Conformal Dynamical Systems and Lipman Bers

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1 Introduction

By the 1920's Fatou and Julia had developed a theory of conformal dynamical systems that arise by iterating a rational map defined on the Riemann sphere. For a rational map f and a positive integer n, the $n-$th iterate f^n of f is the composition $f \circ f \circ \cdots \circ f$; that is, f is composed with itself n times. This discrete system of iterates determines a decomposition of the sphere into two sets, an open set Ω, called the Fatou set, on which the dynamics is predictable and its closed complement J, called the Julia set, on which the dynamics is chaotic. Thus, for all z in a neighborhood of a point in the Fatou set, the orbits $\{f^n(z)\}_{n>0}$ remain close while in a neighborhood of a point in the Julia set the orbits of points are wildly different. Precisely, the Fatou set is defined to be the set of points z which have open neighborhoods on which the family $F = \{f^n(z)\}$ is a normal family of holomorphic functions.

In a similar way, a discrete subgroup F of Möbius transformations also determines a decomposition of the Riemann sphere into a predictable set and a chaotic set. Again, the predictable set Ω is the set of points z that have neighborhoods on which the family $\{\gamma : \gamma \in F\}$ is normal but in this context it is called the regular set; the chaotic set is its complement and is here called the limit set and denoted by Λ. Such a group is now called Kleinian although Poincaré originally reserved the name Kleinian for groups with non-empty Ω.

A discrete system of iterates of a rational function is an example of a dynamical system. A classification theory for the eventually periodic connected components of the Fatou set of such a system was developed by Fatou, Julia, Siegel and others. These are components D of Ω for which, for some positive integers m, n, $f^m(D) = f^n(D)$. All of the evidence suggested that there were no components that were not eventually periodic; that is, there were no "wandering domains".

About 1980 Sullivan turned his attention to these dynamical systems and proved that, indeed, there were no wandering domains [25]. He observed similarities between the theory for the iterates of a rational function and the theory

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for Kleinian groups and sketched a loose "dictionary" that identified concepts in one theory with concepts in the other. For example, the iterates of a rational function correspond to group elements. Moreover, Sullivan saw that the dictionary could be used to advantage on both sides. For example, the proof of the non-wandering domain theorem for rational functions could be converted to a proof of Ahlfors' finiteness theorem for Kleinian groups. Bers was excited about Sullivan's work and recast it in [5].

Given any discrete dynamical system, the iterates of a point form its *orbit*. The Fatou-Julia dichotomy describes the orbit structure for a given system. In addition, one is interested in whether the orbit structure persists if the system is perturbed; that is, if the rational map f or the group G is replaced by a nearby map or group. Two dynamical systems F and G are called conjugate if there is a homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h \circ F \circ h^{-1} = G$. The system F is called *stable* if all small perturbations of it yield conjugate dynamical systems. Of course, the meaning of "stable" depends on the topology used to define "small." In stable systems the orbit structure persists when the system is perturbed.

If a family (F_λ) of dynamical systems varies holomorphically with a parameter λ, there is a natural topology on the functions F_λ. We distinguish between the *dynamical plane* consisting of the arguments $z \in \hat{\mathbb{C}}$ of the iterates of the function F_λ and the *parameter plane* consisting of the values in $\mathbb{C}$ over which λ varies. Mané , Sad and Sullivan [20] realized that a holomorphic dynamical system remains stable if the periodic points of the system — or the fixed points of the group elements — move injectively under perturbation. Moreover, the conjugacies between (F_0) and (F_λ) for $|\lambda| < 1$ are automatically realized by quasiconformal homeomorphisms with dilatation not exceeding $\frac{1+|\lambda|}{1-|\lambda|}$. This famous result is called the λ-lemma.

Motivated by this discovery, Mané, Sad and Sullivan defined a *holomorphic motion* of a closed set J in the extended complex plane $\hat{\mathbb{C}}$ to be a curve $h_t(z)$, defined for every z in J and for every t in the unit disk, such that:

i) $h_0(z) = z$ for all z in J,
ii) $z \mapsto h_t(z)$ is injective as a function from J into $\hat{\mathbb{C}}$ and
iii) $t \mapsto h_t(z)$ is holomorphic for $|t| < 1$ and for each fixed z in J.

In this definition we have switched from the variable λ to the variable t because we wish to think of t as the complex time-parameter for the motion. As t changes the set $J_t = h_t(J)$ moves in $\hat{\mathbb{C}}$. Although J may start out as pure and smooth as a circle and although the points of J move holomorphically, for every $t \neq 0$, J_t can be an interesting fractal set with fractional Hausdorff dimension. Limit sets of Kleinian groups and Julia sets of rational maps are almost always fractals. Computers have made it possible to obtain pictures of them.

That injectivity and holomorphic dependence automatically lead to quasiconformality is surprising because quasiconformality is a geometric idea about distortion. By definition, an arbitrary orientation preserving homeomorphism is quasiconformal if it distorts standard shapes by a bounded amount. It may distort sizes by a large amount; no assumption is made on this point. A beautiful first result of the theory is that K-quasiconformal mappings are necessarily

Hölder continuous with Hölder exponent $1/K$. (This result is due to Lavrentiev or to Ahlfors, although it is difficult to know which of the two deserves credit because each attributed it to the other.) Thus, quasiconformal control of distortion of shape implies some control on distortion of size. Moreover, a normalized family of quasiconformal mappings with a universal bound on the dilatation of every member of the family is equicontinuous. (There is a beautiful expository lecture by Bers on this topic, including the attribution question, preserved in the video-tape collection of Sullivan's Tuesday afternoon dynamical systems seminar at the Graduate Center of CUNY.[3])

While the $\lambda-$lemma showed that mappings in a holomorphic motion automatically have quasiconformal extensions in the dynamical plane, a more remarkable result concerns the extendibility of a holomorphic motion in the parameter plane. The celebrated theorem of Slodkowski [23] states that a holomorphic motion of any closed set J defined for all $|t| < 1$ automatically extends to a holomorphic motion of all of $\hat{\mathbb{C}}$, also defined for all $|t| < 1$. The realization that any holomorphic motion on any set was extendible in the parameter plane was the combined result of several research papers. The first by Màñé, Sad and Sullivan [20] presented the idea of a holomorphic motion and showed that mappings in the motion necessarily have quasiconformal extensions. Then the papers by Sullivan and Thurston [26] and by Bers and Royden [8] showed the local extendibility to a domain $|t| < \epsilon$ where ϵ is universal, not depending on J or the particular holomorphic motion. (Bers and Royden showed that $\epsilon \geq 1/3$.) Finally, Slodkowski [23] showed that any such motion could be extended to the full unit disk, $|t| < 1$.

Slodkowski's result complements nicely the early papers by Bojarski [10] and by Ahlfors and Bers [2] in which singular operators are used to solve the global Beltrami equation

$$h_{\overline{z}}(z) = t\mu(z)h_z, \tag{1}$$

where μ is an arbitrary L_∞ complex-valued function defined on $\hat{\mathbb{C}}$ with $||\mu||_\infty = 1$ and $|t| < 1$. The solutions $h^{t\mu}$ obtained in these papers are quasiconformal homeomorphisms of $\hat{\mathbb{C}}$; they are unique up to post-composition by a Möbius transformation. The emphasis in the paper by Ahlfors and Bers is that $h^{t\mu}(z)$ depends holomorphically on t for $|t| < 1$. When $t = 0$, the normalized solution $h^{t\mu}$ is the identity and, for each fixed $|t| < 1$, $h^{t\mu}(z)$ is an injective function of z. In other words, their paper showed the existence of many holomorphic motions of $\hat{\mathbb{C}}$.

In the following sections we sketch some of the ideas that comprise the synergy of holomorphic dynamical systems and Kleinian groups that so excited Bers in his last years.

[3]Tapes from the collection may be viewed in the mathematics department of the Graduate Center of CUNY.

2 Quasiconformal Dynamical Conjugacies.

In his work, Bers developed techniques to construct quasiconformal conjugacies of conformal dynamical systems corresponding to Fuchsian groups, and later Kleinian groups and then in in his late years, iterates of a rational or meromorphic mapping. From the point of view of partial differential equations, a point of view developed by Ahlfors and Bers in their early paper [2], a quasiconformal selfmapping h of the Riemann sphere $\hat{\mathbb{C}}$ is an orientation preserving homeomorphism of $\hat{\mathbb{C}}$ which is also a solution of the Beltrami equation, equation (1). In this equation μ is called the Beltrami coefficient of h. Since homeomorphic solutions of (1) are unique up to post-composition by a Möbius transformation, for each μ there is a unique homeomorphic solution fixing $0, 1$ and ∞ that we denote by $h = h^{\mu}$.

A *dynamically natural homeomorphism* is a homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that for all elements f in a Kleinian group or f a rational map, $f_1 = h \circ f \circ h^{-1}$ is holomorphic. If h is quasiconformal, we can take the ∂ and $\overline{\partial}-$derivatives of the equation $h \circ f = f_1 \circ h$. Using $f_{\overline{z}} = 0$ and $(f_1)_{\overline{z}} = 0$, we find that the Beltrami coefficient μ of h satisfies the relation

$$\mu(f(z))\overline{f'(z)} = \mu(z)f'(z). \tag{2}$$

We call a Beltrami differential μ satisfying this equation *compatible* with the system and h a *qc-conjugacy*.

A very important observation is that, if μ is compatible with a given system, then so is $t\mu$ for any complex number t. By taking $|t| < 1/||\mu||_{\infty}$ we can solve the Beltrami equation (1) with μ replaced by $t\mu$ and arrive at a topologically conjugate system generated by the mappings $h^{t\mu} \circ f \circ (h^{t\mu})^{-1}(z)$ that are holomorphic in the variables z and t. If f is a Möbius transformation, or a degree d ramified covering of the sphere over itself, then so is the conjugate mapping, $h^{t\mu} \circ f \circ (h^{t\mu})^{-1}$.

Since F is a normal family in a region Ω if and only if $h^{t\mu} \circ F \circ (h^{t\mu})^{-1}$ is normal in the region $h^{t\mu}(\Omega)$, $h^{t\mu}(J)$ or $h^{t\mu}(\Lambda)$ is a holomorphically moving family of Julia or limit sets. Given a rational map or Kleinian group, the set of maps or groups that can be connected to it by a holomorphically moving family is called *holomorphic family* of rational maps or groups.

The construction of a nontrivial dynamically natural quasiconformal mapping for a particular dynamical system F is thus equivalent to the construction of a compatible Beltrami differential. Such a construction can be realized by the "μ-trick"; that is, by lifting a Beltrami differential defined on the Fatou set factored by the dynamical system to the complex plane and setting it to zero on the complementary Julia or limit set.

By an important result of Sullivan [24], if F is a finitely generated Kleinian group, there are no compatible Beltrami differentials supported on the limit set and the "μ-trick" construction on its complement yields all nontrivial qc-conjugacies. If F is generated by a rational map, it is an open and interesting question whether the same is true. See [18] for an overview of these ideas.

The space of all nontrivial qc-conjugacies of the dynamical system constructed by the μ-trick is identifiable infinitesimally with the Banach space dual to the integrable holomorphic quadratic differentials on the Riemann surface formed by factoring the Fatou set by the dynamical system.

This observation is the basis of the finiteness theorems; for Kleinian and Fuchsian groups this theorem is called Ahlfors' finiteness theorem and in the case of rational mappings it is called Sullivan's non-wandering theorem. It is also the basis for the results in [15] where, following an idea suggested by Bers in [6], it is shown that certain classes of tame Kleinian groups form a manifold in the space of representations in $PSL(2,\mathbb{C})$ and that these groups are dynamically stable.

3 Finiteness Theorems

A finitely generated Kleinian group is defined by a finite set of data, for example the coefficients of the generators; any holomorphic family of such groups is therefore inherently finite dimensional. Similarly, the space Rat_d of rational maps of degree d is inherently finite dimensional since a rational map is also determined by the $2d+1$ coefficients. (Just as two Kleinian groups conjugate in $PSL(2,\mathbb{C})$ uniformize the same set of Riemann surfaces, rational maps that are conjugate by a linear fractional transformation determine systems with the same dynamical properties. Therefore we always assume f is normalized in some way and so depends only on $2d-2$ parameters.) The proofs of Ahlfors' finiteness theorem and Sullivan's non-wandering theorem make essential use of this fact. Here is an outline of the proof the non-wandering theorem for rational maps.

If a rational map has a simply connected wandering domain W_0, then its iterates $W_n = f^n(W_0)$ for all $n > 0$, are pairwise disjoint. Since f has $2d-2$ critical points, all but a finite number of the W_n do not contain critical points of f; hence f is injective on W_n for all n greater than or equal to some N and there is a well defined branch $g_k = f^{-k}|_{W_{N+k}} : W_{N+k} \to W_N$. Therefore, if a Beltrami coefficient μ is chosen arbitrarily on W_N it can be compatibly extended to all of $\hat{\mathbb{C}}$ by using the compatibility condition for f to define it on the components $f^k(W_N)$, $k < 0$, the compatibility condition for g_k to define it on the components $f^k(W_N)$, $k > 0$ and setting μ to be zero everywhere else.

Let $M(f)$ be the space of μ's compatible with f. For each $\mu \in M(f)$ there is a global quasiconformal homeomorphism h^μ of $\mathbb{C}$ with Beltrami coefficient μ; moreover, $f^\mu = h^\mu \circ f \circ (h^\mu)^{-1}$ is also a degree d rational mapping. Define μ to be equivalent to ν if f^μ equals f^ν up to conjugation by a Möbius transformation. Let $T(f)$ be the set of equivalence classes of these μ's.

The crux of the proof is that if W_0 and thus W_N exist, the infinite dimensional Teichmüller space of the domain W_N injects into $T(f)$. This implies $T(f)$ is infinite dimensional, which contradicts the fact that it can have dimension at most equal to $2d-2$. Sullivan [25] established the injectivity by using prime ends. Bers' proof [5] uses canonical harmonic Beltrami differentials, which realize the dual space of the integrable holomorphic quadratic differentials on W_N.

The existence of non-simply connected wandering domains is ruled out by similar arguments.

Non-wandering theorems for certain classes of entire and meromorphic functions have been proved by various people [3, 9, 11, 13, 16].

To prove the finiteness theorem for finitely generated Kleinian groups, one has to show (among other things) that there cannot be infinitely many conjugacy classes of components of the regular set. Ahlfors' original proof used an idea based on the μ-trick and "mollifiers" to show that the Teichmüller space of a component of Ω injects into $T(F)$, the space of μ's compatible with the group F modulo equivalence; then dimensionality considerations rule out infinitely many conjugacy classes representing surfaces that are not thrice-punctured spheres. Since components of Ω corresponding to thrice-punctured spheres have trivial Teichmüller spaces, an improved method was required for ruling out infinitely many non-equivalent thrice-punctured spheres. Bers among others, including Ahlfors himself, found such a method, [7, 14], that involved the use of higher order differentials. One of the consequences of this method is an estimate on the total hyperbolic area of all the inequivalent components of Ω. In particular, the number of components corresponding to thrice-punctured spheres is also bounded (see section 5.6 of the article by Kra and Maskit in this volume and the references therein).

The area theorems have an analog in iteration theory as well. The eventually periodic components of the Fatou set fall into five categories. It is possible to attach a critical point to each periodic cycle of components and, using rather crude estimates, to conclude that there are at most $8d-8$ such cycles. This was revised to $4d-4$ using simple methods of algebraic geometry but the conjecture was that the correct number should be $2d-2$, the number of critical points. Bers was very pleased when he learned that Shishikura [22], using the technique of quasiconformal surgery, which we discuss below, proved this conjecture. In fact he asked several of the regular members of his Friday afternoon complex analysis seminar to lecture about it.

4 Polynomial-like maps and surgery

Douady and Hubbard noticed that the μ-trick could be used to take a rational map expanding in a neighborhood and deform it into an expanding holomorphic system that is not necessarily rational. This led them to define a "polynomial-like" map. Roughly speaking, a polynomial-like map f "behaves like" a polynomial on a simply connected domain U in the plane but is more flexible. More precisely, a *polynomial-like map* $f : U \to V$ is a proper holomorphic (degree $d > 1$) map where U and V are topological disks in $\mathbb{C}$ such that $\overline{U}$ is compact and contained in V.

If f is a polynomial, its *filled Julia set* $K(f)$ is defined as the set of points whose orbits are bounded. It is not hard to show that its Julia set $J(f)$ is the boundary of $K(f)$. For polynomial-like maps f, we make a new definition for the *filled Julia set*: it is defined by $K(f) = \cap f^{-n}(U)$ and the *Julia set* is defined as the boundary of $K(f)$. It is easy to see that if f is a polynomial and U is a topological disk containing the polynomially defined $K(f)$ and if V is a topological disk containing $f(U)$, then $f : U \to V$ is polynomial-like and the sets $K(f)$ and $J(f)$ are the same with either definition.

Two polynomial-like maps $f_i : U_i \to V_i$, $i = 1, 2$ of the same degree are *hybrid equivalent* if there exists a quasiconformal map h between neighborhoods of $K(f_i)$ conjugating f_1 to f_2 with $\bar{\partial} h = 0$ a.e. on $K(f_1)$. The *straightening theorem* of Douady and Hubbard [12] says that every polynomial like map $f : U \to V$ of degree d is hybrid equivalent to a polynomial P of degree d and, if $K(f)$ is connected, then P is unique up to conjugation by an affine map.

The proof of this theorem is an example of the *surgery* technique. We give a brief outline here under the assumption that $K(f)$ is connected. First build a model for P: the model is a map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that on a neighborhood N_f of $K(f)$, g agrees with f and so is polynomial-like; on a neighborhood N_∞ of infinity outside $K(f)$ it agrees with $z \mapsto z^d$; it remains to define it on the annulus A between N_f and N_∞. To do this, choose any degree d map ϕ that agrees with f on ∂N_f and with $z \mapsto z^d$ on ∂N_∞. Now apply the μ-trick: define μ on A as the Beltrami differential of ϕ and extend it to the rest of $\hat{\mathbb{C}}$ so that it is compatible with the model map g. Solve the Beltrami equation to obtain a quasiconformal homeomorphism h^μ that conjugates g to a degree d holomorphic map of the sphere — the desired polynomial P. The point here is that the distortion in the annulus A controls the distortion in all of the images of the annulus A under the dynamical system and the orbit of any point intersects A in at most one point.

A periodic cycle of a function f is the set $\{z_0, z_1 = f(z_0), \ldots, z_p = f(z_{p-1}) = z_0\}$; its multiplier is $\lambda = (f^p(z_0))'$. The cycle is called super-attractive if $\lambda = 0$, attractive if $0 < |\lambda| < 1$, repelling if $|\lambda| > 1$ and parabolic if $\lambda = \exp(2\pi i q)$, $q \in \mathbb{Q}$.

The difficulty in proving that a degree d rational map has only $2d-2$ cycles of periodic domains is to show that all its parabolic cycles can be made attractive simultaneously by perturbation. This would prove the theorem since it is not hard to prove that each attractive cycle attracts at least one critical point (and a rational degree d mapping has at most $2d-2$ critical points). Simple algebraic methods show that half the parabolic cycles can be made attractive, but the other half may become repelling. By refining Douady and Hubbard's techniques Shishikura was able to construct a perturbed dynamical system on which all of the parabolic cycles become attractive and those that were already attractive remain so.

5 The Mandelbrot set and Bers slices

Up to affine conjugation, every quadratic polynomial has a representative of the form $p_c(z) = z^2 + c$, $c \in \mathbb{C}$. The dynamics of p_c are determined by the orbit of the critical point at the origin. The Mandelbrot set $\mathcal{M}$ is the set of c such that $p_c^n(0)$ stays bounded for all n.

Because quadratic polynomials have only one critical point, they have at most one attractive periodic cycle. Those that have an attractive or super-attractive periodic cycle are best understood and are called *hyperbolic*. Computer pictures of $\mathcal{M}$ indicate that its interior is made up of infinitely many components. The interior of each of these components consists of hyperbolic maps and except for a single point, called the center, where the corresponding polynomial has a super-attractive cycle, all maps in a given component are quasiconformally conjugate. At the center, the polynomial has a periodic cycle containing the critical point. Choosing inverse branches to move backwards around a periodic cycle defines a combinatorial invariant $\mathcal{P}$ for the cycle. This invariant is the same for all polynomials in a hyperbolic component. It describes how the components of the filled Julia set are permuted under iteration.

In the Sullivan dictionary, $\mathcal{M}$ is the analog of the Bers slice representation of the Teichmüller space in quasifuchsian space. The analogy is not perfect since the interior of $\mathcal{M}$ is a union of Teichmüller spaces, however, the analogy comes from considering a quadratic polynomial as a rational map of degree 2 with one of its critical points fixed (at infinity) and the other varying.

Bers described going to the boundary of a Bers slice as either "squeezing the neck", going to an accidental parabolic, or "wringing the neck", going to a totally degenerate group. For quadratic polynomials, "squeezing the neck" corresponds to varying c so that the attractive cycle of p_c becomes parabolic.

Thurston gave a precise description of the "wringing process" along an open geodesic as a limit of "wringing processes" along closed curves; Bers [4] in turn, redescribed this in terms of quadratic differentials and absolutely extremal Teichmüller maps (see section 6.7 of the article by Kra and Maskit in this volume). Thurston attached an invariant to the "wringing process", the "ending lamination" and conjectured that it determined the boundary point uniquely. This conjecture has recently been proved by Minsky [21] for the Teichmüller space of a punctured torus.

For quadratic polynomials the "wringing process" has an analog in an application of the surgery technique called "tuning". This is again an iterative process, but unlike "wringing", it sends a point from one component of $\mathcal{M}$ to another. The boundary point it determines is, however, invariant under the process. A precise definition of "tuning" is beyond the scope of this article but we give an example that shows some of the ideas involved.

Let $p_{-1} = z^2 - 1$ and let $f = (z^2 - 1)^2 - 1$; then f is a polynomial of degree 4, but restricted to a neighborhood U of the component of its Fatou set containing the origin, it is quadratic-like (polynomial-like of degree 2). The rescaled map, $g = \alpha f(z/\alpha)$, for appropriate α, is again quadratic-like and by the straightening theorem it is hybrid equivalent to a quadratic polynomial; in this case, the polynomial $p_0 = z^2$. p_0 is called a *renormalization* of p_{-1} and if $\mathcal{P}_0$ is the combinatorial invariant of the component of $\mathcal{M}$ containing 0 (the cardioid), then p_{-1} is called the *tuning* of p_0 by $\mathcal{P}_0$.

There are points in $\partial\mathcal{M}$ that are infinitely renormalizable; that is, arbitrarily high iterates have polynomial-like restrictions to neighborhoods of the origin. Since each renormalization is the inverse of tuning with respect to some $\mathcal{P}$, the collection of these tuning invariants are the dictionary analog of the Thurston ending lamination (see [17]).

Applying the tuning process Douady, Hubbard [12] and McMullen [19] proved that there are small copies of the Mandelbrot set near its boundary and that Mandelbrot sets lie inside many different parameter spaces of holomorphic maps.

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