Ergodicity of some classes of meromorphic functions *

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Abstract

We consider the class of meromorphic functions whose set of singular values is bounded and for which the $\omega-$limit set of the post singular set is a compact repeller. We show that if two simples growth conditions are satisfied, then the function is ergodic on its Julia set.

1 Introduction

The dynamical phenomena of systems generated by conformal mappings of the plane are controlled by the behavior of the forward orbits of the singular values. The closure of these orbits is called the post-singular set, or in the case of rational maps, the post-critical set. For rational maps, it is known that outside the post-critical set the function is expanding and this expansion leads to the following dichotomy: a rational map either acts ergodically with respect to Lebesgue measure on the sphere and the Julia set is the full sphere or the postcritical set behaves as a measure theoretic attractor. (See [16], [18]). For transcendental meromorphic functions, the essential singularities make the situation more complicated. For example, in [15] Lyubich proved that for the exponential function, the set of points whose ω-limit set contains the ω-limit set of the post-singular set may have positive but not full measure and in [17] McMullen proved that for the sine family the set of points attracted to infinity always has positive measure and many ergodic components.

In this paper, motivated by our study of the tangent family [14], we find sufficient conditions for a transcendental meromorphic function to be ergodic on its Julia set. We prove

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Theorem 1. *Let f be a transcendental meromorphic function whose singular values lie in a bounded set. If f satisfies two simple growth conditions, if the ω-limit set of the post singular set, ω_f, is compact and if for some $k > 1$, $|f'| > k$ on ω_f, then f is ergodic with respect to Lebesgue measure on its Julia set.*

As examples of the theorem we see first that if the omitted values of the tangent map $\lambda \tan z$ land on repelling periodic cycles, the map is ergodic and second that that there are values of λ such that the map $\lambda e^{-z^2} \sin z$ is ergodic. These special cases also follow from the results in [8].

Our techniques are an adaptation of those developed in [11] to control the set attracted to the essential singularities. The new ideas involve dealing with the poles.

The paper is organized as follows. In section 2 we set our notation and summarize the basic definitions and theory. In section 3 we prove our theorem and in section 4 we discuss applications and examples.

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2 Preliminaries

2.1 Julia sets of meromorphic functions

If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a transcendental meromorphic function, the orbits of points fall into three categories: they may be infinite, they may become periodic and hence consist of a finite number of distinct points, or they may terminate at a pole of f. To study the dynamics, we define the *stable set*, or *Fatou set*, Ω_f as the set of those points z such that the sequence $f^{\circ n}(z)$ is defined and meromorphic for all n and forms a normal family in a neighborhood of z. The *unstable set*, or *Julia set*, J_f is the complement of the stable set. We assume $\infty \in J_f$. Thus, Ω_f is open, J_f is closed. It is easy to see that Ω_f is completely invariant and $z \in J_f$ if and only if $f(z) \in J_f$ or $z = \infty$. As in the rational case $J_f \subset \overline{(\bigcup_{n \geq 0} f^{\circ -n}(z))}$ for all $z \in \hat{\mathbb{C}} - E_f$, where E_f consists of at most two *exceptional values* with finite inverse orbits. For all transcendental entire functions ∞ is an omitted value, so it is also exceptional. For generic transcendental meromorphic functions, the set of prepoles $\mathcal{P}_f = \cup_{n \geq 0} f^{\circ -n}(\infty)$ is infinite and $J_f = \overline{\mathcal{P}_f}$. Moreover, the Julia set J_f is the closure of the repelling periodic points ([1, 3, 9]).

The *singular set* $S = S_f$ of a meromorphic function f consists of those values in $\mathbb{C}$ at which f is not a regular covering. Therefore at a singular value v there is a branch of the inverse which is not holomorphic but has an algebraic or transcendental singularity. If the singularity is algebraic, v is a critical value, whereas if it is transcendental, there is a path $\alpha : [0, \infty) \to \mathbb{C}$ such that $\lim_{t \to \infty} |\alpha(t)| = \infty$ and $\lim_{t \to \infty} f(\alpha(t)) = v$, and v is called an *asymptotic value* for f. If we can associate to a given asymptotic value v an *asymptotic tract*, that is, a simply connected unbounded domain A such that $f(A)$ is a punctured neighborhood of v and $f : A \to f(A)$ is an unramified covering, then v is called a *logarithmic singularity*.

We define the *post-singular set* as

$$PS = PS_f = \overline{\bigcup_{n\geq 0} f^n(S)}$$

and denote its $\omega-$limit set by ω_f.

We distinguish the following classes of meromorphic functions. For f meromorphic and $p \geq 1$ define

$$\mathcal{M}_p = \{f : \mathrm{card}\, S = p\}, \quad \mathcal{M} = \bigcup_{p\geq 1} \mathcal{M}_p, \text{ and } \mathcal{B} = \{f : S \,\text{is bounded}\}.$$

Of course $\mathcal{M} \subset \mathcal{B}$. Some basic examples of entire functions in $\mathcal{M}$ are:

$$\lambda e^z, \quad a\sin(z) + b, \quad f(z) = \int^z h(\eta)\exp(p(\eta))d\eta,$$

where $\lambda, a, b \in \mathbb{C}$ and $h(\eta), p(\eta)$ are polynomials. The class $\mathcal{M}$ also includes meromorphic functions with polynomial Schwarzian derivative. An example is $\lambda\tan(z)$ whose Schwarzian derivative is constant. Examples of functions in $\mathcal{B} - \mathcal{M}$ are:

$$f(z) = \lambda \mathrm{sin}(z)/z \quad \text{and} \quad f(z) = \lambda e^{-z^2}\sin(z);$$

an example of a function not in $\mathcal{B}$ is $f(z) = \lambda z \sin z$.

We call the set ω_f a *repeller* if, for all $z \in \omega_f$ such that $f(z)$ is defined, $f(z) \in \omega_f$ and if there exists $k > 1$ such that, for all $z \in \omega_f$, $|f'(z)| \geq k$. It is a *compact repeller* if $f(z)$ is defined for all $z \in \omega_f$ and if ω_f is compact.

In particular, if $f \in \mathcal{M}$ and all critical and asymptotic values are eventually mapped to repelling periodic points, then ω_f is a compact repeller. Note that for transcendental meromorphic functions with poles, ω_f may include points with finite forward orbits. For example, the singular set S of the function $f(z) = \frac{\pi i}{2}\tan(z)$ consists of the omitted values $\{\pm\frac{\pi}{2}\}$. Since they are poles and have no forward orbit, $\omega_f = S_f$ is a repeller but is not a compact repeller. Clearly, if ω_f is not finite and contains prepoles it cannot be a compact repeller.

2.2 Classification of stable behavior

Let D be a component of the stable set; f will map D to a component, but if the image contains an asymptotic value, the map might not be onto. In any case, we call the image $f(D)$ and note that either there exist integers $m \neq n > 0$ such that $f^{\circ n}(D) = f^{\circ m}(D)$, and D is called *eventually periodic* with period $p = min(|m-n|)$, with the minimum taken over all such m, n, or for all $m \neq n$, $f^{\circ n}(D) \cap f^{\circ m}(D) = \emptyset$, and D is called *a wandering domain.*

The qualitative and quantitative description of the eventually periodic behavior is slightly more complicated than in the rational case because of the transcendental singularity at ∞ and the possibility that f^p may not be defined at some values. As for rational maps, eventually periodic domains may be attracting, parabolic or rotation domains. In addition, however, an eventually periodic domain D may be an an *essentially parabolic or Baker* domain; that is, the boundary of D contains a point z_0 (possibly ∞) such that $f^{np}(z) \to z_0$ for $z \in D$ and f^p is not holomorphic at z_0. If $p = 1$, then the only possible boundary point is ∞.

2.3 Two propositions

We shall need the following proposition proved in [13]

Proposition 2.1. *Let f be a meromorphic function and let D be an open disk such that $D \cap J_f \neq \emptyset$. Suppose that there are branches $g_{n_k} \in f^{-n}$ holomorphic and univalent on D with $n_k \to \infty$. Then $(g'_{n_k}) \to 0$ uniformly on every compact set in D.*

We shall also need the following version of Koebe's distortion theorem.

Proposition 2.2. *If $g : D(0,1) \to \mathbb{C}$ is a univalent map normalized so that $g(0) = 0$, then for every $0 < k < 1$ and $z, w \in D(0,k)$, there is a constant $T(k)$ such that and $|g'(z)|/|g'(w)|$ lies between $1/T(k)$ and $T(k)$.*

2.4 Orbits that tend to infinity

In this section we adapt the discussion of Eremenko and Lyubich [11] on entire functions in $\mathcal{B}$ to meromorphic functions in $\mathcal{B}$. They give a sufficient condition for the measure of the set

$$I_\infty(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$$

to be zero. We need additional condition on the Laurent expansions about the poles to prove this in the meromorphic case.

At each pole p of order m_p, form the Laurent expansion of f about p, $f_p(z) = c_p/(z-p)^{m_p}(1 + \phi_p(z-p))$ where $\phi(z)$ is analytic and $\phi(z-p) = o((z-p)^{m_p})$.

Suppose $f \in \mathcal{B}$. Let $D(q,r)$ denote the disk of radius r centered at q, A_r the annulus $A_r = \{z : r < |z| < \infty\}$.

Because $S(f)$ is bounded, we can find $R_0 > 0$ such that $S(f) \subset D(0, R_0)$. Fix $R > R_0$. Without loss of generality we may assume that f is analytic at 0 and $|f(0)| < R_0/2$.

Suppose $w \in A_R$ and $f(z) = w$. Since there are no singular values in A_R, any branch g of f^{-1} such that $g(w) = z$ can be continued analytically throughout A_R. Let g be some branch of f^{-1} and set $V = g(A_R)$. By [20], ∞ is either a logarithmic branch point of g with asymptotic value a, $V = V_a$ is simply connected and $f : V_a \to A_R$ is a universal covering, or ∞ is an algebraic branch point of g of order $m_p - 1$ for some integer m_p, $V = V_p$ is conformally equivalent to a disk punctured at a pole p of f and $f : V_p \to A_R$ is a regular m_p to 1 covering.

Let m denote Lebesgue measure in $\mathbb{C}$ and let $\Theta_R(r, f)$ be the linear measure of the set $\{\theta : |f(re^{i\theta})| < R\}$.

With this notation we have

Proposition 2.3. *Let $f \in \mathcal{B}$ and suppose that there is a positive integer J and positive constants $b, B, C_1, C_2 > 0$ such that for every p, and $z \in V_p$, the multiplicities m_p are bounded by J and the coefficients c_p and functions $\phi_p(z-p)$ and $\phi'_p(z-p)$ satisfy*

$$b < |c_p| < B, \quad |\phi_p(z-p)| < C_1, \quad |\phi'_p(z-p)| < C_1, \quad C_2 < |1+\phi_p(z-p)| \quad (*)$$

Moreover suppose

$$\liminf_{r\to\infty} \frac{1}{\log r} \int_1^r \Theta_R(t, f)\frac{dt}{t} > 0. \qquad (**)$$

Then $m(I_\infty(f)) = 0$.

Proof: We summarize the proof in [11] that $m(I_\infty) = 0$ for entire functions satisfying condition $(**)$, indicating the necessary changes to account for the poles. We remark that the proof there is valid if there are only finitely many poles. We therefore assume here that our functions have infinitely many poles

The first step in the proof that $m(I_\infty) = 0$ is to obtain a uniform expansion estimate on $|f'|$ restricted to each component V_a or V_p.

For simply connected V_a, the argument is the same as in [11] using Teichmüller's construction of a logarithmic coordinate. We outline that argument here. If ∞ is a logarithmic branch point of g, we first define a logarithmic coordinate $w = \log z$, $z \in V_a$ and set $\mathcal{U}_a = \log V_a$; then $\mathcal{U}_a \subset H = \{w : \Re w > 0\}$ has infinitely many simply connected unbounded components $U_{a,n}$ each contained in a vertical strip of width 2π and the exponential maps each $U_{a,n}$ univalently onto V_a. We have a commutative diagram:

$$\begin{array}{ccccc} & \mathcal{U}_a & \xrightarrow{F} & H_R & \\ \exp & \downarrow & & \downarrow & \exp \\ & V_a & \xrightarrow{f} & A_R & \end{array}$$

where $H_R = \{w : \Re w > \log R\}$ and F maps $U_{a,n} \subset H$ univalently onto H_R.

If $w \in H$, the disk $D = D(F(w), \delta(w))$ centered at $F(w)$ with radius $\delta(w) = \Re F(w) - \log R$ is contained in H_R. Denote the branches of F^{-1} by $G_n : H_R \to U_{a,n}$. By the Koebe $\frac{1}{4}$-theorem $G_n(D)$ contains a disk D_n of center w and radius $\frac{1}{4}\delta(w)|(G_n)'(F(w))|$. Since $U_{a,n}$ cannot contain any vertical segment of length 2π, for $w \in U_{a,n}$ we have

$$|F'(w)| > \frac{1}{4\pi}(\Re F(w) - \log R).$$

It follows there are constants $k, K > 1$ such that if $\Re F(w) > k$, then $|F'| > K$. By the chain rule, we have $|f'(z)| > K|f(z)|/|z|$.

To obtain an estimate for $|f'| = |f'_p|$ on the punctured disk domain V_p we use a different argument.

Note that since ϕ_p and ϕ'_p are analytic in V_p, $|\phi_p|$ and $|\phi'_p|$ achieve their maxima on ∂V_p. Increasing R clearly decreases these maxima so that by taking R large enough, we may assume C_1, C_2 and C_1/C_2 are small relative to R.

If $z \in V_p$, by conditions $(*)$

$$R \leq |f_p(z)| = \left| \frac{c_p}{(z-p)^{m_p}}(1+\phi_p(z-p)) \right| \leq \frac{B}{|z-p|^{m_p}}(1+C_1)$$

so that $|z-p| < M_p = \frac{B(1+C_1)}{R})^{\frac{1}{m_p}} < 1$.

For $t > R$, set $A_{R,t} = A_R \setminus A_t$. Choose $R' >> R$ and such that $R' > C_2 b$; for each pole p set $\tilde{V}_p = f_p^{-1}(A_{R,R'})$. Then if $z \in \tilde{V}_p$, by conditions $(*)$ we have

$$\frac{bC_2}{|z-p|^{m_p}} \leq |f_p(z)| = \left| \frac{c_p}{(z-p)^{m_p}}(1+\phi_p(z-p)) \right| \leq R'$$

so that $|z-p| > \rho_p = \frac{bC_2}{R'}^{1/m_p}$. The annulus $\rho_p < |z-p| < M_p$ is thus contained in $\tilde{V}_p$. Note that since for all p, $1 \leq m_p \leq J$, the moduli of these annuli are uniformly bounded.

Next compute that

$$|f'_p(z)| = |f_p(z)| \left| \frac{m_p}{z-p} + \frac{\phi'_p(z-p)}{1+\phi_p(z-p)} \right|.$$

In $\tilde{V}_p$ we have

$$|f'_p(z)| \leq R' \left| J \left(\frac{R'}{bC_2} \right)^{\frac{1}{m_p}} + \frac{C_1}{C_2} \right|$$

and

$$\left| \frac{m_p}{z-p} + \frac{\phi'_p(z-p)}{1+\phi_p(z-p)} \right| > \left| \left| \frac{m_p}{z-p} \right| - \left| \frac{\phi'_p(z-p)}{1+\phi_p(z-p)} \right| \right| \geq \left| \frac{1}{M_p^{m_p}} - \frac{C_1}{C_2} \right|.$$

Thus by choosing R and R' large enough, we see that there are constants, K', K, independent of p, such that for every $|p| > R$, in $\tilde{V}_p$ we have

$$K' > |f'_p(z)| > K > 1. \tag{1}$$

To show that $m(I_\infty) = 0$, by the Lebesgue density theorem it is sufficient to show that for any point $z \in I_\infty$,

$$\limsup_{\delta \to 0} \frac{m(D(z,\delta) \cap I_\infty)}{m(D(z,\delta))} < 1. \tag{2}$$

In order to do this we first restate condition $(**)$ as follows: Let $\mathcal{V} = f^{-1}(A_R)$. Then there is a $\kappa > 0$ such that

$$\limsup_{t\to\infty} \frac{\text{area}(A_{R,t} \cap \mathcal{V})}{\text{area}(A_{R,t})} \leq 1 - \kappa. \tag{3}$$

In terms of the logarithmic coordinates $\log z = s + i\theta$, let $S_t = \{\log R < s < t, 0 \leq \theta < 2\pi\}$ and set $\mathcal{U} = \log \mathcal{V}$. Then there is a $\delta > 0$ such that

$$\limsup_{t\to\infty} \frac{\text{area}(S_t \cap \mathcal{U})}{\text{area}(S_t)} \leq 1 - \delta. \tag{4}$$

For $z_0 \in I_\infty$, set $z_n = f^n(z_0)$. We may assume without loss of generality that $|z_n| > R$ is so large that the estimates (1) on $|f'|$ hold. Then either there is some N such that for all $n > N$, $z_n \in V_a$ for some simply connected component V_a or there is a subsequence $z_{n_j} \to \infty$ and a sequence of poles $p_j \to \infty$ such that $z_{n_j} \in V_{p_j}$.

In the first case the argument is just as in [11]. For $w_0 = \log z_0$ in a fixed strip of width 2π, let $w_n = F(w_{n-1})$, where F is the lift defined above. We may assume that for all n, $\Re w_n$ is large enough that $|F'(w_n)| > K > 1$. Let $F_n^{-1}(w_n) = w_{n-1}$. Then, there is a constant d, independent of n such that F_n^{-1} is univalent on $D_n = D(w_n, d)$. Moreover, by the expansion of F, we may choose m such that $F(D_n)$ contains a vertical segment of width bigger than 2π. Fix such an m and let $B_j = F^{-j}(D(w_m, d/4))$ for $j = 1, \ldots m$, where F^{-j} is the composition of the appropriate inverse branches. Applying the Koebe distortion theorem to the function F^{-m} we have $B_m \subset D(w_0, d/4K^m)$. Moreover if s_m is the radius of the smallest disk centered at w_0 containing B_m, there is a constant $t < 1$, independent of m such that $D(w_0, ts_m) \subset B_m \subset D(w_0, s_m)$. Clearly $s_m \to 0$ as $m \to \infty$ so that applying (4) we obtain (2).

In the second case we need to modify the argument. Since the moduli of the annulii $\tilde{V}_p$ have the same bounds, there is a constant $0 < \tau < 1$, independent of V_p, such that $\frac{\text{area}(\tilde{V}_p)}{\text{area}(V_p)} \leq 1 - (\frac{bC_2R}{B|1+C_1|R'})^{\frac{2}{m_p}} < 1 - \tau$. We want to show that there is a constant $\mu > 0$ independent of p satisfying

$$\frac{\text{area}(V_p \cap f_p^{-1}(\mathcal{V}))}{\text{area}(V_p)} \leq 1 - \mu. \tag{5}$$

With the constants of inequality (1), set $L = \frac{K'}{K}$. We claim that $\mu = \frac{\kappa}{L^2}\tau$ where κ is the constant in inequality (3).

We rewrite $\frac{\text{area}(V_p \cap f_p^{-1}(\mathcal{V}))}{\text{area}(V_p)}$ and estimate

$$\frac{\text{area}([(V_p \setminus \tilde{V}_p) \cap f_p^{-1}(\mathcal{V})] \cup [\tilde{V}_p \cap f_p^{-1}(\mathcal{V})])}{\text{area}(V_p)} \leq$$

$$\frac{\text{area}([(V_p \setminus \tilde{V}_p) \cap f_p^{-1}(\mathcal{V})])}{\text{area}(V_p)} + \frac{\text{area}(\tilde{V}_p)}{\text{area}(V_p)} =$$

$$\frac{\text{area}([(V_p\setminus\tilde V_p)\cap f_p^{-1}(\mathcal{V})])}{\text{area}([V_p\setminus\tilde V_p])}\times\frac{\text{area}([V_p\setminus\tilde V_p])}{\text{area}(V_p)}+\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}\leq$$

$$\left(1-\frac{\kappa}{L^2}\right)\left(1-\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}\right)+\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}$$

$$=1-\frac{\kappa}{L^2}-\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}+\frac{\kappa}{L^2}\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}+\frac{\text{area}(\tilde V_p)}{\text{area}(V_p)}$$

$$\leq 1-\frac{\kappa}{L^2}+\frac{\kappa}{L^2}(1-\tau)$$

$$=1-\frac{\kappa}{L^2}\tau=1-\mu.$$

Now fix n_j such that $z_{n_j}\in V_{n_j}$ and choose a univalent branch of $f_{p_{n_j-1}}^{-1}$ so that $f_{p_{n_j-1}}^{-1}:z_{n_j}\mapsto z_{n_j-1}$. Let $m=n_j$ and set $D_m=D(z_m,M_m)$. For large m, $|z_m-p_m|<<\rho_m$ and $V_{p_m}\approx D(z_m,M_m)$. We define $B_m=f^{-m}(z_m,M_m/4)$ where again f^{-m} is a composition of the appropriate inverse branches so that $z_0\in B_m$. Then f^{-m} is univalent on V_{p_m} and we can control distortion on D_m. Now for any trajectory from z_n to z_{n+k}, by the chain rule and the estimates on factors $|f'|$, whether the corresponding domain containing z_j is a simply connected V_a or is an annular $\tilde V_p$, we have $|f^k(z_{n+k})'|>K^k\frac{|z_{n+k}|}{|z_n|}$. We apply this with $n=0$ and $k=n_j$ and we may certainly assume $\frac{|z_{n_j}|}{|z_0|}>1$. In the annular domains we also have $|f^k(z_{n_j})'|<K'^k$.

We may assume $|z_{n_j}|>|z_{n_{j-1}}|$ so that the contraction over this part of the orbit is at least $1/K^m$. Then by the Koebe theorems $B_m\subset D(z_0,M_m/K^m)$, its distortion is bounded independent of m and its diameter goes to zero as m goes to infinity. Thus applying (5) to D_m and pulling back we obtain (2).

Thus we have shown that (2) holds at any point in $m(I_\infty)$ and hence that its Lebesgue measure is zero. ■

3 The results

3.1 Two lemmas

For a given f define the sets

$$L=L_f=\{z:f^n(z)\to\omega_f\}$$

$$\mathcal{K}_\epsilon=\mathcal{K}_\epsilon(f)=\{z:\text{dist}(z,\omega_f)<\epsilon\}$$

where we omit the f unless confusion results.

If ω_f is a compact repeller it is obvious that it contains no critical points. In fact, the post-singular set does not accumulate on ω_f but actually lands on it. Precisely,

Lemma 3.1. *If ω_f is a compact repeller then there is an $\epsilon > 0$ and an integer $N > 0$ such that if $c \in S$, $n > N$ and $f^n(c) \in \mathcal{K}_{\epsilon/2}$ then $f^n(c) \in \omega_f$.*

Proof: If PS is finite, the lemma is obviously true so assume PS is not finite. Since ω_f is compact and PS is infinite, ω_f contains no prepoles. Moreover, since it is a repeller, there exist constants $k > 1$ and $\epsilon > 0$ such that $|f'(z)| \geq k > 1$ on the sets $\mathcal{K}_\epsilon$ and $\mathcal{K}_{\epsilon/2}$.

Set $V = f(\overline{\mathcal{K}_{\epsilon/2}})$. Since $\overline{\mathcal{K}_{\epsilon/2}}$ contains no poles V is compact. Moreover, since $f|_{\mathcal{K}_{\epsilon/2}}$ is expanding, $\overline{\mathcal{K}_{\epsilon/2}} \subset \text{int}(V)$ and there is a regular branch g of f^{-1} that maps V to $\overline{\mathcal{K}_{\epsilon/2}}$. Thus, the annuli $A_0 = V - g(V)$, $A_{n+1} = g^n(A_0)$, $n \in \mathbb{N}$ are nested and $mod(A_n) \neq 0$ for all $n \in \mathbb{N}$.

We claim that for each $n \in \mathbb{N}$

$$f^n(S) \cap (\overline{\mathcal{K}_\epsilon} - \omega_f) = \emptyset.$$

If not, there is a sequence $n_k \in \mathbb{N}, n_k \to \infty$ such that for each n_k, there is some $c_k \in S$ (not necessarily distinct) satisfying $v_k = f^{n_k}(c_k) \in \overline{\mathcal{K}_\epsilon} - \omega_f$. By the compactness of $\overline{\mathcal{K}_\epsilon}$, as $k \to \infty$, the sequence v_k accumulates and by definition, any accumulation point belongs to ω_f. For each k, there is an $i(k)$ such that $v_k \in A_{i(k)}$ and $w_k = f^{i_k}(v_k) \in A_0$. Since $\overline{A_0}$ is compact, the sequence w_k has an accumulation point in $\overline{A_0}$ which by definition, also belongs to ω_f, This is a contradiction because A_0 is separated from ω_f by the annuli A_n, $n > 0$ and $PS \cap (\overline{\mathcal{K}} - \omega_f) = \emptyset$ as required. ■

Next, for the set L_f we have,

Lemma 3.2. *If $f \in \mathcal{B}$ and ω_f is a compact repeller then the Lebesgue measure of L_f is 0.*

Proof: Since ω_f is a compact repeller, it is clear by the classification of stable domains that $L_f \subset J_f$.

Let ϵ be chosen as in lemma 3.1 and define

$$\mathcal{L} = \bigcap_{n \geq 0} f^{-n}(\mathcal{K}_{\epsilon/2}).$$

Then $z \in \mathcal{L} - \mathcal{P}_f$ if its full forward trajectory belongs to $\mathcal{K}_{\epsilon/2}$. We will prove that $m(J_f \cap (\mathcal{L} - \mathcal{P}_f)) = 0$. Since $L \subset \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{L})$ and $\mathcal{P}_f$ is countable, this will imply $m(L) = 0$.

Suppose that $m(\mathcal{L} - \mathcal{P}_f) > 0$ and let z_0 be a density point of $\mathcal{L} - \mathcal{P}_f$. Since ω_f is compact, the orbit $\{f^n(z_0)\}$ has a finite accumulation point $y_0 \in \overline{\mathcal{K}_{\epsilon/2}} \subset \mathcal{K}_\epsilon$. It follows that there exists a sequence $n_k \to \infty$ such that $z_k = f^{n_k}(z_0) \to y_0$. For $k \in \mathbb{N}$, let $D_k = D(z_k, \epsilon/4)$ and let g_k be the branch of f^{-n_k} that maps z_k to z_0. Since $z_k \in \mathcal{K}_{\epsilon/2}$, $D_k \subset \mathcal{K}_\epsilon$ and by lemma 3.1 g_k is univalent on D_k.

Now by the definition of a density point and by propositions 2.1 and 2.2

$$\frac{m(g_k(D_k) \cap \mathcal{L})}{m(g_k(D_k))} \to 1.$$

Again by proposition 2.2

$$\frac{m(D_k \cap f^{n_k}(\mathcal{L}))}{m(D_k)} \to 1$$

and $m(D_k \cap f^{n_k}(\mathcal{L})) = m(D_k)$.

Let U be an open set with compact closure contained in $\mathbb{C} - \overline{\mathcal{K}}$. Since $z_k \in J_f$ there exists an integer N such that $f^N(D_k) \supset \overline{U}$ so that $m(f^{N+n_k}(\mathcal{L}) \cap U) > 0$. By definition however, for all $k \in \mathbb{N}$, $f^k(\mathcal{L}) \subset \mathcal{K}_{\epsilon/2}$ so $f^{N+n_k}(\mathcal{L} \cap U) = \emptyset$. This contradiction finishes the proof. ■

3.2 The main theorem

We are now ready to prove the main theorem.

Theorem 3.3. *Suppose $f \in \mathcal{B}$. If ω_f is a compact repeller and if conditions $(*)$ and $(**)$ hold then f is ergodic with respect to Lebesgue measure on its Julia set.*

Proof: Suppose that $E \subset J_f$ is an f-invariant measurable set of positive measure and let z be a density point of E. Since $\mathcal{P}_f$ is countable we may assume $z \notin \mathcal{P}_f$. Under the hypotheses that $f \in \mathcal{B}$ and that both $(*)$ and $(**)$ hold, by proposition 2.3 we may assume $z \notin I_\infty$ and thus that its orbit has a finite accumulation point y; let $z_k = f^{m_k}(z) \to y$, $k \to \infty$.

Under the hypotheses that ω_f is a compact repeller, by lemma 3.2 we may assume $z \notin L$ and hence $y \notin \omega_f$. Now suppose that either $y = f^n(v)$ for some $v \in S$ or that for some sequence $c_k \in S$ and iterates n_k, $f^{n_k}(c_k) \to y$. By lemma 3.1, in the first case $f^N(y) \in \omega_f$ and in the second, the n_k are bounded and $f^{n_k+N}(c_k) \in \omega_f$. In either case there is an N such that $f^N(z_k) \to \omega_f$. Arguing as in the proof of lemma 3.1, we can find a sequence $k_j \to \infty$ such that $w_j = f^{N+k_j}(z_k)$ lie in a compact annulus A_0 separated from ω_f; since $f^{N+k_j}(y) \in \omega_f$, we have a contradiction. Thus $\eta = \mathrm{dist}(y, PS) > 0$ and we can define a univalent branch g_k of f^{-m_k} on $D_k = D(z_k, \eta/4)$ such that $g_k(z_k) = z$.

As above, by the definition of a density point and by propositions 2.1 and 2.2 we obtain

$$\frac{m(g_k(D_k) \cap E)}{m(g_k(D_k))} \to 1 \quad \text{and} \quad \frac{m(D_k \cap f^{m_k}(E))}{m(D_k)} \to 1.$$

Since E is forward invariant we have $m(D_k \cap E) = m(D_k)$.

If f is not ergodic, we can find another forward invariant positive measure set $F \subset J_f$, $m(E \cap F) = 0$, and a density point z' for F for which we can construct disks D'_l of fixed radius such that $m(F \cap D'_{k'}) = m(D'_{k'})$. Because $z_k \in D_k$, $z_{k'} \in D'_{k'}$ defined as above belong to J_f there is some $N \in \mathbb{N}$ such that $f^N(D'_{k'}) \supset D_k$. It follows that $m(E \cap F) > m(f^N(D'_{k'} \cap F) \cap (D_k \cap E)) > 0$ and we obtain a contradiction. ■

4 Examples

1. Examples of functions that the main theorem applies to are found in the class of meromorphic functions with polynomial Schwarzian derivative. By Nevanlinna's theorem, if the Schwarzian of f is a polynomial of degree $p \geq 0$ then $f \in \mathcal{M}_{p+2}$ and all singularities of f^{-1} are logarithmic. Moreover, at least half of the singularities are finite so it is not hard to show directly that condition $(**)$ holds.

Now note from [19] that if the Schwarzian derivative is polynomial is degree $p-2$, then in a sector of width $2\pi/p - 2\epsilon$ about each of the p Julia directions, the function has the asymptotic form

$$\frac{AG_\nu + BG_{\nu+1}}{CG_\nu + DG_{\nu+1}}$$

where

$$G_\nu \approx \exp{(-1)^{\nu+1} z^{p/2}}.$$

Therefore in any given Julia direction the coefficients c_p and functions ϕ_p are all approximately equal and condition $(*)$ holds.

The dynamical properties of these functions were described in [9]. In particular, it is shown that they have no wandering domains and no Baker domains. Thus, if each singular value lands on a repelling cycle, $J = \hat{\mathbb{C}}$ and the function is ergodic.

A special subclass contains the functions with constant Schwarzian derivative. To this class belong the families λe^z, $\lambda \tan(z)$, $\lambda \in \mathbb{C}$ where it is easy to construct examples for which the singular values land on repelling cycles. For instance: if $f_1(z) = \pi i \tan(z)$, then $S(f_1) = \{\pm\pi\}, PS = \{\pm\pi, 0\}$, $\omega_f = \{0\}$ and $|f_1'(0)| = \pi$; and if $f_2(z) = \pi i \exp(z)$ then $S = \{0\}, PS = \{0, \pi i, -\pi i\}$, $\omega_f = \{-\pi i\}$ and $|f_2'(-\pi i)| = \pi$.

2. Functions in the above class where theorem 3.3 does not apply are e^z and $\frac{\pi i}{2}\tan(z)$. For e^z the orbit of the singular value 0 tends to ∞ and for $\frac{\pi i}{2}\tan(z)$ the singular values are the poles $\pm\pi/2$; thus, although ω_f is a repeller, it is not compact. In fact, it is proved in [15] that e^z is not ergodic. The same is probably true for $\pi i/2 \tan(z)$.

Another simple example where theorem 3.3 does not apply is the family $a\sin(z) + b$ studied in [17]. Although the singular set consists of two points, and a, b can be chosen so that the critical values are mapped to repelling periodic orbits, condition $(**)$ fails and $J_f = \overline{I_\infty(f)} = \hat{\mathbb{C}}$ has many ergodic components.

3. It is hard to find examples of maps with infinitely many singular values satisfying the hypotheses of our theorem because it is hard to tell whether ω_f is a compact repeller and to check the conditions on the Laurent expansions. Here is one such example.

Consider the entire function in $f_\lambda = \lambda e^{-z^2} \sin z$. Since it is entire condition $(*)$ is vacuous. To see that it satisfies condition $(**)$ note that for λ real, if $|\arctan z| < 1/2$ then $|f_\lambda'(z)|$ is bounded. Its infinitely many critical values are bounded and accumulate on the asymptotic value 0 hence it belongs to $\mathcal{B}$. Moreover, λ may be chosen so that ω_{f_λ} is a compact repeller. Thus theorem 3.3 applies so that f_λ is ergodic on its Julia set.

In this case we also know there are no eventually periodic domains and, since $m(I_\infty) = 0$, no Baker domains. There are also no wandering domains. If there were a multiply connected wandering domain D, it would contain a homotopically non-trivial loop γ and by [2] the winding number of (a subsequence of) $f^n(\gamma)$ would be non-zero for n sufficiently large and the diameter of $f^n(\gamma)$ would tend to infinity. Thus γ would have to intersect the real axis; since the positive and negative real axes are asymptotic curves such domains cannot exist. Since $m(I_\infty) = 0$, no wandering domain can escape to infinity. Moreover, since by [7] all limit functions of $f^n|_D$ would have to be constants in the compact repeller ω_{f_λ}, we conclude $J_{f_\lambda} = \hat{\mathbb{C}}$.

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