\frac{\nu}{1-|\mu|^2}](\frac{1}{\theta}) \circ f^{-1},$$

where $f = f_t$, $\theta = \frac{\overline{p}}{p}$ and $p = \frac{\partial}{\partial z} f_t(z)$, is the complex derivative of this mapping with respect to s at $s = 0$. $\square$

Combining the previous lemmas and using Schwarz's lemma we obtain the following theorem.

Theorem 1 *Let $f_t(z)$ be a holomorphic motion of $\overline{\mathbb{C}}$ defined for $|t| < 1$ where $f_0 =$ identity. Then, for each t with $|t| < 1$, $z \mapsto f_t(z)$ is quasiconformal with dilatation less than $\frac{1+|t|}{1-|t|}$ and is Hölder continuous with Hölder exponent $\frac{1-|t|}{1+|t|}$.*

2 Quasifuchsian groups

2.1 Quakebends

In his work on simultaneous uniformization, Bers defined a *quasifuchsian group* analytically (see [1], [6]). Any hyperbolic Riemann surface S_0 can be represented as the quotient of the upper half plane U by a Fuchsian group G_0. We assume here for simplicity that S_0 is a compact surface of genus $g \geq 1$ from which at most n, $0 \leq n < \infty$, points have been removed and if $g = 1$, $n > 0$. A *Beltrami differential* μ defined on U, is an L_∞ function on U with $||\mu|| < 1$; it is *compatible* with G_0 if it satisfies the condition $\mu(g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(z)$ for all $g \in G_0$. If μ is extended to the lower half plane L so that the compatibility condition is satisfied, the differential equation $f_{\bar{z}} = \mu f_z$ has a solution f^μ that is a quasiconformal homeomorphism; the compatibility condition implies that $g^\mu = f^\mu \circ g \circ (f^\mu)^{-1}$ for any $g \in G_0$ is a Möbius transformation. The group of Möbius transformations $G^\mu = f^\mu \circ G_0 \circ (f^\mu)^{-1}$ is called *quasifuchsian*. It is discrete and in general, its limit set $\Lambda(G^\mu) = f^\mu(\Lambda(G_0))$ is not a subset of a circle so that it is not Fuchsian; in fact it is usually a fractal with Hausdorff dimension greater than 1.

The group G^μ represents *two* Riemann surfaces, $f^\mu(U)/G^\mu$ and $f^\mu(L)/G^\mu$ each topologically equivalent to S_0. Note however, that any finitely generated Kleinian group G whose domain of discontinuity Ω has exactly two invariant connected components is a quasifuchsian group representing two Riemann surfaces of the same topological type (see [41]).

In the sixties, Fenchel circulated a manuscript that was never published. In it, following on ideas of Poincaré from the turn of the century, he discussed Kleinian groups as discrete groups of isometries of hyperbolic three space $\mathbb{H}^3$. The quotient of $\mathbb{H}^3$ by a Kleinian group is thus a three manifold with a hyperbolic structure. In the mid-seventies, the first paper ([39]) discussing the geometry of the hyperbolic three manifolds and orbifolds arising from Kleinian groups of was published. A few years later, Thurston revolutionized the whole subject of three dimensional geometry and topology using Kleinian groups as a major tool ([**?**]). Among other things he showed how to construct general hyperbolic manifolds by gluing together special ones of the form $S_0 \times (0, 1)$; these building blocks are the manifolds represented by quasifuchsian groups.

In developing his theory, Thurston first considered deformations of the hyperbolic structure of a Riemann surface, and hence of the Fuchsian group representing it. He generalized Dehn twisting along a simple closed curve to a process he called *earthquaking along a measured lamination.* He then extended the earthquake deformation to a deformation from a Fuchsian group into a quasifuchsian group by bending and called the combination of earthquaking and bending *quakebending.* We describe these deformations below and see that the quakebend deformation is actually a holomorphic motion.

A *geodesic lamination* on a Riemann surface S is a generalization of a closed geodesic. It is a closed set that is a union of pairwise disjoint simple geodesics called leaves. A *measured lamination* is a geodesic lamination with a transverse measure invariant under isotopies preserving its leaves.

If the geodesic lamination consists of k simple closed geodesics $\gamma_1, \dots \gamma_k$ and β is a transverse arc, the measure is $\sum_{p \in \beta \cap \gamma_j} t_j \delta_j$ where δ_j is atomic measure and $(t_1, \dots t_k) \in \mathbb{R}^+$. Such a lamination is called *rational.* The weak topology on measures gives a natural topology on the space of measured laminations and the rational laminations are dense in this space (see [18]).

Two measured laminations λ_1 and λ_2 are projectively equivalent if they are supported on the same geodesic lamination and if there exists some $t > 0$ such that for any transverse arc β, $\lambda_1(\beta) = t\lambda_2(\beta)$. The set of projective classes of measured laminations on S is denoted $PML(S)$.

An earthquake is a generalization of a *partial Dehn twist* along an oriented simple closed geodesic γ. A partial Dehn twist may be described roughly as follows: mark a point p on γ, cut the surface along γ and then reglue so that the point p on the right side is glued to a point q on the left, at distance t (the twist parameter) from p measured along γ. For an earthquake γ is a geodesic lamination with a marked point p. One would like to cut along γ and reglue so that the point p on the right side is glued to a point q on the left at a distance t (now called the earthquake parameter) from p. To make this meaningful one has to introduce the measure and work in the universal covering space realized as the hyperbolic plane. See [18] for a detailed discussion of these ideas. This process changes the lengths of geodesics transverse to the lamination and thus the hyperbolic structure of the resulting surface. In the case γ is a closed curve, these lengths can actually be computed from the parameter t (see [26]). In the general case, it is not clear how to compute the new lengths, but the variation $d\ell(\eta)/dt$ with respect to the earthquake parameter where $\ell(\eta)$ is the length of a closed geodesic intersecting γ can be computed (see [50, 33]). In [34] it is shown that the new structure depends real analytically on the earthquake parameter.

Below we describe the earthquake and its extension by bending to a quakebend in the case where the geodesic lamination γ on the base surface S_0 consists of a single simple closed oriented geodesic. The quakebend process is easier to understand in this case and the proof that a small bend results in a quasifuchsian group contains all the geometric ideas of the general proof. After we sketch the proof for this case we indicate how to proceed in the general case. We refer the reader to [21] for a detailed description of quakebends. An alternate approach may be found in [37].

Fix a simple closed geodesic β intersecting γ and let $\tau = t + i\theta$ be a complex number. We define the quakebend as the composition of an earthquake by t and a (pure) bend by θ relative to the choice of β. Represent S_0 by G_0 acting on the disk $\mathbb{D}$ (instead of U) and embed $\mathbb{D}$ as the equatorial plane in the ball model of $\mathbb{H}^3$. Let p be an intersection point of γ and β; let $\tilde{p}$ be a lift of p to $\mathbb{D}$ and let $\tilde{\gamma}$ be a lift of γ to $\mathbb{D}$ with endpoints $\tilde{p}$ and $\tilde{p}_1$. Let V be the element in G_0 such that $V(\tilde{p}) = \tilde{p}_1$. Since γ is geodesic, $\tilde{\gamma}$ is contained in the axis A_V of V. Set $\mathcal{V} = \cup_{g \in G_0} g(A_V)$. These axes divide $\mathbb{D}$ into pieces $\{P_k\}$ that are infinite sided polygons with vertices on $\partial\mathbb{D}$. Let P_0 be a piece that has A_V as one of its boundaries.

Let $\tilde{q}$ be another point on A_V; we determine its signed distance s from $\tilde{p}$ as follows: if, standing on P_0, $\tilde{q}$ is to the left of $\tilde{p}$, we take $s > 0$, whereas if it is to the right, we take $s < 0$.

Let $\tilde{\beta}$ be the lift of β intersecting P_0 with one endpoint at $\tilde{p}$ and the other at a point $\tilde{p}_2$ and let $W(\tilde{p}_2) = \tilde{p}$. Now let P_j be the piece adjacent to P_0 along A_V so that $W(P_0) = P_j$. The group G_0 acts equivariantly on the decomposition of $\mathbb{D}$ into these pieces. In fact, if H_0 is the stabilizer in G_0 of

P_0, then G_0 is generated by H_0 and W.

To construct the earthquake given t, let $\tilde{q}$ be the point on A_V at signed distance t from $\tilde{p}$. Define W_t as the Möbius transformation taking the fixed points of $A_{V'}$, where $V' = W^{-1}VW$, to the fixed points of A_V and taking $\tilde{p}_2$ to $\tilde{q}$. Note that $P_j(t) = W_t(P_0)$ is the same set P_j but sheared in the direction of A_V by a distance t. This construction defines an isomorphism $\Phi_t : G_0 \to G_t$ as follows. Define $\Phi_t(h_0) = h_0$ for all h_0 in the stabilizer H_0 of P_0 and define $\Phi_t(W) = W_t$. Since G_0 is generated by H_0 and W, Φ_t is an isomorphism. Moreover we have $g_t\Phi_t(g_0) = \Phi_t^{-1}(g_t)g_0$ for all $g_0 \in G_0$ and $g_t \in G_t$.

We can form the developing map $\phi_t : \mathbb{D} \to \mathbb{D}$ as the identity on P_0 and for every piece P_k, if $P_k = g_0(P_0)$ for $g_0 \in G_0$, set $\phi_t(P_k) = \Phi_t(g_0)(P_0)$. To determine g_0 uniquely, choose it as W for $P_k = P_j$ as above and develop. It is proved in [26] that the image of the developing map ϕ_t again gives an embedding of $\mathbb{D}$ into $\mathbb{H}^3$ on which G acts discontinuously.

Now to make a pure bend, let $\bar{v}$ be a unit normal vector to A_V at $\tilde{p}$ pointing from P_0 into P_j. Bend P_j along A_V so that the bent vector $\bar{w}$ pointing into the bent P_j makes an angle θ with $\bar{v}$. The tangent $\bar{t}$ to A_V pointing towards its attracting fixed point is orthogonal to both $\bar{v}$ and $\bar{w}$. Make the sign convention that $\theta > 0$ if $(\bar{v}, \bar{w}, \bar{t})$ is a right handed system.

We can define W_θ as the Möbius transformation taking the fixed points of $A_{V'}$, where $V' = W^{-1}VW$, to the fixed points of A_V and taking $W^{-1}(\tilde{p})$ to $\tilde{p}$ and $W^{-1}(\bar{v})$ to $\bar{w}$; W_θ is W followed by a rotation of angle θ with fixed axis A_V. We can define an isomorphism $\Phi_\theta : G_0 \to G_\theta$ as above so that it is the identity on H_0 and sends W to W_θ. We also have an analogous developing map Φ_θ defined on $\mathbb{D}$ whose image $\mathbb{D}_\theta \subset \mathbb{H}^3$ is invariant under G_θ and consists of planar pieces joined at angle θ, along *bending lines* that are the images in $\mathbb{H}^3$ of the axes in $\mathcal{V}$. Again Φ_θ intertwines the actions of G_0 on $\mathbb{D}$ and G_θ on $\mathbb{D}_\theta$. Note that since P_0 and $\phi(P_j)$ are not coplanar, $\Lambda(G_\theta)$ is not necessarily contained in a circle. It is not too hard to see that if $\theta = \pi$, $\Lambda(G_\theta)$ is again a circle but the set of discontinuity of G_θ cannot have two distinct invariant components.

The quakebend by τ is a composition of an earthquake and a pure bend. We have an isomorphism $\Phi_\tau(\gamma) : G_0 \to G_\tau$ and a developing map $\phi_\tau : \mathbb{D} \to \mathbb{D}_\tau$. Unlike the earthquake, however, the group G_τ thus obtained is not necessarily discrete.

We have

Theorem 2 *Let G be a Fuchsian group and let γ be a simple closed geodesic on $S = \mathbb{D}/G$. Let $\tau = t + i\theta$ and let $\Phi_\tau(\gamma)$ be the associated quakebend* *homomorphism. There exists $\epsilon > 0$, depending on the hyperbolic length of γ, such that if $|\theta| < \epsilon$, then $\mathbb{D}_\tau$ is embedded and G_τ is quasifuchsian.*

PROOF. We sketch a proof of this by showing by purely geometric means that the quakebend defines a holomorphic motion of the fixed points of the elements of G_0. A slightly different proof of the theorem, also based on the geometric lemmas below, is carried out in full detail in [31] where it is applied to the case of the punctured torus.

The proof that the quakebend defines a holomorphic motion of the fixed points divides into two parts. First we show that the fixed points of G_τ depend holomorphically on τ; this will be true whether or not G_τ is discrete and so does not use the hypothesis that the bend is small. Second we show that the map ϕ_τ is an embedding and extends to an equivariant map on the fixed points and moreover, that the extension is injective on the fixed points; this part does require the small bending hypothesis.

1. Holomorphicity

Using the method developed in [44], we shall find a formula for the trace $\mathrm{tr}(W_\tau)$ that shows it depends holomorphically on τ. Since the entries of W_τ can be recovered from information about H_0 and $\mathrm{tr}(W_\tau)$, the entries and fixed points are holomorphic in τ. Since any element in G_τ is a word in the fixed elements of H_0 and $W_\tau^{\pm n}$, its entries and fixed points will also be holomorphic in τ.

The trace $\mathrm{tr}(W_\tau)$ and the complex translation length $2\lambda_{W_\tau}$ are related by $\mathrm{tr}(W_\tau) = 2\cosh\lambda_{W_\tau}$; we always take $\Re\lambda_{W_\tau} > 0$. Following the construction in [44], we can form a *skew hexagon* in hyperbolic space by traversing the geodesic arcs l_i, $i = 1, \ldots, 6$ defined as follows:

$l_1 = A_V$ and let l_2 be the the common orthogonal to A_V and $A_{V'}$ with respective endpoints ξ, ξ' on $A_V, A_{V'}$; let l_3 be the orthogonal to P_0 through the midpoint of the line (ξ, ξ').

Let $l_5 = W_\tau(l_3)$, $l_6 = W_\tau(l_2)$ and let l_4 be the common orthogonal to l_3 and l_5.

We claim that l_4 is the axis A_{W_τ} and that we can find its complex translation length using the hyperbolic cosine rule applied to the hexagon formed by the lines $l_1, \ldots l_6$. To prove the claim, recall that any loxodromic element can be written as the product of reflections in a pair of disjoint hyperbolic lines perpendicular to its axis. Denote the reflection in l_k by i_k (see e.g. [25]). Let η be the midpoint of the segment of l_1 joining ξ and $W_\tau(\xi')$ and let l_0 be the geodesic normal to l_1 at η such that $i_0(l_2) = l_6$. Observe that the real hyperbolic length of the segment (ξ, η) is $t/2$ and that l_0 makes an angle of $\theta/2$ with P_0 and with $W_\tau(P_0)$.

Note that $i_0(l_3) = l_5$ so $i_5 = i_0 i_3 i_0$; $i_3(A_V) = A_{V'}$ and if $V'' = W_\tau V W_\tau^{-1}$, then $i_5(A_V) = A_{V''}$. Thus

$$i_0 i_3(A_{V'}) = i_0(A_V) = A_V = W_\tau(A_{V'})$$

$$i_0 i_3(A_V) = i_0(A_{V'}) = A_{V''} = W_\tau(A_V).$$

Therefore $i_0 i_3 = i_5 i_0 = W_\tau$ and A_{W_τ} is perpendicular to l_0, l_3 and l_5; since l_4 was defined as this perpendicular, $l_4 = A_{W_\tau}$ as claimed.

Denote the complex length of the segment of l_2 from l_1 to l_3 by $d_0 + i\psi$. Since d_0 is half the distance between A_V and $A_{V'}$ on P_0 it depends only on G_0. Since l_3 is perpendicular to P_0, $\psi = \pm\pi/2$ where the sign depends on the sign of t and appropriate conventions. These are discussed in detail in [44]. Applying these conventions again, the complex length of the segment $(W_\tau(\xi), \xi)$ is $\pm\tau$ and the complex length of l_4 from l_3 to l_5 is $2\lambda_{W_\tau} + i\pi$. Thus, by the hyperbolic cosine rule applied to the sides $l_1 \dots l_6$ we have

$$\cosh l_4 = \cosh l_1 \cosh l_2 \cosh l_6 - \sinh l_2 \sinh l_6$$

so that

$$\cosh 2\lambda_{W_\tau} = \cosh\tau \cosh^2(d_0) - \sinh^2(d_0).$$

Rewriting, this becomes

$$\cosh \lambda_{W_\tau} = \cosh d_0 \sqrt{\frac{\cosh\tau + 1}{2}}$$

where the root is chosen with sign conventions of [44].

2. Injectivity

The proof of the injectivity is based on two geometric lemmas. Let $\eta : [0,\infty) \to \mathbb{D}$ be a geodesic parametrized by distance; assume it is transverse to a subset of the axes of $\mathcal{V}$ and set $\eta_\tau(s) = \phi_\tau(s)$. If $\eta(s_0)$ intersects an axis, its image $\eta_\tau(s_0)$ is called a *bending point*; denote by ψ the angle between the right and left limits of the forward pointing tangents at $\eta_\tau(s_0)$.

The first lemma says $|\psi| < |\theta|$ and is proved by elementary geometry.

For the second lemma we need the following definitions:

Let $X \in \mathbb{H}^3$ and let $\bar{v}$ be a vector in $\mathbb{H}^3$ at X. The *forward cone* of angle α at X in the direction $\bar{v}$ is the closed solid cone of angle α with vertex at X whose central axis is the $\mathbb{H}^3$ geodesic in the direction of $\bar{v}$. The *shadow* of a cone at X in direction $\bar{v}$ on $\partial\mathbb{H}^3$ is the set of endpoints on $\partial\mathbb{H}^3$ of the semi-infinite geodesics emanating from X and contained inside the forward cone at X.

The second lemma, called the cone lemma, says that given $\alpha \in (0, \pi/2)$, there is an $\epsilon > 0$ depending only on α and the hyperbolic length of γ such that if $|\theta| < \epsilon$, then the developed geodesic $\eta_\tau(s+t)$ is contained inside the forward cone at any point $\eta_\tau(s)$ — whichever of the two directions for the cone is used at the bending points — and moreover, there is a constant d_γ such that the forward cones along η_τ at points d_γ apart are nested. The proof of this lemma is based on estimates using elementary hyperbolic geometry.

From the cone lemma it follows that the developed image η_τ of any oriented geodesic $\eta \subset \mathbb{D}$ is embedded since the forward and backward cones at any point separate the forward and backward trajectories of the geodesic.

Next, assume $\eta(s)$ has as endpoint the fixed point of some element of G_0. The cone lemma applied to a sequence $\eta(s_n) \to \partial\mathbb{D}$ implies that the accumulation set of $\eta_\tau(s_n) \in \partial\mathbb{H}^3$. Since $\{s_n\}$ may be chosen so that the cones at $\eta_\tau(s_n)$ are nested, it follows that the diameters of their shadows go to zero. Using the group invariance, it follows that ϕ_τ extends to the fixed points of elements of G_0, taking them to corresponding fixed points of elements of G_τ.

Now suppose that $x, y \in \mathbb{D}$ are fixed points — of perhaps different elements of G_0 — and let z be a point on the geodesic η joining x and y. As above, the forward and backward cones at $\phi_\tau(z)$ separate the endpoints of η_τ so that $\phi_\tau(x) \neq \phi_\tau(y)$ and ϕ_τ is injective.

3. Application of the λ-lemma

Finally, if Λ_0 is the set of fixed points of elements of G, because the fixed points of all elements of G_τ are holomorphic functions of τ and they move injectively, ϕ_τ defines a holomorphic motion on $\Lambda_0 \times \Delta$ where $\Delta = \{\tau = t + i\theta,\, t \in \mathbb{R}, |\theta| < \epsilon\}$. Moreover, the motion acts equivariantly with respect to G_0 on a dense subset of its limit set. By Slodkowski's version of the λ-lemma, the holomorphic motion can be extended to a quasiconformal homeomorphism f of $\hat{\mathbb{C}}$ and by Theorem 1 of [20] the continuous extension can be made to act equivariantly. Thus, $G_\tau = fG_0f^{-1}$ and G_τ is quasifuchsian. $\square$

If μ is an any rational measured lamination on S with support $\{\gamma_1, \ldots, \gamma_k\}$, $0 < k \leq 3g-3$ and transverse measure $\{t_1\delta_{\gamma_1}, \ldots, t_k\delta_{\gamma_k}\}$, we define the quakebend $\Phi_\tau(\gamma)$ as a composition of quakebends $\Phi_{t_j\tau}(\gamma_j)$, $j = 1, \ldots, k$. Clearly the theorem holds in this case where the condition is that $|\Im(t_j\tau)| < \epsilon$ for all $j = 1, \ldots, k$.

4. Arbitrary laminations

Now suppose γ is an arbitrary geodesic lamination on $S = \mathbb{D}/G$ with a complex valued measure ω supported on γ. In this case, the essential geometric ideas are the same, but carrying out the full details involves a discussion of how to take limits of quakebends along rational laminations. Here we only sketch what has to be done.

In chapter 3 of [21] there is a full description of how to define the *quakebend cocycle* B_ω and the corresponding developed surface $\phi_\omega : \mathbb{D} \to \mathbb{D}_\omega \subset \mathbb{H}^3$ that determine the *quakebend* along ω by a limiting procedure. It follows immediately from section 3.7 of [21] that B_ω determines a group $G_\omega \subset PSL(2, \mathbb{C})$ such that ϕ_ω intertwines the actions of G_0 on $\mathbb{D}$ and G_ω on $\mathbb{H}^3$.

The theorem in the general case is

Theorem 3 *Let G be a Fuchsian group and let γ be a measured lamination on $S = \mathbb{D}/G$. Let $\tau \in \mathbb{C}$ be given and let $B_{\tau\mu}(\gamma)$ be the associated quakebend cocycle. There exists $\epsilon > 0$, depending on γ such that if $|\Im\tau| < \epsilon$, then $\mathbb{D}_{\tau\mu} = \phi_{\tau\mu}(\mathbb{D})$ is embedded and $G_{\tau\mu}$ is quasifuchsian.*

PROOF. The proof follows the same outline as the sketched proof above. Again the holomorphicity does not depend on small bending but follows from the general quakebend construction applied to complex measures of the form $\omega = \tau\mu$ where μ is a transverse measure supported on γ in the usual sense and $\tau \in \mathbb{C}$. Theorem 3.9.1 of [21] can be carried out for this measure and it says that the trace (or equivalently the complex translation length) of any geodesic on S transverse to the lamination γ is a holomorphic function of τ. It follows that the fixed points of G_τ also depend holomorphically on τ.

The main difference in the proof of the extendibility and injectivity of ϕ_τ is that the cone lemma needs to be generalized. In the rational case, because there is a minimum to the lengths of geodesic arcs joining leaves of the lamination, elementary hyperbolic geometric arguments can be used to prove the cone lemma. In the general case however, there is no minimum and a new argument is required. This is given in detail for punctured tori in [31]; the arguments there extend without difficulty. We outline the steps here.

First, one needs to get a uniform control of the approximations to the quakebend cocycle. This is done using area estimates on S_0: one proves that there are constants K_μ and d_μ such that if β is any geodesic segment on S of length less than d_μ, then $\mu(\beta) < K_\mu$.

Second, the cones have to be properly defined. To do this, an appropriate metric D_T on the tangent space to $\mathbb{H}^3$ is introduced. Next the geodesics

$\eta(s) \subset \mathbb{D}$ and $\eta_\tau(s) \subset \mathbb{D}_\tau$ are approximated. Using tangents to the approximations and taking limits in D_T, one can prove that tangent vectors to $\eta_\tau(s)$ can be properly defined even though the bending points form a Cantor set in $\eta_\tau(s)$,

Finally, using the constant d_μ one can prove that the cone lemma still holds; that is, $\eta_\tau(s+t)$ is entirely contained in forward cone at $\eta_\tau(s)$ and that the cones at $\eta_\tau(s+t)$ are nested for $t > d_\mu$.

The same argument as in the rational case shows that the map ϕ_τ extends injectively to Λ_0 and defines a holomorphic motion. Again, extending to obtain an equivariant quasiconformal homeomorphism of $\hat{\mathbb{C}}$, we conclude that G_τ is quasifuchsian. $\square$

2.2 The convex hull boundary

One of the most elegant aspects of the study of Fuchsian and Kleinian groups is the interplay between the analytical structure of the Riemann surfaces and their parameter spaces and the geometric structure of the quotient spaces. For a Fuchsian group G_0, the conformal structure and the hyperbolic structure of the disk descends to the quotient $S_0 = \mathbb{D}/G_0$. Traces of elements of G_0 can be used as parameters and they also have a geometric meaning in terms of the lengths of closed geodesics on S_0.

If S_0 is a Riemann surface of finite type but not finite volume — a surface of finite genus from which finitely many points and disks have been removed — the limit set $\Lambda(G_0)$ is a Cantor set in the boundary of the hyperbolic plane. Nielsen realized that if one takes the hyperbolic convex hull of $\Lambda(G_0)$ (the Nielsen convex region) and forms the quotient, the result is a surface $\bar{S}_0$ of finite volume that contains all of the interesting geometric information; in particular, it is completely determined by trace parameters for G_0 or by the lengths of geodesics on S_0.

For quasifuchsian groups and more generally for Kleinian groups, the relation between the conformal structure of the quotient surfaces and the traces of group elements is very mysterious. By contrast however, the traces can be interpreted in terms of the hyperbolic structure of the quotient three manifold. Thurston pointed out that again, the convex hull of the limit set $\mathcal{C}(G)$, and its quotient $\mathcal{C}/G$, the convex core, contain all of the relevant geometric information.

A *pleated surface* is a pair S, ϕ where S is a complete hyperbolic surface and $\phi : S \to M$ is an isometry into a hyperbolic 3-manifold such that at every point $p \in S$ there is at least one geodesic segment σ such that $\phi(\sigma)$ is geodesic and such that ϕ induces an injective map on fundamental groups.

The set of $p \in S$ such that there is only one such σ is called the *pleating locus* (see [21] for a full discussion). For example, the pair $(\mathbb{D}, \phi_\tau)$, where ϕ_τ is the developing map defined above, is a pleated surface, and the bending lines are the pleating locus. For simplicity we also call the immersed image surface $\mathbb{D}_\tau = \phi_\tau(\mathbb{D})$ pleated.

The quotient surfaces $\partial\mathcal{C}/G$ have a hyperbolic structure and are pleated surfaces. Thurston conjectured that the pleating locus of the convex hull boundary with its natural bending measure should completely determine the group. This has been proved for Schottky groups in the case that the pleating locus is rational by Otal [43] and for arbitrary laminations by Bonahon [11]. These proofs use the deep machinery of Kerckhoff-Hodgson and Thurston-Morgan-Shalen.

Using the quakebend construction and Theorem 3 it follows that

Theorem 4 *For a quasifuchsian group G, the boundary surface components of $\partial\mathcal{C}/G$ are pleated and the pleating locus for one component can be prescribed arbitrarily.*

Proof. Again we begin with the rational case. To prove the boundary surfaces are pleated, we observe that by convexity, every bi-infinite geodesic in a component of $\partial\mathcal{C}$ extends to Λ. We can thus define an isometry from each component of $\partial\mathcal{C}$ to $\mathbb{D}$ by unfolding.

That the pleating locus of one side can be prescribed follows from

Proposition 1 *Suppose μ is a rational measured lamination on the surface S with support $\{\gamma_1, \ldots, \gamma_k\}$ and measure $\{t_1\delta_{\gamma_1}, \ldots, t_k\delta_{\gamma_k}\}$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_\tau$ be the developed surface associated to the quakebend $\Phi_\tau(\gamma)$. If ϵ is as in the cone lemma and if $0 \leq \Im(t_j\tau) < \epsilon$ for all j, then one of the half spaces cut out of $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ by $\overline{\mathbb{D}_\tau}$ is convex.*

Proof. Since $\partial\mathbb{D}_\tau$ is a Jordan curve on $\hat{\mathbb{C}}$, $\mathbb{D}_\tau$ does indeed divide $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ into two half spaces, H_i, $i = 1, 2$. Suppose H_1 is not convex so there are points $x_1, y_1 \in H_1$ and a geodesic σ_1 joining them that leaves H_1. Let Π_1 be a plane through σ_1 intersecting H_1. Then $\Pi_1 \cap (\sigma_1 \cup \partial H_1)$ is a hyperbolic n-gon, the sum of whose interior angles is less than $(n-2)\pi$. By hypothesis, the angles from ∂H_1 all have the same sign and are less than π in absolute value. If H_2 is also not convex, we form another polygon by taking points $x_2, y_2 \in H_2$, an arc σ_2, and a plane Π_2 through σ_2 and H_2; since this time the angles are measured on the opposite side of $\mathbb{D}_\tau$, they again all have the same sign, but their absolute values are all greater than π. Such a polygon cannot exist and this gives a contradiction. $\square$

Corollary 1 *The pleated surface $\mathbb{D}_\tau$ is a component of $\partial\mathcal{C}(G_\tau)$.*

PROOF. On one hand, H_2 is convex so $H_2 \supset \mathcal{C}$. On the other, $\mathbb{D}_\tau$ is contained in the convex hull of its bending lines so that $\mathbb{D}_\tau \supset \partial\mathcal{C}$ and $\mathbb{D}_\tau = \partial H_2$ by definition. □

Since the projective rational laminations are dense in $PML(S)$, we can extend these results to arbitrary laminations by approximation. In [32] it is proved that the map from quasifuchsian space to $PML(S)$ defined by sending a group to the pleating locus of one component of the convex hull boundary is continuous. This coupled with the observation that the limit of convex sets is convex completes the proof of theorem 4. □

Appendix

Much has happened mathematically since this paper was written. In this appendix we indicate some of this progress that builds heavily on Bers work.

- In his study of the geometric structures of all closed three manifolds with finitely generated fundamental group, [**?**, **?**], Thurston made fundamental use of Bers work. The Thurston Geometrization Program first stated in [**?**] says that there is a uniformization theory for three manifolds analogous to the uniformization of surfaces. One has, however, to use eight rather than three different geometries. Like surface theory, most three manifolds are hyperbolic. Thurston began with a very general class of closed three manifolds: that can be cut apart into pieces that are manifolds whose boundary components are incompressible surfaces; such manifolds are called *Haken manifolds*. With certain restrictions on the fundamental group of the Haken manifold it is called *atorioidal*. Thurston proved that these manifolds admit a hyperbolic structure. The idea is that the pieces of the manifold can be represented by Kleinian groups, which in turn, give hyperbolic structures to their interiors. The main step is to glue the pieces together in such a way that the resulting closed manifold has a hyperbolic structure on its interior. Finally, by Mostow rigidity, one knows this structure is unique.

  The first step in the gluing process is to use Bers simultaneous uniformization. By hypothesis there is a surface in the manifold whose fundamental group injects into the fundamental group of the manifold. Cutting along this surface we obtain two copies X and Y. Each of these surfaces can be uniformized by a quasifuchsian group that represents it and another surface, $\sigma(X), \sigma(Y)$. The next step is to define a homeomorphism from Y to $\sigma(X)$. This determines a holomorphic map from the Teichmüllerspace of X to itself, the *skinning map* which is holomorphic with respect to the complex analytic structure coming from the Bers embedding. As such, it is weakly contracting. The next step is to prove the skinning map is strongly contracting so that under iteration the resulting points in Teichmüller space converge to a fixed point, the desired manifold. (See Minsky's lecture in this volume). The full geometrization theory involves understanding pieces that dont have Kleinian group representations, and proving that manifolds either admit the cutting process or have covering spaces that do. Combining the work of many others, the last step in the full proof of Thurston's geometric theory was given in 2012 by Ian Agol, based on work of Wise and Kahn-Markovic. A good expository account can be found in [9, 30].

- An important part of the Thurston theory is the study of the manifolds with boundary and in particular those that can be represented by the quasifuchsian groups obtained by simultaneous uniformization. The point is that these groups have a natural representation as a subset of $\mathbb{C}^m$ for appropriate m: Let S_0 be a fixed base surface of genus g with n punctures such that $m = 3g-3+n > 0$ and fix a presentation for its fundamental group $G_0 = \pi_1(S_0)$. Let $\mathcal{D} = \{\rho : \pi_1(S_0 \to PSL(2,\mathbb{C})\}$ such that ρ is a discrete and faithful representation. The subspace $\mathcal{QF} \subset \mathcal{D}$ is the connected component of the identity. Using holomorphic motions, the groups in the image of $\mathcal{QF}$ are all quasifuchsian groups; that is, for each $\rho \in \mathcal{QF}$ there exists a Beltrami differential μ, compatible with G_0, such that $f^\mu(U)/\rho(G_0)$ and $f^\mu(L)/\rho(G_0)$ are oppositely oriented homeomorphic Riemann surfaces. By the measurable Riemann mapping theorem, (see [4] and the discussion in [51] in this volume), the traces of a finite set of elements of $\rho(G_0)$ define a complex analytic embedding into $\mathbb{C}^{6g-6+2n}$. The boundary of this embedding is a very complicated fractal set.

  The *Bers Slice* consists of those $\rho \in \mathcal{QF}$ or, by abuse of notation, its $3g-3+n$ dimensional image in $\mathbb{C}^{6g-6+2n}$ such that $f^\mu(L)/\rho(G_0) = \overline{S_0}$. The boundary of the Bers slice in $\mathbb{C}^{6g-6+2n}$ is called the *Bers boundary.* A point on the boundary of this image such that the image $\rho(A)$ of a loxodromic element $A \in G_0$ is parabolic ($\text{tr}^2\text{A} = 4$) is called a *cusp*. Bers conjectured that the cusps were dense in this boundary. In 1991 McMullen [36] proved this conjecture. It is also now known that the space $\mathcal{QF}$ has many different components and the boundaries of these components have very interesting properties. See for example, [12, 16]

- Thurston [24, 47] gave his own description of points in Teichmüller space. The hyperbolic structures on surfaces are determined by geodesic laminations that admit transverse measures invariant under translation along the geodesic. (See the discussion in the article above). The boundary of this space consists of the projective measure classes of these laminations. Given a point $G = \rho(G_0) \in \mathcal{QF}$, the quotient manifold $\mathbb{H}^3/G$ has a convex core. The boundary of the core consists of two pleated surfaces. These have hyperbolic structures determined by the pleating locus and bending measure. Thus, each point is determined by the boundary data of its convex core. Thurston called this boundary data the *ending laminations* of the manifold If the pleating locus is rational, that is, consists of closed geodesics, one can deform by a holomorphic motion so that the lengths of these geodesics goes to zero and the bending goes to zero. The corresponding point on $\partial\mathcal{QF}$ is a cusp. Other deformations can lead to other kinds of boundary points. He conjectured that in all circumstances, the limit of ending laminations makes sense as an ending lamination of the limit hyperbolic manifold and that the manifold could be recovered from this data. This is called the *Ending Lamination Conjecture*. This was proved in a long series of papers: see [13, 14, 42] and the papers cited there. Also see Wolpert's article and the expository articles [9, 30].

- As part of this large program to understand hyperbolic three manifolds, Marden conjectured that the structure of the interior of the manifold as one moves outwards in the convex core is not too complicated: that all three manifolds are *tame*. Describing this is beyond the scope of this article. We mention it though, because this tameness conjecture was proved true in [17, 22] and this has as a corollary a theorem that Bers would have very much liked. To wit, the Lebesgue measure of the limit set of any finitely generated Kleinian group is either zero or full.

  There was a similar conjecture for the measure of the limit set of a rational map. This conjecture, however, is false. There is a construction by Buff and Cheritat of a quadratic polynomial whose limit set has positive measure but the limit set is not the whole sphere. [15].

- Thurston applied the technique of iterating a map from a Teichmüller space to itself to another problem, that of determining when a finite degree topological covering map f of the sphere to itself can be *realized* by a rational map R, (See for example, [DH, Ji] and there references in those papers.) It is easy to find a topological conjugacy; such a map preserves the critical values and the branching structure. What isn't clear is how to make this conjugacy invariant under the dynamics; that is, to make it preserve P_f, the closure of the set of forward orbits of the critical values of f. As in the case of Kleinian groups, to expect to control convergence of the iteration procedure one wants to work with finite dimensional Teichmüller spaces. Thus the Kleinian groups were assumed finitely generated. The analogous assumption here is that P_f is a finite set and the Riemann surface is $S = \hat{\mathbb{C}} \setminus P_f$. The equivalence relation defining the Teichmüller space is del P_f and the Thurston map is from this Teichmüller space to itself. Thurston proved that either the map is strongly contracting the iteration converges to a point representing the desired rational map or there is a topological obstruction and the iteration doesn't converge to an interior point.

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