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RESEARCH ARTICLE

Boundaries of Bounded Fatou Components of Quadratic Maps

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In this paper we characterize those external rays that land on the bounded Fatou components of hyperbolic and parabolic quadratic maps. For those maps not in the main cardioid of the Mandelbrot set, we prove that the arguments of these rays form a Cantor subset of the circle at infinity. Our techniques involve both the orbit portraits of Goldberg and Milnor that relate the dynamic and parameter planes and the Thurston theory of laminations for quadratic maps.

Keywords: Fatou Components, External Rays, Laminations, Orbit Portraits

AMS Subject Classification: 30D05, 32H50, 37F45

1. Introduction

When the Julia set of rational map is locally connected it can be divided into two subsets, those points that are accessible boundary points of a stable component of the complement and those that are not. The latter form the residual Julia set. This paper grew out of an investigation into a full characterization of points, and in particular buried points, in the Julia set of a rational map arising from a self-mating of a generalized star-like quadratic polynomial in the first author's thesis [6]. The crux of that characterization was to determine which external rays of the generalized star-like quadratic actually land on a boundary point of a bounded component of the Fatou set. The aim of this paper is to provide this characterization for all quadratic polynomials.

The study of the combinatorial structure of the external rays of the Julia set goes back to the seminal work of Douady and Hubbard on the dynamics of quadratic polynomials [3–5]. They originated a number of techniques to characterize the combinatorics of Julia sets including "Hubbard trees" , the "Spider algorithm" and "tuning" . This was developed further by Hubbard's students including Schleicher [17]. Thurston and Rees [15, 16, 19] used the concept of "laminations" to study the combinatorics. A different, more purely combinatorial approach, via the "devil's staircase algorithm", was presented by Bullet and Sentenac in [1]. Milnor and Goldberg [12, 13] developed a theory of "orbit portraits" that complements the Douady-Hubbard theory. The combinatorics of the dynamics of quadratic polynomials is also described in [10]. Each of these works gives insight from a different perspective and together they give a pretty full picture.

The characterization we give here is not in this literature and adds to this picture. We use a variation of the "laminations" and the "orbit portrait" techniques.

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In this paper, we consider arbitrary hyperbolic and parabolic quadratic polynomials. For these, every external ray lands at some point on the Julia set but, in general, only certain rays land on the boundary of a bounded component of the Fatou set. For example, the zero dynamic ray always lands at a repelling fixed point; however, outside the main cardioid of the Mandelbrot set, it never lands on a bounded component. Our main result is a full description of the external rays that land on boundary points of the bounded components of the Fatou set. For quadratics in the main cardiod, the problem is trivial: the Julia set is a Jordan curve and every point is a landing point of an external ray. We therefore remove these polynomials from our discussion.

Our methods use the combinatorial structure given by the external rays to the Mandelbrot set M and the corresponding structure of the Julia set as described by Goldberg and Milnor in their papers [7, 8, 12, 17]. We look only at points in the interiors of the (non-cardiod) hyperbolic components of M and their roots. The problem is clearly much more difficult for other points and our methods don't extend.

The combinatorial structure of the hyperbolic components of the Mandelbrot set has been well-studied; the original description is in [5]. A more recent account is given in [10]. Another excellent exposition and history is contained in [12] and we will take our terminology from that paper. A full bibliography can be found in that article. More recent results can be found in [18].

The root points of M are parameter values $c_{\mathcal{P}} = \mathbf{r}_{\mathcal{P}}$ such that the corresponding quadratic $f_{c_{\mathcal{P}}} = z^2 + c_{\mathcal{P}}$ has a parabolic periodic cycle of period p for some integer $p > 0$ and whose dynamics are described by an *orbit portrait* $\mathcal{P}$ (see section 2.5). They are called root points because they lie on the boundary of a hyperbolic component $H_{\mathbf{r}_{\mathcal{P}}}$ with the property that for any $c \in H_{\mathbf{r}_{\mathcal{P}}}$, f_c has an attracting periodic cycle of period vp for some positive integer v.

With the exception of the point $c = 1/4$, there are always two external parameter rays of M, $\mathcal{R}_{t_-}(\mathbf{r}_{\mathcal{P}}), \mathcal{R}_{\mathbf{t}_+}(\mathbf{r}_{\mathcal{P}})$ (measured in turns), landing at $\mathbf{r}_{\mathcal{P}}$. These rays cut off a region of the plane called the *wake* of $\mathbf{r}_{\mathcal{P}}$ and denoted by $W_{\mathbf{r}_{\mathcal{P}}}$. There are external dynamic rays $R_{t_\pm}(\mathbf{r}_{\mathcal{P}})$ of the filled Julia set $K_{\mathbf{r}_{\mathcal{P}}}$ of $f_{\mathbf{r}_{\mathcal{P}}}$ landing at the parabolic periodic point on the boundary of the Fatou component containing the critical value. The angles $t_\pm$ of the dynamic rays are the same as the angles of the parameter rays.

Wakes of two roots $\mathbf{r}_{\mathcal{P}}$ and $\mathbf{r}_{\mathcal{P}'}$, $\mathcal{P} \neq \mathcal{P}'$ are either disjoint or one is contained inside the other.

We can define a partial order on the roots of M as follows: $\mathbf{r}_{\mathcal{P}} \prec \mathbf{r}_{\mathcal{P}'}$ if $W_{\mathbf{r}_{\mathbf{P}'}} \subset W_{\mathbf{r}_{\mathcal{P}}}$.

A hyperbolic component $H_{\mathbf{r}_{\mathcal{P}}}$ is *primitive* if $\mathbf{r}_{\mathcal{P}}$ is not a boundary component of any other hyperbolic component. Otherwise it is called a satellite. If $\mathbf{r}_{\mathcal{P}_\mathbf{0}}, \mathbf{r}_{\mathcal{P}_\mathbf{2}}, \ldots \mathbf{r}_{\mathcal{P}_\mathbf{n}}$ is a sequence of roots such that $H_{\mathbf{r}_{\mathbf{P}_\mathbf{0}}}$ is primitive and $\mathbf{r}_{\mathcal{P}_{\mathbf{i+1}}}$ lies on the boundary of $H_{\mathbf{r}_{\mathcal{P}_\mathbf{i}}}$, $i = 1, \ldots, n$, then the components $H_{\mathbf{r}_{\mathbf{P}_\mathbf{i}}}$ are satellites of the same primitive component $H_{\mathbf{r}_{\mathbf{P}_\mathbf{0}}}$. We say that $\{H_{\mathbf{r}_{\mathcal{P}_\mathbf{i}}}\}_{i=k}^{i=n}$ forms a *chain* from $H_{\mathbf{r}_{\mathcal{P}_\mathbf{k}}}$ to $H_{\mathbf{r}_{\mathcal{P}_\mathbf{n}}}$ for any $0 \leq k < n$.

Our main result is

Main Theorem *If $c \in H_{\mathbf{r}_{\mathcal{P}}}$ and $f_{\mathbf{r}_{\mathcal{P}}}$ has a parabolic orbit of period p, then the rational rays landing on the boundary points of the bounded Fatou components of f_c are precisely all the rays in the union of the sets $\{\cup_n f_{\mathbf{r}_{\mathcal{P}}}^{-n}(R_{t_\pm}(r_{\mathcal{P}}))\}$ and of $\{\cup_n f_{\mathbf{r}_{\mathcal{Q}}}^{-n}(R_{t_\pm}(\mathbf{r}_{\mathcal{Q}}))\}$ for all $\mathbf{r}_{\mathcal{Q}} \succ \mathbf{r}_{\mathcal{P}}$ such that there is a chain from $H_{\mathbf{r}_{\mathcal{P}}}$ to $H_{\mathbf{r}_{\mathbf{Q}}}$.*

We also prove that the arguments of the rays that land on any bounded Fatou component form a Cantor set in the circle at infinity.

The paper is organized as follows. In section 2 we give the basic definitions and set notation. In section 3 we define and describe the portrait laminations that are our main technique to describe the external rays that land on the boundary of the Fatou set, and in section 4 we prove our main theorem.

2. Basic Theory and Notation

In this section we recall basic definitions and set our notation. Proofs may be found in standard texts, for example [2, 11]. We work with quadratic polynomials $f_c = z^2 + c$.

2.1 The Fatou set and the Julia set

The *orbit* of a point z_0 is the set $\{z_n = f_c(z_{n-1})\}_1^\infty$. We use the notation f^n to denote the n^{th} iterate of the function f. The *grand orbit* is the set $\{f_c^n(z_0)\}$ for all $n \in \mathbb{Z}$.

As usual, the *filled Julia set* K_c of f_c is the set of points in the dynamic plane whose orbits under f_c are bounded and the Julia set $J_c = \partial K_c$. The *Fatou set* is the complement of the Julia set. The complement B_∞ of K_c is the attracting basin of infinity. It is *completely invariant*; that is, it is forward invariant, $f_c(B_\infty) = B_\infty$, and it is backward invariant $f_c^{-1}(B_\infty) = B_\infty$.

For readability, in statements where the specific parameter is not relevant, we omit the subscript.

The *Mandelbrot set* M is the set of values in the parameter plane such that for each $c \in M$ the orbit of the origin under f_c is bounded. The filled Julia set K_c is connected if and only if $c \in M$. The connected components of the Fatou set are simply connected if and only if $c \in M$.

A point z_0 is a *periodic* point for f_c if for some positive integer p, $f_c^p(z_0) = z_0$ and the smallest such p is called the *period* of the orbit; z_0 is *pre-periodic* if $f_c^{p+n}(z_0) = f_c^n(z_0)$ for some integers $p, n > 0$. If z_0 is a periodic point, the set of points $\{z_0, z_1, \ldots, z_{p-1}\}$ are all periodic and form a *periodic cycle*. The cycle is called *attractive, neutral or hyperbolic* as the multiplier $\lambda = df^p(z_0)/dz$ has absolute value less than, equal to or greater than 1. If $\lambda = e^{2\pi ij/k}$ the cycle is *parabolic*; if $\lambda = 0$ the cycle is *super-attracting*. Note that if f_c is considered as a rational function on the Riemann sphere, the point at infinity is always a super-attracting fixed point.

A bounded component F of the Fatou set is *periodic* if $f^p(F) = F$ for some some positive integer p and is *eventually periodic* if $f^{n+p}(F) = f^n(F)$ for some integers $p, n > 0$. As usual, the *period* of the component is the smallest p for which its orbit is periodic and the periodic components in its orbit form a cycle. We also speak of the grand orbit of a component.

If f_c has an attractive or parabolic periodic cycle, the critical point is contained in one of the components of the Fatou cycle. We reserve the notation F_0 for this component. The set of components $\{F_0, f(F_0) = F_1, \ldots, f(F_{p-2}) = F_{p-1}\}$ are all periodic and form what we call the *Fatou cycle* of components.

If f_c has a non-parabolic neutral cycle and a non-empty Fatou cycle of p components, the map f_c^p is holomorphically conjugate to an irrational rotation and the orbit of the critical point accumulates on the boundaries of the components in this cycle. By the general classification theory, these are the only possibilities for non-empty bounded Fatou components. Therefore there is at most one cycle of bounded Fatou components.

In this paper we will be concerned only with those quadratic polynomials with bounded critical orbit and an attracting or parabolic periodic cycle. The situation for quadratics with non-parabolic neutral cycles is much more delicate and our techniques do not apply.

2.2 Dynamic External rays and Böttcher Coordinates

If $c \in M$, the basin of infinity is simply connected and the orbit of the critical point is bounded. There is unique holomorphic change of coordinate, $\psi_c : B_\infty \to \mathbb{C} \setminus \Delta$ called the *Böttcher coordinate at infinity* that conjugates the map $f_c|_{B_\infty}$ to the map z^2 on the complement of the closed unit disk Δ; the derivative at infinity is one.

The pre-image under ψ_c of a circle of radius $\rho > 1$ is a simple closed curve in B_∞ called an *equipotential* of potential $\log \rho$. It is mapped by f_c 2 to 1 onto the equipotential of potential $2 \log \rho$. The pre-image of the ray $\rho e^{2\pi i t}$ for fixed $t \in \mathbb{R}/\mathbb{Z}$ and $\rho > 1$ is an orthogonal trajectory of the equipotential curves. It is called an *external ray of angle* t and denoted by R_t. It is mapped one to one onto the external ray $R_{2t \text{ mod } 1}$ under the map f_c.

In the dynamic plane, as $\rho \to \infty$, R_t is asymptotic to the ray with argument $2\pi i t$. Written this way, the ray is measured in *turns*. We say R_t meets the *circle at infinity* at t. The action of f_c on the circle at infinity is thus a doubling mod 1. The equipotentials and external rays determine a chart of orthogonal coordinates in the basin at infinity.

If $\lim_{\rho \to 1} R_t$ exists, the ray is said to *land* on the Julia set. It is standard [12] that there are external rays landing at repelling and parabolic periodic points in the Julia set. Moreover, whenever there is a parabolic or attracting periodic cycle, the Julia set is locally connected and the inverse of the Bottcher coordinate is uniformly continuous and extends to the unit circle as a semi-conjugacy between the doubling map and the quadratic polynomial. The bounded Fatou components are also Jordan domains, and a similar property holds as well for them.

Böttcher coordinates may also be defined in a neighborhood of any super-attracting fixed point. If f_c has a Fatou cycle with p components that contains a super-attracting cycle, then $f_c^p(F_i) = F_i$ for any $i = 0, \ldots, p-1$. In particular, $0 \in F_0$ is a super-attracting fixed point for f_c^p. In this case, the rays defined by the Böttcher coordinate $\psi_c : F_0 \to \mathbb{C} \setminus \Delta$ that conjugates the map f^p to z^2 are called *internal rays*.

The boundary ∂F of a Fatou component in this case is precisely the set of endpoints of the internal rays. We will show which external rays land at these points.

2.3 Roots and components of the Mandelbrot set

The interior of the Mandelbrot set M contains simply connected components H such that for each $c \in H$, f_c has an attracting periodic cycle of the same period. There is a holomorphic map from the parameter c to each of the periodic points in the attracting cycle of f_c. These components are the *hyperbolic components*. The *period* of a hyperbolic component is the period of this attracting cycle.

The map $\mu : H_c \to \Delta$ defined by sending the point $c' \in H_c$ to the multiplier λ of the attracting cycle of $f_{c'}$ is a holomorphic homeomorphism. It extends continuously to the boundary, and in particular to the points where the argument of λ is a rational multiple of π.

Definition 2.1. Let H be a component of M. The *root point* $\mathbf{r}$ of H is the point $\mu^{-1}(1)$ on ∂H. The *center* $\mathbf{c}$ is the point $\mu^{-1}(0)$.

At the root there is a parabolic cycle and at the center there is a super-attractive cycle.

Hyperbolic components are divided into two types:

Definition 2.2. Suppose the period of H is p and let $\hat{z}$ be a point in the parabolic cycle corresponding to the root point $\mathbf{r}$. Let R_t be an external ray landing at $\hat{z}$. If $f_{\mathbf{r}}^p(R_t) = R_t$ the component H is *primitive*. Otherwise it is a *satellite*.

This notation comes from the fact that the root of a satellite component of period p is a boundary point of another hyperbolic component of lower period q where $q|p$. A primitive component is not.

Definition 2.3. Define a *chain* of hyperbolic components $\{H_i\}$, $i = k, k+1, \ldots, n$, $0 \leq k < n$, as a finite collection of components with the property that the root $\mathbf{r_i}$ of H_{i+1} is on the boundary of H_i. We say the chain goes from H_k to H_n. [1]

The component H_k may or may not be a primitive. If it is not, we can extend the chain to a chain $\{H_i\}$, $i = 0, \ldots, n$ such that H_0 is primitive.

2.4 Parameter External rays

In the parameter plane there is a Riemann map $\Psi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \Delta$ such that $\Psi(\infty) = \infty$ and the derivative at infinity is one. This is the parameter analog of the function ψ_c that defines the Böttcher coordinate in the dynamic plane. As for the Böttcher coordinate, Ψ too has external rays, defined in the same way : for each t, $\mathcal{R}_t$ is the pre-image under Ψ of the ray $\rho e^{2\pi i t}$, $\rho > 1$. To differentiate between dynamic and parameter rays, we will use different fonts and denote the former by R_t and the latter by $\mathcal{R}_t$.

In analogy with dynamic external rays, we say an external parameter ray *lands* at a point of M if $\lim_{\rho \to 1} \Psi^{-1}(\rho e^{2\pi i t})$ exists.

2.5 Orbit Portraits

We refer the reader to [12] for a full discussion and only give the definitions and results we will need here. Where we state the theorems as they appear we refer to the theorem number in [12]. In several instances, however, we have taken results from several theorems there and combined them; these are marked only with the citation [12].

Let $\mathcal{O} = \{z_0, \ldots, z_{p-1}\}$ be a periodic orbit for f. If R_t is a ray landing at z_0 then R_{2t} lands at $z_i = f(z_0)$ and $R_{2^n t} = R_t$ for the smallest n such that $2^n t \equiv t \bmod 1$. Note that this implies that if R_t lands on a periodic point, or a pre-image of a periodic point, t must be rational.

The collection $\{R_t, R_{2t}, \ldots, R_{2^{n-1}t}\}$ is called the *ray cycle* of the orbit and n is called the ray period. There is a natural equivalence relation on the ray cycle where two rays are identified if they land on the same point of the orbit; the equivalence classes are denoted by A_i.

Definition 2.4. The *orbit portrait* is the collection

$$\mathcal{P} = \mathcal{P}(\mathcal{O}) = \{A_0, \ldots, A_{p-1}\}.$$

[1] In [18] H_n is called *visible* from H_k.

The number of elements is the same in each A_i and is called the *valence* v. Denote the period of the angles in each A_i under the map $t \mapsto 2^p t \mod 1$ by r so that the period of each ray in the orbit is rp.

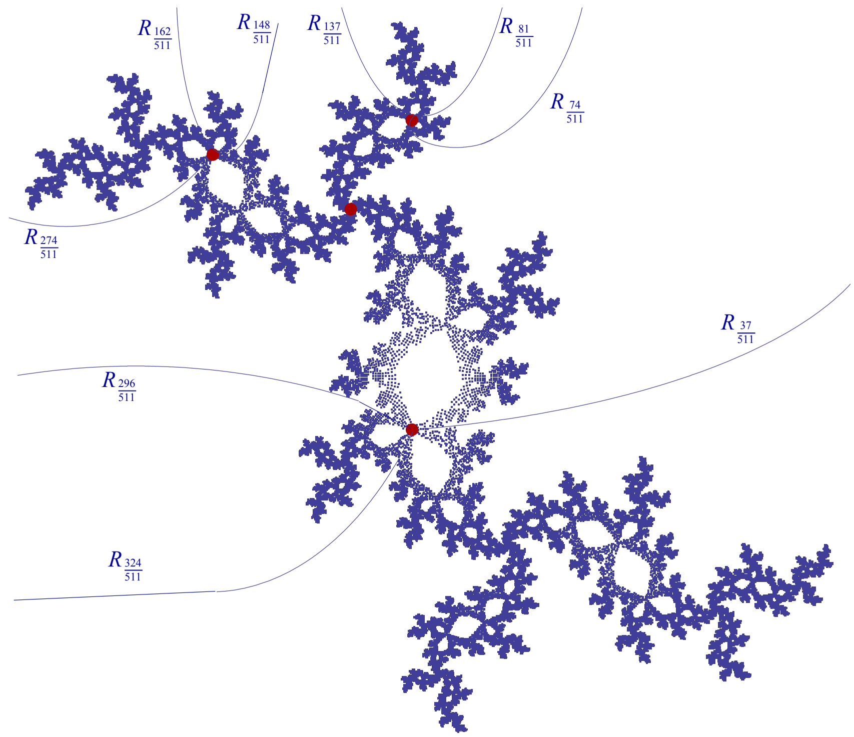

Figure 1. Orbit Portrait for the dynamic root with parameter value $c = -0.03111 + 0.79111i$

The period is 3 and the valence is 3, $\mathcal{P} = \{(\frac{74}{511}, \frac{81}{511}, \frac{137}{511}), (\frac{148}{511}, \frac{162}{511}, \frac{274}{511}), (\frac{296}{511}, \frac{324}{511} . \frac{37}{511})\}$

If $v \geq 2$ or if the only angle in the portrait is zero, $\mathcal{P} = \{\{0\}\}$, then the orbit portrait is called *non-trivial*. For a non-trivial orbit with $v \geq 2$, the v rays in each A_i divide the dynamic plane into v sectors. They divide the full circle of angles at infinity into v arcs so that the sum of the arc lengths of the sectors is $+1$.

Under minimal assumptions satisfied in all cases here, the orbit portrait has the following properties:

- Each A_j is mapped onto A_{j+1} under the doubling map $t \mapsto 2t \bmod 1$
- All the angles in $A_1 \cup \ldots \cup A_p$ are periodic under doubling with common period rp, and
- For each $i \neq j$ the sets A_i and A_j are contained in disjoint sub-intervals of the circle.

Given a rational number t, we can form its *formal* orbit portraits by forming the ray cycle using the doubling map and form the various possible partitions into subsets A_i.

We will need the following theorems:

THEOREM 2.5 ([12], Theorem 1.1) *Let $\mathcal{O}$ be an orbit of period $p \geq 1$ for $f = f_c$. If there are $v \geq 2$ dynamic rays landing at each point of $\mathcal{O}$, then there is one and only one sector S_1 based at some point $z_1 \in \mathcal{O}$ which contains the critical value $c = f(0)$, and whose closure contains no point of $\mathcal{O}$ other than z_1. Among all of the pv sectors based at all the points of $\mathcal{O}$, S_1 is the unique sector of smallest angular width.*

We call S_1 the *critical value sector*.

Suppose $\mathcal{O}$ is given with valence $v \geq 2$ and suppose f_c admits an orbit with portrait $\mathcal{O}$. Let $R_{t_\pm}$ be the dynamic rays defining the critical value sector of f_c, $0 < t_- < t_+ < 1$.

THEOREM 2.6 ([12], Theorem 1.2) *The two corresponding parameter rays $\mathcal{R}_{t_\pm}$ land at a single point $\mathbf{r}_\mathcal{P}$ of the parameter plane. These rays, together with $\mathbf{r}_\mathcal{P}$ divide the plane into two open subsets, $W_\mathcal{P}$ and $\mathbb{C} \setminus \overline{W_\mathcal{P}}$ such that: a quadratic map f_c has a repelling periodic orbit with portrait $\mathcal{P}$ if and only if $c \in W_\mathcal{P}$, and has a parabolic orbit with portrait $\mathcal{P}$ if and only if $c = \mathbf{r}_\mathcal{P}$.*

Definition 2.7. The set $W_\mathcal{P}$ is called the $\mathcal{P}$-*wake* in parameter space and $\mathbf{r}_\mathcal{P}$ is the *root point* of the wake. The set $M_\mathcal{P} = M \cap W_\mathcal{P}$ is the $\mathcal{P}$-*limb* of the Mandelbrot set.

The open arc $\mathbf{I}_{S_1} = (t_-, t_+)$ consisting of all angles of dynamic rays R_t contained in S_1 is the *characteristic arc* $\mathbf{I}_\mathcal{P}$ for the orbit portrait $\mathcal{P}$.

An immediate corollary to theorem 2.6 is

COROLLARY 2.8 [12] *If $\mathcal{P}$ and $\mathcal{P}'$ are two distinct non-trivial orbit portraits and if $\mathbf{I}_\mathcal{P} \subset \mathbf{I}_{\mathcal{P}'}$ with $\mathcal{P} \neq \mathcal{P}'$ then $\overline{W}_\mathcal{P} \subset W_{\mathcal{P}'}$.*

The following theorem tells us that all non-trivial formal orbit portraits are realized as roots of hyperbolic components.

THEOREM 2.9 [12] *There is a one to one correspondence between the set of non-trivial formal orbit portraits and the root points of the Mandelbrot set. If $\mathcal{P}$ is the portrait, we denote the corresponding root $\mathbf{r}_\mathcal{P}$.*

As part of the proof of this theorem, it is necessary to distinguish the orbit portraits of primitive and satellite components . For primitive components, there are two distinct cycles of rays that land on the parabolic orbit. The valence is 2 and the rays that land have the same period as the orbit so that $r = 1$. For satellite components, there are $v = r$ rays landing at each point in the orbit and these are permuted under doubling. The number of rays is vp and there is only one distinct cycle.

For points in the component with root $\mathbf{r}_\mathcal{P}$ we have

THEOREM 2.10 [12] *If c is in the hyperbolic component with root $\mathbf{r}_{\mathcal{P}}$, then f_c has a repelling periodic cycle with orbit portrait $\mathcal{P}$ and the points in this cycle lie on the boundaries of the Fatou cycle components.*

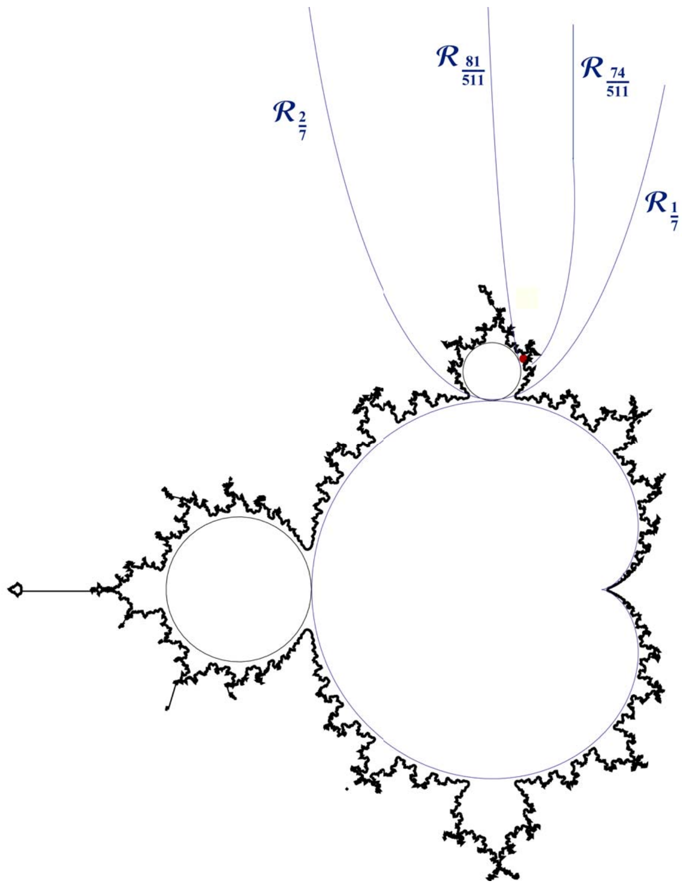

Figure 2. Rays in the Parameter plane: the inner rays land at the root $\mathbf{r}_{\mathcal{P}}$ for $\mathcal{P}$ as in Figure 1

Figure 1 shows the Julia set and the orbit portrait of the repelling cycle on the boundary of the Fatou set. Figure 2 shows the corresponding rays in the parameter plane that define the corresponding wakes.

It is possible to form chains of components using the following deformation theorems. Suppose $\mathcal{P}$ is an orbit portrait of period p and ray period $rp \geq p$.

THEOREM 2.11 ([12] Theorem 4.1) *Let H_c be a hyperbolic component with root $\mathbf{r}_{\mathcal{P}}$. Then we may form a smooth path $c(t)$ in parameter space ending at $\mathbf{r}_{\mathcal{P}}$ such that $f_{c(t)}$ has a repelling orbit of period p with portrait $\mathcal{P}$ whose periodic points lie* *on boundaries of the Fatou cycle $f_{c(t)}$. Furthermore, $f_{c(t)}$ has an attracting orbit of period rp and, as $c(t) \to \mathbf{r}_{\mathcal{P}}$, both orbits converge to the orbit of $f_{\mathbf{r}_{\mathcal{P}}}$.*

THEOREM 2.12 ([12], Lemma 4.4) *Under the same hypothesis as above, there also exists a smooth path $c(t)$ in parameter space ending at $\mathbf{r}_{\mathcal{P}}$ such that $f_{c(t)}$ has an attracting orbit of period p, and a repelling orbit of period rp whose points lie on the boundaries of the Fatou cycle. Furthermore, the dynamic rays with angles in $\mathcal{A}_{\mathcal{P}} = A_1 \cup \cdots \cup A_p$ all land on this repelling orbit.*

It follows that given a satellite component H_c, we can form a chain of components $\{H_{c_i}\}$, $i = 0, \ldots, n$ as follows: c_{i+1} is on the boundary of H_{c_i} for $i < n$, $c_n = c$ and H_{c_0} is primitive. We say that each H_{c_i} is a satellite of the primitive H_{c_0} for all $i = 1, \ldots, n$.

The periods of the attracting cycles of all components in the chain are multiples of the period of the attracting cycle of the primitive H_{c_0}. The wakes $W_{\mathbf{r_i}}$, $i = 0, \ldots, n$ of the roots of the components in the chain are nested, $\overline{W}_{\mathbf{r_{i+1}}} \subset W_{\mathbf{r_i}}$. We call the wake $W_{\mathbf{r_0}}$ the *pre-wake* of any of the wakes $W_{\mathbf{r_i}}$, $i = 1, \ldots, n$ provided there exist nested formal orbits of the given periods.

It also follows from this discussion that given any component H_c with root $\mathbf{r}_{\mathcal{P}}$ of period p_0, and a set of integers, $r_1, \ldots r_n$, we can form a chain H_{c_i}, such that $c_0 = \mathbf{r}_{\mathcal{P}}$, the root of H_{c_i} is on the boundary of $H_{c_{i+1}}$ and the period p_i of the attracting cycle in H_{c_i} is $r_i p_{i-1}$. The wakes and the characteristic arcs of the portraits are nested.

3. Portrait Laminations

Let $\mathcal{P} = \{A_0, \ldots, A_{p-1}\}$ be the orbit portrait of a repelling or parabolic periodic cycle of period p such that $\mathbf{r}_{\mathcal{P}}$ is not a primitive root. For the rest of the paper we shall tacitly assume, unless we say otherwise, that the portraits we consider are non-trivial and are not the $\{0\}$ portrait.

For each ray $R_{t_j} \in A_i$, $i = 0, \ldots p-1$, mark the point $e^{2\pi i t_j}$ on the unit circle. Join the points corresponding to the rays in each equivalence class A_i by hyperbolic geodesics so that they form a convex hyperbolic polygon $Q(A_i)$. The number of sides is the valence v. Since the period of the portrait is p we have p polygons.

Note that if there are only two rays in an equivalence class, the polygon is degenerate and is just a line. This happens, for example, for the portrait $\mathcal{P} = (\{\frac{1}{3}, \frac{2}{3}\})$ and in particular, for portraits corresponding to roots of primitive components. The discussion below is valid for degenerate polygons. We will assume, however, that our polygons are not degenerate and leave it to the reader to make the necessary adjustments when they are.

Definition 3.1. The *length* of a side of a polygon is the length of the minor arc of the unit circle that the side subtends.

We assume the polygons are labeled so that the vertices of $Q(A_{i+1})$ are obtained from those of $Q(A_i)$ by doubling (mod 1). That is, we double the arguments of the vertices and connect to form a convex polygon again.

Definition 3.2. The doubling map on the circle at infinity thus induces a map on the polygons that we call the *lamination map* and which we denote by $\Lambda(\mathcal{P})$ or simply Λ if there is no confusion. The name is justified below.

LEMMA 3.3. *The polygons $Q(A_i)$ for a given orbit portrait $\mathcal{P}$ are mutually disjoint.*

Proof. Let p be the period of the repelling or parabolic cycle with portrait $\mathcal{P}$. The rays in a given class A_i all land at the same point, z_i, fixed under the p^{th} iterate. Since the external rays to the Julia set cannot intersect outside the Julia set, the rays landing at z_j cannot intersect the rays landing at z_i for $i \neq j$. Thus the rays landing at z_j lie between two rays landing at z_i and all the vertices of $Q(A_j)$ must lie between two adjacent vertices of $Q(A_i)$.

By the same considerations, it follows that adjacent vertices go to adjacent vertices so that we can say $Q(A_{i+1})$ is obtained from $Q(A_i)$ by the lamination map induced by the doubling map on the circle at infinity. □

Under the lamination map Λ these polygons form a forward invariant cycle.

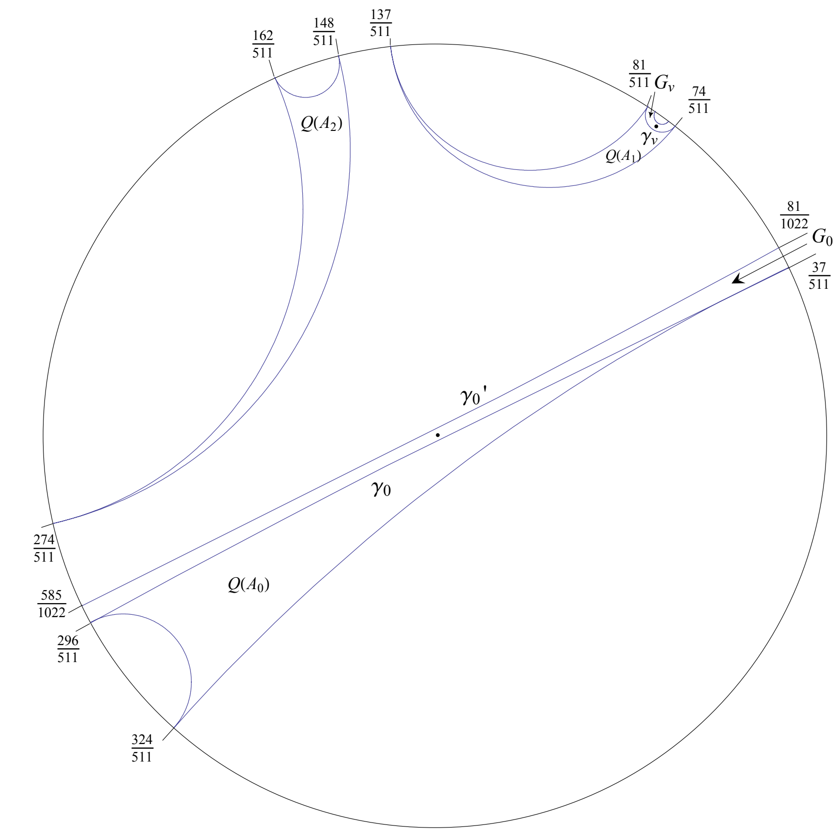

Figure 3. **The Portrait Lamination and the Gaps for f_c, $c = -0.03111 + 0.79111i$**

Definition 3.4. Suppose $\mathcal{P} = \{A_0, \ldots, A_{p-1}\}$ is a non-trivial orbit portrait of a repelling or parabolic periodic cycle such that $\mathbf{r}_{\mathcal{P}}$ is not a primitive root. We define the *portrait lamination* $\mathfrak{L}(\mathcal{P})$ as the collection of polygons $Q(A_i)$, $i = 0, \ldots, p-1$ together with all their pre-images under the lamination map $\Lambda(\mathcal{P})$.

PROPOSITION 3.5. *Let H_i, $i = 0, \ldots, n$ be a chain such that H_0 is primitive and $H_n = H_{\mathbf{r}_\mathcal{P}}$. Suppose $f \in H_{\mathbf{r}_\mathcal{P}}$. Then f has $n+1$ repelling periodic cycles with non-trivial orbit portraits $\mathcal{P}_0, \ldots \mathcal{P}_n$ and the characteristic arcs satisfy $\mathbf{I}_{\mathcal{P}_j} \subset \mathbf{I}_{\mathcal{P}_{j-1}}$; that is, they are nested. There is a portrait lamination $\mathfrak{L}(\mathcal{P}_j)$ for each $j = 1, \ldots, n$ and the polygons in each lamination are mutually disjoint. Moreover, the same is true for the union of the polygons in all the laminations.*

Proof. That f has $n+1$ repelling periodic cycles with non-trivial nested orbit portraits $\mathcal{P}_0, \ldots \mathcal{P}_n$ is a consequence of the discussion following theorems 2.11 and 2.12. That the portrait laminations exist and the polygons in each are disjoint is a consequence of applying the argument of lemma 3.3 to each lamination. That the union of the polygons is disjoint as well follows because the external parameter rays cannot intersect. □

The above proposition also holds if $f = f_{\mathbf{r}_\mathcal{P}}$ is the root of $H_{\mathbf{r}_\mathcal{P}}$ but the $n+1$-st cycle with portrait $\mathcal{P}$ is parabolic instead of repelling.

There is a repelling or parabolic periodic point corresponding to the orbit portrait $\mathcal{P}_n$ on the boundary of the Fatou component F_0 containing the critical point. This is called the *dynamic root point* by Schleicher in [17]. We will thus call the portrait $\mathcal{P}_n$ the *root portrait* for f.

We will focus on this root portrait and by abuse of notation set $\mathcal{P} = \mathcal{P}_n$ in what follows.

Definition 3.6. We define the *Gaps of* $\mathfrak{L}(\mathcal{P})$, denoted by $\mathcal{G} = \mathcal{G}(\mathcal{P})$, to be the complementary components of $\mathfrak{L}(\mathcal{P})$ in the unit disk Δ.

The gaps are open infinite sided ideal polygons in the disk. The boundary of a gap consists of geodesic sides of polygons in the portrait lamination and points on $\partial\Delta$.

PROPOSITION 3.7. *The lamination map induces a map on $\mathcal{G}$ that sends gaps to gaps.*

Proof. Since $\mathfrak{L}(\mathcal{P})$ is completely invariant under the lamination map, sides and vertices of gaps are mapped to sides and vertices of gaps. We need to show that all the sides of a given gap are mapped to sides of the same gap. Suppose γ and γ' are sides of a given gap G and $\Lambda(\gamma)$ and $\Lambda(\gamma')$ belong to different gaps. Then there is at least one leaf of the $\mathfrak{L}$ that separates them; call its endpoints s and t. Since the vertices are mapped by the doubling map their order is preserved. This means that there is a pre-image of s between an endpoint of γ and an endpoint of γ'. Then the pre-images of t either lie between the other endpoints of γ and γ' or are separated from G by γ or γ'. In the first case a leaf joining s to one of these pre-images divides G and it cannot be a gap; in the second, the leaf intersects another leaf of the gap which cannot happen. □

Note that this map, like the map on polygons, is not defined pointwise. By abuse of notation we call both the lamination map and its extension to gaps, Λ.

The endpoints of a given side of a gap subtend two complementary arcs in $\partial\Delta$; one contains vertices of the gap and the other contains vertices of the polygon the side belongs to. We say the side *bounds* the latter arc.

We want to focus on two particular gaps. One of the gaps has a side γ_v whose endpoints (t_-, t_+) bound the characteristic arc $\mathbf{I}_{S_1}$. We call this gap the *critical value gap* and denote it G_v. The side γ_v belongs to one of the polygons $Q(A_i)$. We may label the polygons so this one is $Q(A_1)$.

Proposition 3.8. *The two pre-images of γ_v under Λ are symmetric with respect to the origin. Moreover, no side of any polygon in the lamination separates these pre-images.*

Proof. Assume $0 < t_- < t_+ < 1$. By Theorem 2.6, the characteristic arc bounds the smallest sector among all sectors determined by the orbit $\mathcal{O}$ so that

$$t_+ - t_- < 1/2.$$

The two pre-images of this arc under the doubling map are the arcs, taken counterclockwise,

$$(\frac{t_- + 1}{2}, \frac{t_+}{2}) \text{ and } (\frac{t_-}{2}, \frac{t_+ + 1}{2}).$$

These are clearly symmetric with respect to the origin and each has arc length greater than $\frac{1}{2}$.

Denote the sides of the lamination polygons that bound the complements of these arcs by γ_0 and γ_0'. One of these is a side of $Q(A_0)$; call this one γ_0. The arc bounded by each of these sides is mapped one to one onto the complement of the characteristic arc (t_+, t_-).

Recall that γ_v is a side of a polygon of $\mathfrak{L}$ with endpoints (t_+, t_-). Suppose there were a side γ of one the polygons in the lamination with ends (s, t) where

$$\frac{t_-}{2} < s < \frac{t_+}{2} \text{ and } \frac{1 + t_+}{2} < t < \frac{1 + t_-}{2}.$$

Under the lamination map, the image, $\Lambda(\gamma)$ would have one endpoint at $2s$ where $t_- < 2s < t_+$, and the other endpoint at $2t$ where $t_+ < 2t < t_-$. This would imply that $\Lambda(\gamma)$ intersects γ_v, however, contradicting the fact that the polygons in the lamination are disjoint. □

By the second statement in the proposition the two sides γ_0 and γ_0' bound a common gap. This gap is the unique pre-image of G_v under Λ; we call it the *critical gap* and denote it by G_0. We denote by $Q(A_0)'$ the other pre-image of $Q(A_1)$ under Λ. We saw above that each of arcs on $\partial\Delta$ between γ_0 and γ_0' maps one to one onto the critical value sector.

The grand orbit of G_v under Λ is $\mathcal{G} = \mathcal{G}(\mathcal{P})$.

As an immediate corollary to the proposition we have

Corollary 3.9. *The sides γ_0 and γ_0' of $Q(A_0)$ and $Q(A_0)'$ are the longest sides among all the polygons in the lamination $\mathfrak{L}(\mathcal{P})$.*

It follows from this discussion that if a polygon $Q \neq Q(A_0), Q(A_0)'$ of the lamination has a side in the boundary of G_0, this side has both endpoints in one or the other of the intervals $(\frac{t_-}{2}, \frac{t_+}{2})$ and $(\frac{1+t_+}{2}, \frac{1+t_-}{2})$. Any other side of Q must have endpoints in the same interval. Moreover, the side of Q that is in the boundary of G_0, must be the longest side of Q.

Recall that the period of the orbit of $\mathcal{P}$ is p but the period of a component in the Fatou cycle is vp where v is the valence of the portrait.

Proposition 3.10. *None of the sides $\{\Lambda^{-j}(\gamma_0')\}$, $j = 1, \ldots, vp-1$ is a side of G_0 but there are two sides in $\{\Lambda^{-vp}(\gamma_0'\}$ that are; we denote them by $E_+(\gamma_0')$ and $E_-(\gamma_0')$. These are the second longest sides of G_0 and γ_v and the side $\Lambda(E_+(\gamma_0')) = \Lambda(E_-(\gamma_0'))$ of G_v are the longest and second longest sides of G_v.*

Proof. The vp^{th}-iterate of the lamination map fixes the sides and vertices of $Q(A_0)$ so that $\Lambda^{vp}(\gamma_0) = \gamma_0$. Any lower iterate either maps $Q(A_0)$ onto one of the other polygons in $\mathcal{L}$ or permutes its sides. Moreover $\Lambda^{vp-1}(Q(A_1)) = Q(A_0)$ so that $\Lambda^{vp}(Q(A_0)') = Q(A_0)$ and $\Lambda^{vp}(\gamma_0') = \gamma_0$.

Considering the extension to gaps we see that $\Lambda^{vp}(G_0) = G_0$ and γ_0 has exactly two pre-images in G_0, γ_0' and itself. It follows that γ_0' also has two pre-images in G_0. Call these $E_+(\gamma_0')$ and $E_-(\gamma_0')$. Because G_0 is symmetric, they will be a pair of symmetric sides of G_0, distinct from γ_0, γ_0', and both endpoints of each will lie in one of the symmetric arcs of $\partial\Delta$ between γ_0 and γ_0'. Since γ_0, γ_0' are the longest sides of G_0, these are the second longest. It then follows that γ_v and the side $\Lambda(E_+(\gamma_0')) = \Lambda(E_-(\gamma_0'))$ of G_v are the longest and second longest sides of G_v. □

We use argument of the preceding proof to construct two "inverse branches" of the map Λ^{vp} on G_0: let $E_+ : G_0 \to G_0$ be the map that sends γ_0 to γ_0' and γ_0' to $E_+(\gamma_0)$; and let $E_- : G_0 \to G_0$ be the map that sends γ_0 to γ_0' and γ_0' to $E_-(\gamma_0)$. If we insist that $\Lambda^{vp} \circ E_+$ and $\Lambda^{vp} \circ E_-$ be the identity map on the sides of G_0 these maps are well defined on every side of G_0. We will use these maps in the next section.

We note that if we iterate either of the maps Λ^{vp} or E_+ on G_0, the endpoints of γ_0, which are fixed, are accumulation points of all the vertices of the sides of G_0. The same is therefore true for any point in the grand orbit of these vertices and thus for any vertex of a polygon in the lamination.

We now prove that the boundary of G_0 in $\partial\Delta$ is a Cantor set.

Lemma 3.11 Cantor Set Lemma *Let G_0 be the critical gap of the polygon lamination $\mathcal{L}(\mathcal{P})$. Then the $\partial G_0 \cap \partial\Delta$ is a Cantor set.*

Proof. Denote the length of the characteristic arc of (t_-, t_+) of the $\mathcal{P}$-limb by

$$\ell = |t_+ - t_-|.$$

Label the arcs on $\partial\Delta$ between γ_0 and γ_0' by I_0 and I_1. Since Λ doubles the lengths of the sides (mod 1), it is straightforward to compute that for $k = 1, 2$ the length of I_k is $\ell_1 = \ell/2$. Remove the complementary arcs to I_0 and I_1.

Let $\tilde{\gamma}$ be a side of G_0 and a side the of the polygon $\tilde{Q} = E(Q(A_0))$ where the branch E of Λ^{-vp} is chosen to fix G_0. Suppose the endpoints of $\tilde{\gamma}$ are $\tilde{t}_\pm$. Denote the length of $\tilde{\gamma} = |\tilde{t}_+ - \tilde{t}_-|$ by $|\tilde{\gamma}|$. Since the other sides of $\tilde{Q}$ are not sides of G_0, their endpoints must lie in the interval $(\tilde{t}_+, \tilde{t}_-)$. It follows that $\tilde{\gamma}$ is the longest side of $\tilde{Q}$. Since Λ doubles the lengths of the sides (mod 1), we see that

$$|\tilde{\gamma}| = |E(\gamma_0)| = \frac{1}{2^{vp+1}}\ell.$$

The endpoints of $\tilde{\gamma}$ must lie in only one of the intervals I_k, $k = 0, 1$. Let $\tilde{\gamma}' = E'(\gamma_0)$ be the pre-image of γ_0 under the other choice E' of Λ^{-vp} fixing G_0. By symmetry, it will have the same length as $\tilde{\gamma}$ and its endpoints will lie in the other I_k.

Now remove the intervals spanned by $\tilde{\gamma}$ and $\tilde{\gamma}'$ leaving four intervals, $I_{00}, I_{01}, I_{10}, I_{11}$. Computing, we see that each I_{jk} has length

$$\ell_2 = |I_{jk}| = \frac{2^{vp} - 1}{2^{vp+1}}\ell_1.$$

We iterate this process: at stage n we have 2^n intervals of length ℓ_n; we remove an interval whose length is $\frac{\ell_n}{2^{vp}}$ from each of these and obtain 2^{n+1} intervals of length

$$\ell_{n+1} = \frac{1}{2}\ell_n(1 - \frac{1}{2^{vp}}) = \frac{2^{vp} - 1}{2^{vp+1}}\ell_n.$$

This is a standard Cantor set construction and from the formulas it follows easily that the sum of the lengths of the removed intervals tends to 1. □

We have the following relation between gaps of polygon laminations for portraits of chains.

LEMMA 3.12. *Suppose that there is a finite chain $H_{\mathbf{r}_{\mathcal{P}_i}}$, $i = 0\ldots, n$ from $H_{\mathbf{r}_{\mathcal{P}}}$ to $H_{\mathbf{r}_{\mathcal{Q}}}$ with $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_n = \mathcal{Q}$ and $\mathbf{r}_{\mathcal{Q}} \succ \mathbf{r}_{\mathcal{P}}$. Denote the lamination polygons of $\mathfrak{L}(\mathcal{P}_i)$ by $Q_i(A_j)$ and the critical gaps and critical value gaps of $\mathcal{P}_i$ by $G_{0,i}$ and $G_{v,i}$. Then the critical gaps and the critical value gaps are nested, $G_{0,i+1} \subset G_{0,i}$ and $G_{v,i+1} \subset G_{v,i}$. Moreover, if ζ is a boundary point in Δ of $G_{0,n}$, it is a boundary point in Δ for all $G_{0,i}$ for all $i = 0, \ldots, n-1$.*

Proof. By Proposition 3.5, the polygon laminations are mutually disjoint for all the $\mathcal{P}_i$. It follows that we can fit polygons from each of the portraits $\mathcal{P}_i$ into the gap $G_{0,0}$. Because the characteristic arcs of the portraits in the chain are nested, the polygons $Q_1(A_0)$ and $Q_1(A_0)'$ separate the polygons $Q_0(A_0)$ and $Q_0(A_0)'$ from the origin and the gap $G_{0,1} \subset G_{0,0}$. Continuing in this way, we obtain a nested sequence of critical gaps. Taking the images of the critical gaps under the appropriate lamination maps the critical value gaps are also nested. It follows immediately that the boundary points on Δ are also nested. □

4. The Relation between F_0 and G_0

4.1 Centers of Hyperbolic Components

Fix a hyperbolic component H with root $\mathbf{r}_{\mathcal{P}}$ and assume the period of the parabolic cycle at $\mathbf{r}_{\mathcal{P}}$ is p. Let $c_{\mathcal{P}}$ be the unique point in H for which the attracting cycle is super-attractive. This is the center of H. The period of the super-attractive cycle is vp where v is the valence of the portrait. For simplicity of notation denote this map by $f = f_{c_{\mathcal{P}}}$ for now. By Theorem 2.10, there is a repelling cycle of period p with portrait $\mathcal{P}$ whose orbit lies in the boundary of the Fatou cycle. Let z_0 be the point in this orbit on the boundary of the component F_0, containing the critical point; then $f^p(z_0) = z_0$ is a repelling fixed point on ∂F_0 and is the dynamic root for f. Let $z_k = f(z_{k-1})$, $k = 1, \ldots, p$ be the orbit of z_0. Each point is a fixed point of f^p on the boundary of v Fatou components.

We can define a Böttcher coordinate in the component F_0 such that the internal ray in F_0 with argument 0 lands on the fixed point z_0 of f^p. There is another point z_0' on ∂F_0 such that $f(z_0') = z_1$; it will have internal ray 1/2. Using branches of the inverse of f^{nvp} that fix F_0, we find that for each $n = 1, 2, \ldots$ there are n points $\zeta_{k,n}$ with internal ray $2^k/2^n$, $k = 0, \ldots, n-1$ on ∂F_0. The orbit portrait tells us there are exactly v external rays landing at z_0. This immediately implies v external rays land at each endpoint on ∂F_0 of an internal ray of argument $2^k/2^n$, $k = 0, \ldots, n-1$. These endpoints are the pullbacks of the v external rays that land at the periodic point z_0 by the branches of the inverses of f^{nvp} that fix F_0, $n = 1, 2, \ldots$.

4.2 Defining the map Ψ

We continue to consider the dynamics of the map $f = f_{c_{\mathcal{P}}}$ at the center of a hyperbolic component. We fix notation as follows. We let $m = vp$ denote the period of F_0 and let $g_{\pm} = f^{-m}$ be the branches of the inverse that fix F_0. We denote by $E_{\pm}$ the branches defined in section 3 of the inverse of the first return of the lamination map, Λ^m, that fix the gap G_0. We use the fact that as sets in the plane, both F_0 and G_0 are symmetric with respect to the origin.

THEOREM 4.1. *There is a continuous map $\Psi : G_0 \to F_0$ with the following properties:*

- Ψ *preserves the dynamics; that is,* $\Psi(G_0) = f^m \circ \Psi(G_0)$
- Ψ *has a continuous extension* $\bar{\Psi} : \partial G_0 \cap \partial\Delta \to \partial F_0$*.*

Proof. Draw a ray in G_0 from the origin to each point in $\partial G_0 \cap \partial\Delta$. These rays remain inside G_0 because, by construction, it is starlike with respect to the origin. Set $\Psi(0) = 0$.

To each side γ of G_0, there is a triangle T_γ formed by the γ and the two rays from the origin to the endpoints of the side. Since the endpoints of the sides are disjoint, these triangles are disjoint. Define $\Lambda^m(T_\gamma)$ as the triangle formed by $\Lambda^m(\gamma)$ and the rays of from the origin to endpoints of this side. The maps $E_{\pm}$ are also defined on the triangles in the same way.

If T_0 is the triangle with side γ_0, parameterize the ray sides of T_0, $t_0(s), t_0'(s)$ in Δ, by hyperbolic length s; join points at equal distance from the origin by a geodesic $l_s(t)$, $0 \le t \le 1$, where $l_s(0) = t_0(s)$ and $l_s(1) = t_0'(s)$. The component F_0 admits a hyperbolic metric. Parameterize the internal ray with argument 0 it by its hyperbolic length $r_0 = r_0(s)$.

Define the map $\Psi_{|T_0}$ from the interior of T_0 into F_0 as follows. For all $t \in [0, 1]$ and all $s \in [0, \infty]$ set

$$\Psi(l_s(t)) = r_0(s).$$

Note that the side γ_0 of T_0 and the endpoints of the other two sides map to a single point on the boundary of F_0, the dynamic root.

Use $E_{\pm}$ and $g_{\pm}$ to extend $\Psi|_{T_\gamma}$ as a map from the remaining triangles to the internal rays $r_{2^k/2^n}(s)$, $k = 0, \dots n-1$. We have

$$E_+(T_0) = T_0\ , g_+(r_0) = r_0.$$

Set $T_{01} = E_-(T_0)$. If $\mathbf{x}_n = x_0 x_1 \dots x_n$ where $x_i \in \{0, 1\}$ we define inductively,

$$T_{0\mathbf{x}_n} = E_-(T_{\mathbf{x}_n}) \text{ and}$$

$$T_{1\mathbf{x}_n} = E_+(T_{\mathbf{x}_n}).$$

We can use the same scheme to code the internal rays:

$$r_{01} = g_-(r_0) \text{ has internal argument } 1/2$$

$$r_{10} = g_+(r_0) = r_0$$

$$r_{0\mathbf{x}_n} = g_-(r_{\mathbf{x}_n}) \text{ and}$$

$$r_{1\mathbf{x}_n} = g_+(r_{\mathbf{x}_n}).$$

We extend the maps $\Psi|_{T_\gamma}$ to the interiors of the triangles in the same way we did for T_0.

It follows from lemma 3.11 that the endpoints of the ray sides of the triangles form a dense set in $\partial G_0 \cap \partial\Delta$. Therefore, given a point $t \in \partial G_0 \cap \partial\Delta$ that is not a vertex of a triangle, we can find a sequence T_{γ_n} of triangles whose ray sides converge to the ray with endpoint t. We extend the map Ψ to this ray by continuity. This gives us an extension of Ψ to all of G_0 and to its closure in $\overline{\Delta}$. Since the images of the triangles are precisely the collection of all internal rays whose argument has denominator 2^n for some n, and this set is dense in the set of all internal rays, the image of the extended map consists of $F_0 \cup \partial F_0$.

□

We can use this theorem to characterize the external rays that land on ∂F_0.

Theorem 4.2. *The external ray R_t lands on ∂F_0 if and only if $t \in \partial G_0 \cap \partial\Delta$.*

Proof. The map Ψ assigns the argument of one or two points in $\partial G_0 \cap \Delta$ to each point of ∂F_0, depending whether or not they are endpoints of a side of G_0. By the construction of the lamination $\mathcal{L}$, if t is not in $\partial G_0 \cap \Delta$, a ray in Δ with this argument must pass through a side of some polygon in the lamination. This means that the external ray R_t is blocked from landing on ∂F_0 because it lies between two rays that land at the same point.

If $t \in \partial G_0 \cap \Delta$, then either t is a vertex of a polygon or a limit of such points, $t = \lim_{n\to\infty} t_n$. Because we have a Böttcher coordinate in F_0, the boundary of F_0 is locally connected and every internal ray lands on ∂F_0. The points $\Psi(t_n)$ lie on ∂F_0 and are endpoints of the external rays R_{t_n}. Since Ψ is continuous $\Psi(t_n)$ converges to $\Psi(t)$ and the Hausdorff limit of the external rays R_{t_n} is a ray R_t whose endpoint is $\Psi(t)$ and so it lands on ∂F_0. □

As a corollary we have

Corollary 4.3. *The rational external ray R_t lands on ∂F for any component F of the Fatou set if and only if $2^n t$ mod $1 \in \partial G_0 \cap \partial\Delta$ for some n.*

Proof. This follows directly from the fact that the map Ψ preserves the dynamics. □

We are now ready to characterize all the rational rays landing on ∂F.

Theorem 4.4. *At a point on ∂F, for the polynomial $f_{c_\mathcal{P}}$ at the center of the hyperbolic component $H_{\mathbf{r}_\mathcal{P}}$, either*

- *there are exactly v external rays that land, and this point is in the grand orbit of the dynamic root z_0 of period p and portrait $\mathcal{P}$ with valence v or,*
- *there is exactly one ray landing at the point. In this case, if R_t lands on ∂F and t is rational, then t, or some iterate of it under doubling, belongs to some orbit portrait $\mathcal{Q} \neq \mathcal{P}$ whose root $\mathbf{r}_\mathcal{Q}$ is in the wake of $W_{\mathbf{r}_\mathcal{P}}$ and can be reached by a chain from $H_{\mathbf{r}_\mathcal{P}}$ to $H_{\mathbf{r}_\mathcal{Q}}$. The period of $\mathcal{Q}$ is a multiple of vp.*

Proof. We may assume, by taking iterates if necessary, that R_t lands at a point ζ_t on ∂F_0 for $f_{c_\mathcal{P}}$. Consider the gap G_0 for this portrait. By Theorem 4.2, t is on the boundary of the G_0 corresponding to $\mathcal{P}$. It is either a vertex of a lamination polygon in $\mathfrak{L}(\mathcal{P})$, in which case it belongs to the grand orbit of the ray cycle of the portrait $\mathcal{P}$ or not, in which case only one ray lands at ζ_t.

Assume now that only one ray lands at ζ_t. The same is therefore also true for any image under the lamination map. Let $\zeta_{t'}$ be an iterate of ζ_t on ∂F_1. By the definition of the characteristic arc $\mathbf{I}_{\mathcal{P}}$, since t' is not one of its endpoints, it lies in $\mathbf{I}_{\mathcal{P}}$. Since t is rational we may assume t' is periodic of some period, say k, under doubling. It follows that t' lies on the boundary of $G_v = G_{0,v}$ for $\mathcal{P}$. We can find a root $\mathbf{r}_{\mathcal{P}_1}$ on the boundary of $H_{\mathbf{r}_{\mathcal{P}}}$ such that $t' \in \mathbf{I}_{\mathcal{P}_1}$. Draw in the polygons of $\mathfrak{L}(\mathcal{P}_1)$ that lie inside $G_{1,v}$; by Proposition 3.5, these polygons are disjoint from those bounding $G_{0,v}$. If t' is a vertex of one of these polygons, it belongs to the ray cycle of the portrait $\mathcal{P}_1$, and so, by Theorem 2.10 the angles of $\mathcal{P}_1$ are a trivial portrait for $f_{c_{\mathcal{P}}}$. If it is not such a vertex, it, (or one of the points in its periodic cycle) belongs to the boundary of $G_{1,v}$ for $\mathcal{P}_1$ and we repeat this process. We find a root $\mathbf{r}_{\mathcal{P}_2}$ such that $t' \in \mathbf{I}_{\mathcal{P}_2}$ and draw its polygon lamination $\mathfrak{L}(\mathcal{P}_2)$ inside $G_{1,v}$. We claim that after repeating this process n times, for some finite n, t' must be a vertex of a polygon of $\mathfrak{L}(\mathcal{P}_n)$.

If the above procedure does not stop, because the periods of the portraits are increasing multiples of p, for some n, the period of $\mathcal{P}_n$ will be greater than k, the period of t'. This would mean, however, that there is a point on the boundary of $G_{n,v}$ for $\mathcal{P}_n$ whose period is less than the period of $G_{n,v}$ and hence a point on the boundary of $G_{n,0}$ for $\mathcal{P}_n$ whose period is less than the period of $G_{n,0}$. Applying Theorem 4.1, this means that there is a point on ∂F_0 for $f_{c_{\mathcal{P}_n}}$ with lower period than the period of F_0. This is clearly a contradiction.

The points on the boundary of ∂F that are endpoints of rational rays are thus the grand orbits of rays whose arguments are in the non-trivial portraits obtained from roots reached by finite chains in parameter space. □

Remark 1. We note that this theorem implies that for any primitive component $H_{\mathbf{r}_{\mathcal{Q}}}$ inside the wake $W_{\mathcal{P}}$, arguments of parameter rays landing at $\mathbf{r}_{\mathcal{Q}}$ cannot be arguments of dynamic rays landing on any Fatou component of $f_{c_{\mathcal{P}}}$.

Remark 2. By corollary 2.8, we know that the only periodic or pre-periodic points in the full Julia set where more than one point may land correspond to cycles whose portraits lie in the pre-wake of $\mathcal{P}$. The above theorem shows that the landing points for these cycles do not lie on boundaries of Fatou components.

In the next section we will need to refer to the set we have described here of external rays that land on the boundary of the Fatou set of f so we introduce the following notation.

Definition 4.5. Denote by Land(f) the set of arguments of external rays that land on the boundary of the Fatou set of f.

4.3 Non-centers of hyperbolic components

Suppose now that c is either the root $\mathbf{r}_{\mathcal{P}}$ or an arbitrary point in the component $H_{\mathbf{r}_{\mathcal{P}}}$ with center $c_{\mathcal{P}}$. It is well-known, see for example, [14], that for all points in $c \in H_{\mathbf{r}_{\mathcal{P}}}$, there is a conjugacy Φ_c between the Julia sets J_c and $J_{c_{\mathcal{P}}}$ that preserves dynamics. That is, $\Phi_c : J_{c_{\mathcal{P}}} \to J_c$ and $f_c\Phi_c = \Phi_c f_{c_{\mathcal{P}}}$. At the root, the dynamics of the orbit of the dynamic root point are preserved by Corollary 2.8.

This immediately gives us

Theorem 4.6 . *The set of external rays that land on the boundary of the Fatou set of any $f \in H_{\mathbf{r}_{\mathcal{P}}}$ or its root depends only on $\mathcal{P}$. In other words, let $H_{\mathbf{r}_{\mathcal{P}}}$ be any hyperbolic component and let c be either its root or an interior point. Then*

$$\mathrm{Land}(f_c) = \mathrm{Land}(f_{c_{\mathcal{P}}}) = \mathrm{Land}(\mathcal{P}).$$

Thus, Land($\mathcal{P}$) is precisely described in Theorems 4.2 and 4.4 and Corollary 4.3.

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