Coverings of Cantor sets

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Abstract. Let X be the Riemann sphere from which a Cantor set $\mathcal{C}$ quasiconformally equivalent to the middle thirds Cantor set has been removed and let G be a Fuchsian group representing X. We prove that the universal covering map has a well defined extension as a projection from the non-conical subset of the limit set $\Lambda(G)$ onto the boundary $\mathcal{C}$ of X.

1. Introduction

Suppose S is a conformally finite Riemann surface represented by a Fuchsian group G acting on the upper half plane $\mathbb{H}$. Because the punctures on S form a finite set and the group is finitely generated, it is natural to extend the projection map $\pi : \mathbb{H} \to \mathbb{H}/G \equiv S$ to the parabolic fixed points of G by mapping any such fixed point to a puncture. The parabolic fixed points have a dynamical characterization used by Patterson and Sullivan [**12, 16**] and by Beardon and Maskit [**4**] as the set of non-conical limit points $NC(G)$; that is, the set of points such that no cone based at the point contains infinitely many points congruent under the group.

Many Riemann surfaces of infinite type arise in the context of holomorphic dynamics as the complement in the Riemann sphere of a Cantor set of bounded geometry, that is, of a set invariant under a dynamical system formed from a collection of expanding maps. Using the dynamics, one can find bounds on the conformal geometry of such surfaces. Studying the relation between the conformal and hyperbolic geometries of these surfaces, it is natural to ask whether the uniformizing map extends to a map of a natural subset of the limit set onto the boundary of the surface. The answer in general is yes and the subset is precisely the non-conical limit set.

In this paper we consider a particular instance of this situation. Our Riemann surface, X, is the complement in the Riemann sphere of the standard middle thirds Cantor set $\mathcal{C}$. We prove there is a natural map from $NC(G)$ to $\mathcal{C}$. The techniques we develop clearly extend to the more general dynamically defined Riemann surfaces

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mentioned above as well as to an even larger class. We plan to carry out this extension in future work.

As a consequence of our constructions, we also prove that there is a canonical lift ω_0 of X to $\mathbb{H}$ and that the map from the boundary of ω_0 in $\partial\mathbb{H}$ to $\mathcal{C}$ is Hölder continuous. The crucial point is that there are sequences of geodesics defining the ends of X whose hyperbolic lengths are uniformly bounded.

Let D be the disk centered at $z = 1/2$ of radius $7/9$, $D_0 = E_0^{-1}(D), D_1 = E_1^{-1}(D)$, where $E_0(z) = 3z, E_1(z) = 3z - 2$. Then, the middle thirds Cantor set $\mathcal{C}$ is the accumulation set of all possible backwards images of points in D under the expanding dynamical system defined by

$$E(z) = \begin{cases} E_0(z) & \text{if } z \in D_0 \text{ and} \\ E_1(z) & \text{if } z \in D_1. \end{cases} \tag{1}$$

The map E restricted to $\mathcal{C}$ defines a dynamical system topologically conjugate to the shift on two symbols. Using all of the different possible images of the boundary of the disk D under backwards branches of E we obtain a collection of simple closed curves on X. These are round circles in the Euclidean geometry of X. In the free homotopy class of each one of these curves there is a unique hyperbolic geodesic with respect to the hyperbolic metric on the complement of $\mathcal{C}$. We call these circles, or the corresponding hyperbolic geodesics (resp. conformal or hyperbolic) *waist curves.* Nested sequences of these waist curves characterize the points in $\mathcal{C}$. Employing the conformality of E we obtain upper and lower bounds on the lengths of the conformal waist curves. These upper bounds, in turn, give upper bounds on the lengths of the hyperbolic waist curves that allow us to identify points in $\mathcal{C}$ with certain nested sequences of lifts of the hyperbolic waist curves to $\mathbb{H}$ that we call *allowable chains of axes.* These chains limit at points of the non-conical limit set $NC(G)$.

Although our constructions are carried out for the special system of curves based on the middle thirds Cantor set $\mathcal{C}$, they apply immediately to any quasiconformal image of $\mathcal{C}$. This follows because the crucial bounds on the lengths of the waist curves in our proof are quasiconformally quasi-invariant. Thus our main result is that for any surface representing a point in the Teichmüller space of X and a Fuchsian group representing it, there is a projection from the non-conical limit set to the boundary of the surface.

With more work, but using similar techniques, the same is true when the Riemann surface is the complement in the sphere of any Cantor set of bounded geometry.

The organization of the paper is as follows. In section 2 we set up notation for coding the middle thirds Cantor set $\mathcal{C}$ and the conjugacy from the action of E on $\mathcal{C}$ to the action of the one-sided shift on the space of one-sided sequences on two symbols. In section 3 we discuss the dynamics on the Riemann surface X of the conformal system E and obtain the upper bound on the lengths of the hyperbolic waist curves. The action of the system $E(z)$, mapping subsurfaces of X onto X implies that the lengths of the hyperbolic waist curves are also bounded below. Although this bound is not needed in the proof of the main theorem, the proof that it exists involves the interaction of the conformal and hyperbolic structures on X in an interesting way. We therefore include it in an appendix. In section 4, we discuss the boundary of X in terms of sequences of hyperbolic waist curves on X and in section 5 we define the fundamental domain ω_0 and the allowable chains of axes.

In section 6, which is the heart of the paper, we show that the set of limit points of allowable chains of axes $\mathcal{L}_X(G)$ coincides with the non-conical limit set $NC(G)$ of the Fuchsian covering group G of X. In section 7 we show that our results hold true under quasiconformal deformation and in section 8 we describe Cantor sets of bounded geometry and their complements in the Riemann sphere and indicate how our theorem generalizes to them. Section 9 is the appendix.

2. The middle thirds Cantor set

Let $\mathcal{C}$ be the standard middle thirds Cantor set in the unit interval. It is obtained inductively: first remove $(1/3, 2/3)$ from $[0,1]$ and at the n^{th} step remove the middle thirds of the remaining intervals. The points $x \in \mathcal{C}$ have unique representations of the form $x = s_1 s_2 \ldots$ where $s_i \in \{0,1\}$ and every sequence in $\{0,1\}^\infty$ occurs. This representation arises as follows: represent all points in the interval $[0,1]$ by a triadic expansion

$$x = \sum_{j=1}^{\infty} \frac{s_j}{3^j},$$

where s_j equals $0, 1$ or 2. Now remove from the unit interval $[0,1]$ all of those points for which the the "digit" s_j is equal to 1 anywhere in the expansion. The remaining set of points is the middle thirds Cantor set $\mathcal{C}$. Note that any point which admits two distinct expansions, $s_1 \ldots s_n 111\ldots$ and $s_1 \ldots s_n 2000\ldots$ and $s_i \neq 1$ for $i = 1 \ldots n$ belongs to $\mathcal{C}$.

Now replace 2 by 1 in every expansion and drop the "decimal" point. Then the correspondence i between codes $x = s_1 s_2 \ldots$, where each s_j is either zero or one, and the points $i(x) = \sum_{j=1}^{\infty} \frac{2s_j}{3^j}$ in the Cantor set $\mathcal{C}$ contained in the complex plane $\mathbb{C}$ is bijective. If we use the metric $d_{\mathcal{C}}(x,y) = \sum_k |s_j - t_j|/2^j$ on the space of codes, then i is a homeomorphism from the space of codes onto $\mathcal{C}$. Moreover, the shift map $\Sigma(s_1 s_2 s_3 \ldots) = s_2 s_3 \ldots$ on the space of codes is conjugate to the expanding map E on $\mathcal{C}$ in the sense that

$$i(\Sigma(x)) = E(i(x)).$$

For readability, often we simply write x instead of $i(x)$; in other words we identify the code x with the corresponding point $i(x)$ in $\mathcal{C}$.

3. The Riemann surface X

Let X be the Riemann surface $X = \hat{\mathbb{C}} \setminus \mathcal{C}$.

3.1. The dynamics and the conformal structure. The map E defined in (1) has two inverse branches $\phi_0(z) = E_0^{-1}(z)$ and $\phi_1(z) = E_1^{-1}(z)$. As maps of $\mathbb{C}$ they are conformal and contracting homeomorphisms. Let α_{-1} be the annulus $\{z | 13/18 < |z - 1/2| < 15/18\}$. We label images of this annulus under successive applications of the mappings ϕ_0 and ϕ_1 in the following way: $\alpha_0 = \phi_0(\alpha_{-1})$ and $\alpha_1 = \phi_1(\alpha_{-1})$, $\alpha_{00} = \phi_0(\alpha_0), \alpha_{01} = \phi_0(\alpha_1), \alpha_{10} = \phi_1(\alpha_0)$ and $\alpha_{11} = \phi_1(\alpha_1)$. For $x_n = s_1 s_2 \ldots s_n$, inductively define

$$\alpha_{0x_n} = \phi_0(\alpha_{x_n}), \quad \alpha_{1x_n} = \phi_1(\alpha_{x_n}).$$

These annuli are contained in X and are mutually disjoint.

Since ϕ_0, ϕ_1 are conformal, the modulus of α_{x_n} is the same for all truncations x_n. Moreover, the same is true of the extremal length $m(\Sigma'_{x_n})$ of the family Σ'_{x_n}

of curves separating the boundary curves of α_{x_n}; we easily compute $m(\Sigma'_{x_n}) = 2\pi/\log(15/13)$.

Take the circle $|z-1/2| = 14/18$ as the core curve of α_{-1}; its images in the annuli α_{x_n} are their core curves and we denote them by Γ_{x_n}. The curves Γ_0 and Γ_1 both separate (in $\hat{\mathbb{C}}$) the points with expansions $0s_2s_3\ldots$ from the points with expansions $1t_2t_3\ldots$, where $s_i, t_j \in \{0,1\}$.

Using the map ϕ_0 it follows that the curve Γ_{00}, separates (in $\hat{\mathbb{C}}$) the point with code $00s_3s_4\ldots$ from all points with codes whose first or second entry is not 0, that is, points with codes of the form $1s_2s_3\ldots$ or of the form $01s_3s_4\ldots$. It also follows that the curve Γ_{01} separates the points with codes $01s_3s_4\ldots$ from all points with codes whose first entry is not 0 and whose second entry is not 1. Analogously, using the map ϕ_1, the curves Γ_{10} and Γ_{11} separate, respectively, points with codes $10s_3s_4\ldots$ and $11s_3s_4\ldots$ from points whose codes disagree in one of their first two entries.

Applying the maps ϕ_0 and ϕ_1, according as $s_i = 0$ or 1 we see inductively that the curve $\Gamma_{s_1\ldots s_n} = \phi_{s_n}(\Gamma_{s_1\ldots s_{n-1}})$ separates points with codes $s_1s_2\ldots s_ns_{n+1}s_{n+2}\ldots$ from all points whose codes disagree in any one of their first n entries.

It is clear that the curves Γ_{x_n}, for all truncations x_n of all possible codes x, divide X into pairs of pants (three holed spheres). These curves are the *Euclidean waist curves.* These pairs of pants are glued together with the structure of a binary tree to form X. For a given $x \in \mathcal{C}$ the curves Γ_{x_n} are nested and their Euclidean diameters go to zero; therefore the intersection of their interiors is the unique point $x \in \partial X$.

3.2. The hyperbolic structure. By the Riemann mapping theorem, the universal cover of X is conformally equivalent to the upper half plane $\mathbb{H}$ and X inherits a hyperbolic structure under projection. We denote the distance in this metric by d_X to distinguish it from the Euclidean distance d on X inherited from X as a subset of $\mathbb{C}$.

Each free homotopy class of simple closed curves on X has a unique geodesic representative with respect to the hyperbolic structure. Except for Γ_0 and Γ_1, the curves Γ_{x_n} of the previous section are all in distinct free homotopy classes. Denote by γ_{x_n} the geodesic representative in the free homotopy class of Γ_{x_n}. We call γ_{x_n} a *hyperbolic waist curve of level* n. For readability, if the code x_n is not relevant we write γ_k for an arbitrary waist curve. Also, we omit the modifier Euclidean or hyperbolic for a waist curve if it is clear from the context.

Since Γ_0 and Γ_1 are freely homotopic, $\gamma_0 = \gamma_1$. Note that by symmetry $\gamma_0 = 1/2 + iy, y \in \mathbb{R}$. In general, Γ_{x_n} and γ_{x_n} are not the same curve.

Let $l(\gamma_k)$ denote the hyperbolic length of γ_k. Then we have the following easy upper bound.

PROPOSITION 1. *There is a constant $B > 1$ such that, for all waist curves of every level, $l(\gamma_k) < B$.*

Proof: Fix γ_k. Let Σ_k be the set of curves on X freely homotopic to γ_k and let $m(\Sigma_k)$ be the extremal length of this family. The family Σ'_k of closed curves in the annulus α_k is a subset of Σ_k so that $m(\Sigma_k) \leq m(\Sigma'_k) = 2\pi/\log(15/13)$. Using hyperbolic geometry it is easy to prove (see e.g. [**9**]) that $l(\gamma_k) \leq \pi m(\Sigma_k)$. □

Remark. There is also a lower bound on the lengths of the waist curves. Since its proof involves an interesting interaction between the conformal and hyperbolic structures of X, but is not necessary for our main theorem we defer it to an appendix.

On each pair of pants $P_k \subset X$ there are three unique geodesic arcs η_i^k, $i = 1, 2, 3$ orthogonal to each distinct pair of hyperbolic waist curves. Using the hyperbolic cosine laws and properties of right-angled hyperbolic hexagons, [**3**], we can express the length of each η_i^k as a function of the lengths of the waist curves of P_k. Thus the following corollary follows from Proposition 1 and also from Proposition 13 in the appendix.

COROLLARY 2. *There are uniform upper and lower bounds on the lengths of waist curves γ_j and the orthogonal arcs η_i^j, $i = 1, 2, 3$ on all pair of pants in X.*

Remark. The bound we really use is the lower bound on the orthogonals to the waist curves. Thus for the remainder of the paper we let ϵ_0 be a positive number less than the lengths of all of these orthogonals.

4. Characterizing the boundary

In this section we describe non-recurrent and adjacent sequences in general. Then we use the waist curves of X to specify a marking for the fundamental group $\pi_1(X)$.

4.1. Allowable chains of waist curves. In section 3.2 we considered the special sequences of waist curves γ_{x_n} that characterize points in $\mathcal{C}$. Here we discuss more general sequences of waist curves that determine points of $\mathcal{C}$.

Definition: A sequence of waist curves γ_k has the *adjacency property* if each consecutive pair γ_k and γ_{k+1} are boundary curves of a common pair of pants P_k.

Definition: The sequence γ_k is *non-recurrent* if for every level L there is k_0 such that for every $k > k_0$, the level of γ_k is greater than L. Obviously, any subsequence of a non-recurrent sequence is non-recurrent.

Definition: A non-recurrent sequence satisfying the adjacency property is called *an allowable chain of waist curves* or simply *an allowable chain.*

It is clear that given $x \in \mathcal{C}$ the sequence γ_{x_n} constructed in section 3.2 is an allowable chain and that, in particular, allowable chains exist. In the next proposition we prove a converse.

PROPOSITION 3. *An allowable chain γ_k on X uniquely determines a point $x \in \mathcal{C}$.*

Proof: Without loss of generality, we may assume γ_1 is at level L_1 and, for $k > 1$, γ_k is at level L_1 or higher. Because the sequence γ_k is non-recurrent, there is a largest value of k such that γ_k is at level L_1. Let k_1 be this largest value. Because of the adjacency property, γ_{k_1+1} is at level $L_1 + 1$. We let k_2 be the largest value of k for which γ_k is at level $L_1 + 1$ and, inductively, we let k_j be the largest value of k for which γ_k is at level $L_1 + j - 1$.

Then γ_{k_j} is an allowable chain of waist curves which is minimal with respect to the adjacency property in the following sense. It has the adjacency property but none of its subchains do. Moreover, any subchain with the adjacency property eventually visits the same sequence of pairs of pants.

The pairs of pants P_{k_j} are adjacent and form a nested sequence in the Euclidean representation of $X \subset \mathbb{C}$ and therefore uniquely determine a point $x \in \mathcal{C}$. Because any other subchain must visit the same sequence of pairs of pants, no other subchain can nest on a different point in $\mathcal{C}$. $\square$

4.2. The marking for X. In order to define a presentation for $\pi_1(X, q)$ we first canonically choose a point on each pair of pants.

Denote by P_{x_n} the pair of pants in X bounded by the three waist curves $\gamma_{x_n}, \gamma_{0x_n}$ and γ_{1x_n}. Below we will need to distinguish among these curves so we will call γ_{x_n} the *waist* of P_{x_n} and γ_{0x_n} and γ_{1x_n} its *cuffs.* On each pair of pants P_{x_n} we orient the cuffs so that for any choice of base point, the geodesic $\gamma_{0x_n}\gamma_{1x_n}$ in the free homotopy class of a curve formed by going around γ_{0x_n} and then going around γ_{1x_n} is simple. The geodesic in the free homotopy class of $\gamma_{0x_n}\gamma_{1x_n}^{-1}$ then has a single self-intersection point on P_{x_n} which we denote by p_n.

For each x_{n+1} equal to $0x_n$ or to $1x_n$, the curve $\gamma_{x_{n+1}}$ is a cuff of P_{x_n} and the waist of $P_{x_{n+1}}$. There is a family of geodesic arcs on the four holed sphere $Y = P_{x_n} \cup P_{x_{n+1}}$ joining p_{x_n} and $p_{x_{n+1}}$. All arcs in this family intersect $\gamma_{x_{n+1}}$ but do not intersect any of the other waists or cuffs of Y. Among all arcs in this family there are at most two whose length is equal to $d_X(p_{x_n}, p_{x_{n+1}})$. There may be two if each of the points p_{x_n} and $p_{x_{n+1}}$ lies on the common orthogonal to the cuffs of its pair of pants. Let δ_{x_n} be the shortest among these arcs and if there are two such arcs, choose one. Parameterize δ_{x_n} by $t \in [n, n+1]$.

Analogously, let σ be the shortest geodesic arc joining $p_0 \in P_0$ to $p_1 \in P_1$. Let q be the intersection point of σ and $\gamma_0 = \gamma_1$ and set $\sigma_0 = \sigma \cap P_0$, $\sigma_1 = \sigma \cap P_1$. Parameterize σ_i by $t \in [0, 1]$, $i = 0, 1$, $\sigma_i(0) = q$.

To each $x_n = s_1 \dots s_n$ let σ_{x_n} be the arc $\sigma_{x_n} = \sigma_{s_1} \cup \delta_{x_1} \cup \delta_{x_2} \dots \delta_{x_{n-1}} \cup (\delta_{x_n} \cap P_{x_n})$ and set

$$\hat{\gamma}_{x_n} = \sigma_{x_n}\gamma_{x_n}\sigma_{x_n}^{-1}.$$

We have

PROPOSITION 4. *The fundamental group of X with basepoint q is isomorphic to the free group on the generators*

$$\langle \gamma_0, \hat{\gamma}_{x_n}; x_n = s_1 \dots s_n, s_i \in \{0, 1\}, n = 1, \dots \infty \rangle$$

factored by the relations that the words $\hat{\gamma}_{x_n}^{-1}\hat{\gamma}_{0x_n}\hat{\gamma}_{1x_n}$ are equal to the identity.

Proof: On each pair of pants P_{x_n}, for $t = x_n, 0x_n, 1x_n$, let $\bar{\gamma}_t = \hat{\gamma}_t \cap P_{x_n}$. Then $\pi_1(P_{x_n}, p_{x_n}) = \langle \bar{\gamma}_{x_n}, \bar{\gamma}_{0x_n}, \bar{\gamma}_{1x_n}; \bar{\gamma}_{x_n}^{-1}\bar{\gamma}_{0x_n}\bar{\gamma}_{1x_n} \rangle$. By Van Kampen's theorem $\pi_1(X)$ is formed by taking free products with amalgamation along the common boundary of the fundamental groups of adjacent pairs of pants. The curves $\hat{\gamma}_{x_n}$ and $\bar{\gamma}_{x_n}$ are pointed curves freely homotopic to the geodesics γ_{x_n} by construction and thus generate the amalgamating subgroups. $\square$

5. Lifting the boundary

Let $\rho : \pi_1(X, q) \to PSL(2, \mathbb{R})$ be a representation with discrete torsion free image G such that $\mathbb{H}/G$ with conformal structure inherited from $\mathbb{H}$ is holomorphically equivalent to X. As usual, we identify an element in $PSL(2, \mathbb{R})$ with its corresponding Möbius transformation.

We use the marking defined above to obtain a specific set of generators for G. Let $g_{x_n} = \rho(\hat{\gamma}_n)$ and let A_{x_n} denote the axis (invariant geodesic) of g_{x_n}. Then $\pi(A_{x_n}) = \gamma_{x_n}$. To specify the representation uniquely we normalize as follows. We assume that the axis A_0 is the imaginary axis and that zero is the repelling fixed point of g_0. We also assume that the attracting fixed point of g_{01} is -1. Because of the symmetry of the surface X, the attracting fixed point of g_{10} is $+1$.

5.1. A canonical fundamental polygon. We can canonically choose lifts $\tilde{P}_{x_n}$ of pairs of pants so that as x_n varies through all truncations of points $x \in \mathcal{C}$ we obtain a canonically determined fundamental domain ω_0 for G. More specifically, consider the three axes $A_{x_n}, A_{0x_n}, A_{1x_n}$ together with the three common perpendiculars joining A_{x_n} to A_{0x_n}, A_{0x_n} to A_{1x_n}, and A_{1x_n} to A_{x_n}. This configuration forms a hyperbolic hexagon Q_{x_n} in which each interior angle is 90^o and the side lengths are all bounded above and below by $1/\epsilon_0$ and ϵ_0. Let $\bar{Q}_{x_n}$ be the reflection of Q_{x_n} in the orthogonal joining A_{0x_n} and $A_{x_{n}1}$. Then $\tilde{P}_{x_n} = Q_{x_n} \cup \bar{Q}_{x_n}$ is a canonical octagon which covers the pair of pants P_{x_n}. The union of these octagons for all truncations x_n and all $x \in \mathcal{C}$ is the canonical fundamental domain ω_0 for G.

We claim that ω_0 is connected. More precisely, $\tilde{P}_{x_n}$ and $\tilde{P}_{x_n0}$ both contain boundary arcs on the axis A_{x_n0} and these arcs intersect; the same is true of course, for $\tilde{P}_{x_n}$ and $\tilde{P}_{1x_n}$. We carry out the argument only in the first case. Note that the axes of the products $g_{0x_n}g_{1x_n}^{-1}$ and $g_{0x_n}^{-1}g_{1x_n}$ intersect in a point $\tilde{p}_{x_n}$ that projects to p_{x_n}. Moreover, by consideration of the separation properties of these axes and the axes $A_{x_n}, A_{0x_n}, A_{1x_n}$, one sees that $\tilde{p}_{x_n}$ lies inside the closure of $\tilde{P}_{x_n}$. There is a lift $\tilde{\delta}_{0x_n}$ at $\tilde{p}_{x_n}$ which determines the marked generators g_{00x_n} and g_{10x_n} and therefore ends at $\tilde{p}_{0x_n}$. Further, for any integer $k \neq 0$, the length of $\tilde{\delta}_{0x_n}$ is shorter than the length of any geodesic joining $\tilde{p}_{x_n}$ to $g_{0x_n}^k(\tilde{p}_{0x_n})$. This implies either the boundary arcs of $\tilde{P}_{x_n}$ and $\tilde{P}_{0x_n}$ on A_{1x_n} overlap or $\tilde{\delta}_{0x_n}$ is orthogonal to A_{0x_n} and forms part of the boundary of $\tilde{P}_{x_n} \cup \tilde{P}_{0x_n}$.

5.2. Allowable Chains of Axes. Let W be the totality of all hyperbolic waist curves on X and set $\mathcal{W} = \pi^{-1}(W)$. Note that any lift A_k of γ_k is an element of $\mathcal{W}$ and is an axis of an element of G conjugate to some generator. Because the waist curves are pairwise disjoint on X it follows that the axes in $\mathcal{W}$ are also disjoint.

Each sequence A_k in $\mathcal{W}$ determines a sequence of intervals I_k on the boundary of $\mathbb{H}$. For any A_k except the generator A_0, let I_k be the interval on $\partial\mathbb{H}$ between the fixed points of A_k. For that special generator we take I_0 to be either $[-\infty, 0]$ or $[0, \infty]$.

We make the following definitions for chains of axes which are parallel to the definitions given at the beginning of the previous section for chains of waist curves.

Definition: A sequence of axes is called *nested* if the corresponding sequence of intervals I_k is nested. A sequence of axes has the *adjacency property* if the sequence of waist curves $\gamma_k = \pi(A_k)$ on X has the adjacency property. A sequence of axes is *non-recurrent* if the corresponding sequence of waist curves $\gamma_k = \pi(A_k)$ on X is non-recurrent.

Definition: A nested non-recurrent sequence of axes satisfying the adjacency property is called an *allowable chain of axes.*

Proposition 5. *Any nested sequence of axes can be extended to have the adjacency property.*

Proof: Let A_k and A_{k+1} be elements of the nested sequence of axes. Denote the endpoints of the intervals I_k by a_k, b_k. Since the sequence is nested, we assume $a_k < a_{k+1} < b_{k+1} < b_k$. Set $L = a_{k+1} - a_k, M = b_{k+1} - a_{k+1}, R = b_k - b_{k+1}$ and $T = b_k - a_k$. Let d be the hyperbolic distance between A_k and A_{k+1}; it satisfies

$$\cosh d - 1 = \frac{2RL}{MT}. \tag{2}$$

Since there is a lower bound ϵ_0 on the lengths of the orthogonals to the waist curves, there is an ϵ_1 such that $\min\{R/T, L/T\} \geq \epsilon_1 > 0$. It follows that there are only finitely many axes in $\mathcal{W}$ between A_k and A_{k+1}; that is, axes with one endpoint in (a_k, a_{k+1}) and one endpoint in (b_{k+1}, b_k). Thus we can extend any nested chain by adding in finitely many axes between every pair until the adjacency property is satisfied. $\square$

Remark. Another way to obtain a bound on the number axes of hyperbolic elements of a group joining two disjoint intervals is given in [**6**]. The bound applies even to hyperbolic axes which may correspond to curves with self-intersection on X.

Lemma 6. *If A_k is an allowable chain of axes, then the intervals I_k intersect in a single point $x \in \partial\mathbb{H}$.*

Proof: We use the notation in the proof of proposition 5 above for the endpoints of the intervals and their lengths. The lemma will follow from the claim that if ϵ_0 is the lower bound on the lengths of the orthogonals to the waist curves, then for all k, $|I_{k+1}/I_k| < 1 - \epsilon_0/2$, which implies the lengths of I_k decrease geometrically to 0.

In the above notation $|I_{k+1}/I_k| = M/T$ and $\frac{L}{T} + \frac{M}{T} + \frac{R}{T} = 1$. Then equation (2) yields $\frac{d^2}{4} \leq \frac{RL}{MT}$. This implies either $\frac{R}{T}$ or $\frac{L}{T}$ is greater than or equal to $d/2$ and hence greater than $\epsilon_0/2$. This, in turn, implies the upper bound $\frac{M}{T} \leq 1 - \epsilon_0/2$. $\square$

We now show that allowable chains of axes exist.

Proposition 7. *Let $x \in \mathcal{C}$ and let A_{x_n} be the chain of axes of the generators g_{x_n}. Then A_{x_n} is an allowable chain.*

Proof: We use the notation of sections 4.2 and 5.1. We use the points $\tilde{p}_{x_n}$ in the lifts $\tilde{P}_{x_n}$ forming the canonical polygon ω_0.

There is a unique geodesic $\tilde{\sigma}$ that joins the points $\tilde{p}_0$ and $\tilde{p}_1$ in $\mathbb{H}$. Its intersection $\tilde{q} = it$ with A_0 projects to q and $\tilde{\sigma}$ is a uniquely defined lift of σ at $\tilde{q}$.

To show the A_{x_n} are nested we proceed by induction on n. The geodesic $A_0 = A_{x_1}$ divides $\mathbb{H}$ into two hyperbolic half planes, one containing $\tilde{p}_0$ and the other containing $\tilde{p}_1$. Denote by H_1 the half plane containing $\tilde{p}_{x_1}$; it contains the connected lift $\tilde{P}_{x_1}$ containing $\tilde{p}_{x_1}$ and the boundary of this lift contains an arc of each of the axes $A_0 = A_{x_1}$, A_{0x_1} and A_{1x_1}. Thus A_0 does not separate A_{0x_1} and A_{1x_1} and they both lie in H_1.

We saw above that the lifted arc $\tilde{\delta}_{x_{n-1}}$ is a geodesic arc that joins $\tilde{p}_{x_{n-1}}$ and $\tilde{p}_{x_n}$ and crosses the geodesic A_{x_n}. Thus A_{x_n} divides $\mathbb{H}$ into two hyperbolic half planes, one containing $\tilde{p}_{x_{n-1}}$ and the other, which we denote by H_n, containing $\tilde{p}_{x_n}$.

We take as our induction hypothesis that $H_k \subset H_{k-1}$ for all $k \leq n$. Suppose now that $H_{n+1} \not\subset H_n$. Then $\tilde{\delta}_{x_n}$ joining $\tilde{p}_{x_n}$ and $\tilde{p}_{x_{n+1}}$ has to intersect $A_{x_{n+1}}$ an even number of times. This is impossible because it must intersect it once by definition, and geodesics in $\mathbb{H}$ intersect exactly once or not at all. Thus $H_{n+1} \subset H_n$ and the induction is complete.

The adjacency property follows directly from the construction. Moreover since the level of A_{x_n} is precisely n the chain is non-recurrent. □

Definition: The point $\cap_k I_k$ for an allowable chain A_k is its *limiting point.*

We remark that there are infinitely many different allowable chains of axes with the same limiting point. Using the special chains A_{x_n}, however, we obtain the following result.

Proposition 8. *There is a well-defined map $\tilde{\pi}$ from the limiting point $\tilde{x}$ of the allowable chain of axes A_{x_n} determined by the code of $x = s_1 s_2 \ldots$ to the corresponding point $x = \tilde{\pi}(x)$ of $\mathcal{C}$. Restricted to the boundary of the canonical polygon ω_0, this map is Hölder continuous.*

Proof: Consider two points $\tilde{x}$ and $\tilde{y}$ on the $x-$axis which are on the boundary of the canonical fundamental polygon ω_0. Without loss of generality, assume $\tilde{x}$ and $\tilde{y}$ are both positive. We let A_{x_n} and A_{y_n} be the allowable chains of axes described at the beginning of section 5 associated to the points $\tilde{x}$ and $\tilde{y}$. Also, let I_{x_n} be the interval on the real axis whose endpoints are the endpoints of the axis A_{x_n}. $I_{x_{n+1}}$ is a subinterval of I_{x_n} and the four endpoints of these intervals divide I_{x_n} into three intervals, L, $M = I_{x_{n+1}}$ and R. We let $T = I_{x_n}$. Using equation (2), we see that the lower bound ϵ_0, on the distances between A_{x_n} and A_{x_n+1} guarantees the existence of a number $\epsilon_1 > 0$ such that

$$\min\{L/T, M/T, R/T\} > \epsilon_1,$$

for any x_n and any $\tilde{x}$ on the boundary of ω_0. Therefore, there is a positive number C such that the distance from $\tilde{x}$ to either of the endpoints of I_{x_n} is greater than or equal to $C\epsilon_1^n$. Now suppose $\tilde{y}$ is in the boundary of ω_0 and that $|\tilde{y} - \tilde{x}| < C\epsilon_1^n$. Then $I_{x_k} = I_{y_k}$ for $0 \leq k \leq n$. This implies the waist curves γ_{x_k} and γ_{y_k} are equal for $0 \leq k \leq n$. In particular, the points $x = \tilde{\pi}(\tilde{x})$ and $y = \tilde{\pi}(\tilde{y})$ in the Cantor set $\mathcal{C}$ corresponding to $\tilde{x}$ and $\tilde{y}$ satisfy $|x - y| < (\frac{1}{3})^n$. A simple computation shows that if we let $\alpha = (\log(1/3)/\log(\epsilon_1)$ the above inequalities imply

$$|x - y| \leq C|\tilde{x} - \tilde{y}|^\alpha$$

. □

Note that any element $g \in G$ determines an image $g(\omega_0)$ and allowable chains $g(A_{x_n})$ and that the same result holds for these chains and the boundary of $g(\omega_0)$.

Definition: The lifted boundary $\mathcal{L}_X(G)$ is the set of limiting points of all allowable chains of axes.

It is clear from the definitions that the image of an allowable chain of axes under an element of the group G is another allowable chain and hence that $\mathcal{L}_X(G)$ is G-invariant.

6. Conical and non-conical limit points

Suppose G is a Fuchsian group acting on $\mathbb{H}$ with limit set $\Lambda(G)$. Let c be a point in $\Lambda(G)$ and let r be a geodesic ending at c. Given $d > 0$, a *cone* C_d based at c is the set of points in $\mathbb{H}$ whose hyperbolic distance from the ray r is at most d.

The conical limit points of a Fuchsian group are defined as follows, see [**13**], where they are called radial limit points, or [**11**] or [**10**], where they are called points of approximation:

Definition: A point $x \in \Lambda(G)$ is called a *conical limit point* if there is point p in $\mathbb{H}$ and a cone $C_d \subset \mathbb{H}$ based at x such that x is an accumulation point of the intersection of the G-orbit of p and the cone C_d. The *conical limit set* is the set of conical limit points and the *non-conical limit set* $NC(G)$ is its complement in $\Lambda(G)$.

Note that if x is a conical limit point then given any $p' \in \mathbb{H}$ there is some d' such that a cone $C_{d'}$ contains infinitely many points of the G-orbit of p'. It is obvious that the attracting fixed point x of a hyperbolic element $g \in G$ is in the conical limit set because any cone C based at x and centered on A_g contains infinitely many points of $\{g^n(z)\}$ for $z \in A_g$ as $n \to \infty$.

On the other hand if g is parabolic with fixed point x then x is in the non-conical limit set. To see this let D_x be a disk tangent to the real axis at x and precisely invariant under the cyclic subgroup generated by g. By precisely invariant, we mean $g(D_x) = D_x$ and, if g' is any element of G not equal to g^n for some integer n, then $g'(D_x) \cap D_x = \emptyset$. Such a disk always exists for any primitive parabolic element of a Fuchsian group. In fact, such a disk can be chosen so that its image on the Riemann surface $\mathbb{H}/G$ has hyperbolic area equal to one, [**15**]. Let C_d be any cone at x and let $p \in D_x \cap C_d$. By the precise invariance, if g is not a power of g, then $g'(p) \notin D_x$. By the parabolic action $g^n(p)$ remains in D_x but leaves C_d as $n \to \infty$. It follows that x is in the non-conical limit set.

It is clear from the definitions that the conical and non-conical limit sets are G-invariant.

We are now ready to prove our main result.

THEOREM 9. *If X is the Riemann sphere from which the Cantor set $\mathcal{C}$ is removed and G is a Fuchsian group representing X, then the set of non-conical limit points coincides with the set of limiting points of all allowable chains of axes, that is,*

$$NC(G) = \mathcal{L}_X(G).$$

Proof: Suppose that $\tilde{x} \in \mathcal{L}_X(G)$ and let A_k be an allowable chain of axes limiting to $\tilde{x}$. Let C_d be any cone based at $\tilde{x}$. Denote by R_k the intersection of C_d and the region between A_k and A_{k+1}. By the adjacency property of the chain, the uniform lower bound on the distances between axes projecting to waist curves and the fact that the width of the cone is finite, there is a uniform upper bound on the areas of the R_k. The projection $\pi(R_k)$ onto X is contained in a pair of pants P_k. By the uniform upper bound on the lengths of the waist curves there is a uniform upper bound on the the areas of the P_k. Since R_k is compactly contained in $\mathbb{H}$, the projection π, restricted to R_k, has image contained in P_k and is finite to one. Since the chain is non-recurrent, for any k, $P_{k'} = P_k$ for only finitely many k'. It follows that any point in the projection of C_d onto X can have only finitely many preimages in C_d and therefore that $\tilde{x} \in NC(G)$.

Now suppose $\tilde{x} \in NC(G)$ and let C_d be a cone based at $\tilde{x}$. Let C_d consist of the set of points in the upper half plane with hyperbolic distance less than d from a geodesic ray r starting at a point z_0 in $\mathbb{H}$ and ending at $\tilde{x}$ on $\mathbb{R}$. Assume $2d$ is less than ϵ_0, the uniform lower bound on the lengths of the orthogonals to the waist curves.

Consider the arc $\pi(r)$ on X and let P_k be the sequence of pairs of pants in X traversed, in order, by the arc $\pi(r)$. If γ_k of level m_k is the waist of P_k, we claim that m_k goes to infinity with k. If not, for some k_0 there is a sequence $k_j \to \infty$ such that $P_{k_j} = P_{k_0}$. The lower bound ϵ_0 on the lengths of the orthogonals implies the areas of the pairs of pants is uniformly bounded above. Thus, there must be a neighborhood N_ϵ of diameter ϵ in P_{k_0} with infinitely many disjoint preimages in C_d, which contradicts the assumption that $\tilde{x} \in NC(G)$.

Each waist curve γ_k lifts to a unique axis A_k that intersects C_d; these axes determine a nested chain which, by the above argument, is non-recurrent. The cone C_d is simply connected, and the order of the axes is determined by the ray r. Since the ray projects onto adjacent pairs of pants, the chain is allowable. The point $\tilde{x}$ is therefore in $\mathcal{L}_X(G)$. □

7. Quasiconformal deformations

So far we have dealt only with the middle thirds Cantor set, the Riemann surface obtained by removing this Cantor set from the Riemann sphere and a normalized Fuchsian group representing this surface. In this section we vary all of these objects. To fix notation we now denote the middle thirds Cantor set by $\mathcal{C}_0$, the corresponding Riemann surface by $X_0 = \mathbb{C} \setminus \mathcal{C}_0$ and the normalized Fuchsian group by G_0.

The Teichmüller space $T(X_0)$ is defined as follows. Consider the set of pairs (f, X) where $f : X_0 \to X$ is a normalized quasiconformal homeomorphism to a variable Riemann surface X. Define an equivalence relation on this set by stipulating that (f, X) is equivalent to (g, Y) if there exists a conformal homeomorphism $h : X \to Y$ such that $g^{-1}hf : X_0 \to X_0$ is homotopic to the identity through a homotopy which fixes pointwise the points of $\mathcal{C}_0$. By definition the Teichmüller space $T(X_0)$ consists of the set of these pairs factored by this equivalence relation. If (f, X) represents a point in $T(X_0)$, then the map f induces an isomorphism $\Theta_f : G_0 \to G = fG_0f^{-1}$. The group G is Fuchsian and $\mathbb{H}/G$ is holomorphically equivalent to X.

We have the following facts:

- Let (f, X) be any representative of a point in $T(X_0)$ and suppose f is a K-quasiconformal homeomorphism. Then f extends uniquely to a K-quasiconformal homeomorphism of the full Riemann sphere. This follows because for each annulus α_{x_n}, (see [**2**])

$$1/K \leq \frac{\mathrm{mod}(f(\alpha_{x_n}))}{\mathrm{mod}(\alpha_{x_n})} \leq K.$$

One uses this inequality to show f extends to a homeomorphism defined everywhere and then one shows that f is absolutely continuous on almost every horizontal and vertical line. Denote the extended map again by f.

- The symbolic dynamics carries over as well. If $x = s_1 s_2 \ldots \in \mathcal{C}_0$ we assign $f(x) = s_1 s_2 \ldots$. Let $\mathcal{C} = f(\mathcal{C}_0)$. The expanding map fEf^{-1} commutes with the shift map on $\mathcal{C}$. Moreover, since it is defined, it generates a (not necessarily conformal) dynamical system fEf^{-1} acting on X conjugate to the original system E acting on X_0.
- The construction of the set $\mathcal{L}_{X_0}(G_0)$ was based only on the length bounds. If a waist curve γ on X_0 has length $l(\gamma)$ and the waist curve in the free homotopy class of $f(\gamma)$ has length $l(f(\gamma))$ on X, then the following inequality is well-known:
$$l(\gamma)/K \leq l(f(\gamma)) \leq Kl(\gamma),$$
(see for example [**5**], page 159). It follows that we can define the set $\mathcal{L}_X(G)$. The proof of the theorem $NC(G) = \mathcal{L}_X(G)$ holds mutatis mutandi. This is summarized by the following theorem.

THEOREM 10. *For any point $[(f, X)] \in T(X_0)$ represented by a Fuchsian group G, the non-conical limit set coincides with the set of limiting points of allowable chains of axes. Moreover, the natural mapping from the boundary of the canonical fundamental polygon ω for G maps surjectively onto $\mathcal{C}$ and is Hölder continuous.*

8. A Further Generalization

In this section we define a conformal, expanding dynamical system that generalizes the construction of $\mathcal{C}$. Take the interiors of a disk D and m subdisks $D_1, \cdots, D_m$ of D. We assume the subdisks are mutually disjoint and that the distances between their boundaries and between the boundary of any subdisk D_j and the the boundary of D is bounded below by a positive number. We let E_j be any affine transformation mapping D_j onto D and E be the join of the mappings E_j. Thus, E is an m to 1 mapping with domain $\bigcup_j D_j$ and range D and E restricted to D_j is equal to E_j. On D_j, E' is constant and $|E'| > 1$.

We say the $\tilde{E}$ is *a locally homeomorphic, expanding conformal dynamical system* if there is a a plane quasiconformal mapping f with the property that $\tilde{E} = f \circ E \circ f^{-1}$ is conformal on $f\left(\bigcup_j D_j\right) = \bigcup_j \tilde{D}_j$.

Now let E be a locally homeomorphic, expanding conformal dynamical system with range D and let $\Lambda = \bigcap_k E^{-k}(D)$. Written in this form, the Cantor set Λ is seen as the decreasing intersection the sets $E^{-k}(D)$ each of which consists of m^k components.

If $Y = \hat{\mathbb{C}} \setminus \Lambda$, then, like X, it is a (no longer binary) tree of pairs of pants. The boundaries of the disks D_j and their inverse images under E are conformal waist curves. By construction, there is a uniform upper bound on the extremal length of the free homotopy class of any such waist curve. It follows that the hyperbolic lengths of the hyperbolic waist curves is also bounded above.

The definitions of allowable chains of waist curves and allowable chains of axes for a Fuchsian uniformization G for Y clearly extend to this case. Therefore the limit set $L_Y(G)$ is well defined. Moreover, the construction of the fundamental polygon ω generalizes and from the the proof above we have

THEOREM 11. *Fix some locally homeomorphic expanding conformal dynamical system E_0 with invariant Cantor set Λ_0 and let $Y_0 = \hat{\mathbb{C}} \setminus \Lambda_0$. For any point $[(f,Y)] \in T(Y_0)$ represented by a Fuchsian group G, the non-conical limit set coincides with the set of limiting points of allowable chains of axes. Moreover, the natural mapping from the boundary of the canonical fundamental polygon ω for G maps surjectively onto $\mathcal{C}$ and is Hölder continuous.*

9. Appendix

The existence of a lower bound on the hyperbolic lengths of the waist curves is a corollary to a stronger result valid for complements of Cantor sets generated by conformal dynamical systems. That result is called the hyperbolic distortion lemma; it measures the extent to which long compositions E^n of E deviate from being isometries in the hyperbolic metric on the Riemann surface close to its Cantor set boundary. For consistency of presentation, we prove this distortion lemma in the special case where the surface X is the complement in the sphere of the middle thirds cantor set $\mathcal{C}$. We keep the notation set above.

Let $ds = \lambda_X(z)|dz|$ denote the hyperbolic metric on X and for $x_n = s_1 \ldots s_n$ set $\phi^{x_n}(z) = \phi_{s_n} \circ \phi_{s_{n-1}} \circ \ldots \phi_{s_1}(z)$. Then we have

LEMMA 12 (Hyperbolic Distortion lemma). *For every $\epsilon > 0$ there exists a neighborhood $U = U(\epsilon) \supset \mathcal{C}$ such that for every $z \in U \setminus \mathcal{C}$ and every x_n,*

$$\frac{1}{1+\epsilon} \leq \frac{\lambda_X(\phi^{x_n}(z))|(\phi^{x_n})'(z)|}{\lambda_X(z)} \leq 1+\epsilon.$$

Proof: A similar result is proved in lemma 3 of [**7**].

We begin the proof with three observations. Suppose ρ and λ are infinitesimal forms of metrics on domains which contain $\mathcal{C}$. We say λ and ρ are asymptotic at $\mathcal{C}$ if, for every $\epsilon > 0$, there is a neighborhood U of $\mathcal{C}$, such that if z is in $U \setminus \mathcal{C}$ then

$$\frac{1}{1+\epsilon} \leq \frac{\lambda(z)}{\rho(z)} \leq 1+\epsilon.$$

The first observation is that this lemma is true for a metric λ if and only if it is true for any metric ρ asymptotic to λ at $\mathcal{C}$.

For the second observation, suppose $\Omega_1 \supset \Omega_2$ are plane domains containing $\mathcal{C}$ and ρ_1 and ρ_2 are the Poincaré metrics for the domains $\Omega_1 \setminus \mathcal{C}$ and $\Omega_2 \setminus \mathcal{C}$. We claim ρ_1 and ρ_2 are asymptotic at $\mathcal{C}$. Moreover,

$$\rho_1(z) < \rho_2(z) < \frac{1+e^{-2d}}{1-e^{-2d}}\rho_1(z),$$

where d is the Poincaré distance from z to the boundary of Ω_2 measured in the ρ_1-metric. The first inequality follows from Schwarz's lemma applied to the inclusion map from $\Omega_2 \setminus \mathcal{C}$ into $\Omega_1 \setminus \mathcal{C}$. To see the second inequality, represent the fundamental group for $\Omega_1 \setminus \mathcal{C}$ as a Fuchsian group G acting on the unit disk, $\mathbb{D} = \{w : |w| < 1\}$, in such a way that the universal covering π projects 0 to z. Among all holomorphic functions f defined on the unit disk and mapping 0 to z, the minimum of the ratio $\frac{1}{|f'(0)|}$ and $\frac{1}{|\pi'(0)|}$ is 1. It is therefore realized by the function π and $\rho_1(z) = \frac{1}{|\pi'(0)|}$. From the definition of d, $\pi^*\rho_2(z) = \rho_2(\pi(z))|\pi'(z)|$ is a complete, Riemannian metric of constant negative curvature equal to -1 on a domain contained in the unit disk and containing the disk of radius r, where r is related to d by the formula

$$\frac{1}{2}\log\frac{1+r}{1-r} = d.$$

Since the formula for the infinitesimal form of Poincaré's metric on a disk of radius r is

$$\frac{r|dz|}{r^2 - |z|^2},$$

we conclude $\rho_2(\pi(0))\|\pi'(0)| \leq \frac{1}{r}$, which implies

$$\rho_2(z) \leq \frac{1}{r}\frac{1}{|\pi'(0)|}.$$

Therefore,

$$\rho_2(z) \leq \frac{e^{2d}+1}{e^{2d}-1}\rho_1(z). \tag{3}$$

To see that ρ_1 and ρ_2 are asymptotic at $\mathcal{C}$ we only have to observe that the set of z such that the distance from z to the boundary of Ω_2 measured in the Poincaré metric ρ_1 is larger than a constant M determines a neighborhood U of the set $\mathcal{C}$.

For the third observation, let $\Delta = \{z : |z - 1/2| < 11/18\}$ be the interior of Γ_{-1} and let $Y_0 = \Delta \setminus \mathcal{C}$. Next let $Y_1 = (\phi_{s_1}(\Delta \setminus \mathcal{C}), Y_k = \phi_{s_k} \circ \cdots \circ \phi_{s_1}(\Delta \setminus \mathcal{C})$ and let $\lambda, \lambda_0, \lambda_1, \lambda_2, \ldots$ be the infinitesimal forms of the Poincaré metrics on $X, Y_0, Y_1, Y_2, \ldots$, respectively. Let δ_1 be the minimum $\lambda-$distance from ∂Y_0 to ∂Y_1 and, in general, δ_k be the minimum $\lambda-$distance from ∂Y_{k-1} to ∂Y_k. We claim

$$\delta_1 < \delta_2 < \cdots < \delta_n.$$

First we show $\delta_1 < \delta_2$. We may assume $Y_1 = \phi_0(\Delta \setminus \mathcal{C})$. Let $\tilde{\lambda}$ be the infinitesimal form of the Poincaré metric on the $\tilde{Y} = \overline{\mathbb{C}} \setminus \phi_0(\mathcal{C})$. Then the map $z \mapsto 3z$ from $\tilde{Y}$ to X is an isometry and it maps Y_1 to Y_0. Therefore, the distance δ_1 from the boundary of Y_0 to the boundary of Y_1 in the $\lambda-$metric is equal to the distance $\tilde{\delta_2}$ from the boundary of Y_1 to the boundary of Y_2 in the $\tilde{\lambda}-$metric. But since $\tilde{\lambda}(z) < \lambda(z)$, $\tilde{\delta_2} < \delta_2$. To show $\delta_k < \delta_{k+1}$, note that $Y_{k+1} = \phi(Y_k)$, where $\phi(z) = \frac{1}{3}z + (const)$. Now let $\tilde{\lambda}$ be the Poincaré metric for the extended complex plane with $\phi(\mathcal{C})$ removed and apply the same argument as before.

To begin the proof of the lemma, first note that

$$\lambda < \lambda_0 < \lambda_1 < \cdots < \lambda_n.$$

Now assume z is in a small neighborhood of $\mathcal{C}$ and contained in Y_n and let d_k be the $\lambda_{k-1}-$distance from z to the boundary of Y_k. Then from the second observation above

$$\lambda_k(z) < \lambda_{k-1}(z)\frac{1+e^{-2d_k}}{1-e^{-2d_k}}.$$

Since $\lambda_k(\phi_{k-1}(z))|\phi'_{k-1}(z)| = \lambda_k(z)$ and $\lambda_n(\phi^{x_n}(z))|(\phi^{x_n})'(z)| = \lambda_0(z)$, we find that

$$\lambda_0(z) < \lambda_n(z) = \lambda_0(\phi^{x_n}(z))|(\phi^{x_n})'(z)| =$$

$$(\lambda_n)/\lambda_{n-1})\cdots(\lambda_1/\lambda_0)(\lambda_0) < \lambda_0(z)\prod_{k=1}^{n}\frac{1+e^{-2d_k}}{1-e^{-2d_k}}. \tag{4}$$

However, each d_k is bigger than or equal to $\sum_{j=k}^{n} \delta_j$ and each $\delta_j \geq \delta_1$. Therefore the product in (4) is dominated by

$$(\frac{1+e^{-2d_1}}{1-e^{-2d_1}}) \prod_{k=2}^{n} \frac{1+\alpha^{2k}}{1-\alpha^{2k}}, \tag{5}$$

where $\alpha = e^{-\delta_1} < 1$. Since the second factor in (5) is bounded independently of n and the first factor can be made less than $1+\epsilon$ by choosing the neighborhood U sufficiently small, the lemma is proved. □

Using this proposition we can prove

PROPOSITION 13. *There is a constant $M > 0$ such that for all waist curves γ_k, $l(\gamma_k) > M$.*

Proof: Fix $\epsilon > 0$. Since the lengths of the hyperbolic waist curves are bounded above we can fix $k = x_m$ such that $\gamma_{x_m} \subset U(\epsilon)$. By the hyperbolic distortion lemma, for every truncation x_n we have

$$\begin{aligned} l(\gamma_{x_m x_n}) &= \int_{\gamma_{x_m x_n}} \lambda(z)|dz| \\ &\geq \frac{1}{1+\epsilon} \int_{\gamma_{x_m x_n}} \lambda(\phi^{x_n}(z))|(\phi^{x_n})'(z)||dz| \\ &= \frac{1}{1+\epsilon} \int_{E^{x_n}(\gamma_{x_m x_n})} \lambda(z)|dz| \\ &\geq \frac{1}{1+\epsilon} \int_{\gamma_{x_m}} \lambda(z)|dz| \\ &= \frac{1}{1+\epsilon} l(\gamma_{x_m}). \end{aligned}$$

To find the constant M pick a neighborhood U corresponding to the choice of $\epsilon = 1/2$ in Lemma 12. All of the waist curves at level m for m bigger than or equal to some m_0 are contained in U. Let M be less than $2/3$ multiplied by the minimum of the hyperbolic lengths of all the waist curves γ_{x_k} for $k \leq m_0$. Then from the above inequality the hyperbolic lengths of all the waist curves γ_{x_n} for $n \geq m_0$ is greater than or equal to M. Moreover, each of the hyperbolic lengths of waist curves γ_{x_k} for $k \leq m_0$ is also greater than or equal to M and so this inequality holds for all waist curves. □

Remark. There is more direct way to obtain a uniform lower bound on the lengths of the hyperbolic waist curves. This also generalizes to complements of Cantor sets of bounded geometry. One explicitly constructs a planar Riemann surface Ω which is a binary tree of pairs of pants. One arranges for Ω to have one hole "at the top" and to be such that all of its waist curves have hyperbolic length equal to one. Also, one constructs Ω so that there are conformal maps G_0 and G_1 from the right and left subtrees of Ω onto Ω mapping pants to pants. By using compositions of the mappings G_0, G_1, E_0 and E_1, one can construct a quasiconformal mapping from Ω to a subdomain of X. Combining this result with inequality (3) and the Teichmüller inequality bounding the distortion of the length of a closed hyperbolic geodesic in a given homotopy class by a quasiconformal mapping (Lemma 2, page 159, of [**5**]), one obtains the uniform bound.

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