BOUNDED GEOMETRY AND CHARACTERIZATION OF SOME TRANSCENDENTAL MAPS

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ABSTRACT. We define two classes of topological infinite degree covering maps modeled on two families of transcendental holomorphic maps. The first, which we call exponential maps of type (p, q), are branched coverings and is modeled on transcendental entire maps of the form Pe^Q, where P and Q are polynomials of degrees p and q. The second is the class of universal covering maps from the plane to the sphere with two removed points modeled on transcendental meromorphic maps with two asymptotic values. The problem we address is to give a combinatorial characterization of the holomorphic maps contained in these classes whose post-singular sets are finite. The main results in this paper are that a post-singularly finite topological exponential map of type $(0, 1)$ or a certain post-singularly finite topological exponential map of type $(p, 1)$ or a post-singularly finite universal covering map from the plane to the sphere with two points removed is combinatorially equivalent to a holomorphic same type map if and only if this map has *bounded geometry*.

1. INTRODUCTION

Thurston proved that a post-critically finite degree $d \geq 2$ branched covering of the sphere, with hyperbolic orbifold, is either combinatorially equivalent to a rational map or there is a topological obstruction, now called a "Thurston obstruction" (see [T, DH, J] for the definition). In this paper, we address the problem of characterizing some post-singularly finite covering maps of infinite degree that can be realized as meromorphic maps. Thurston's proof uses an iteration scheme defined for an appropriate finite dimensional Teichmüller space. It is divided into two parts. Given an initial point, the iteration scheme is used to obtain a sequence of points in the Teichmüller space. The first part of the proof is to show that the sequence is contained in a compact subset of Teichmüller space if and only if the map is combinatorially equivalent to a rational map (a limit map of the iteration). The second part is to give a topological criterion equivalent to non-compactness. The criterion is called a "Thurston obstruction" to the existence of a combinatorially equivalent rational map.

Thurston's proof is presented in [DH]. There the main idea for the first part of the proof is to reduce the sequence in Teichmüller space to a sequence in the associated moduli space and to prove that compactness of the sequence in moduli

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space is equivalent to non-existence of the Thurston obstruction. Then, the finiteness of the degree is used to lift this argument to the Teichmüller space. The key lemma, [DH, Lemma 5.2], guarantees that because the degree is finite, the sequence in the Teichmüller space is contained in a compact subset (in the Teichmüller space) provided the corresponding sequence in moduli space is contained in a compact subset (in moduli space). Because finiteness of degree is a crucial element in this proof it does not extend to infinite degree maps.

Another framework to prove the first step for branched coverings of finite degree is outlined in [J]. It also uses Thurston's iteration scheme and the finite dimension of the associated Teichmüller space but it avoids the key lemma [DH, Lemma 5.2] in Thurston's original proof and does not depend on the finiteness of the degree of the given map. Instead, it uses the concept of "bounded geometry". This will be defined precisely in Section 9 but the idea is that under iteration the images of the points of the post-critical set stay bounded and remain a bounded distance from one another. At a limit point of the iteration, the images form a discrete set of the same multiplicity. The main purpose of this paper is to show that this framework works for topological transcendental maps too.

We define two topological classes of covering maps of the plane that model transcendental maps with finitely many singular values. The model maps form a subspace of the topological space with a natural parameterization. Recall that over a singular value a map fails to be a covering. Transcendental maps have two types of singular values, critical values over which they are branched, and asymptotic values. We give precise definitions of algebraic and asymptotic singular values for topological maps in Section 2 that agree with the classical definitions when the maps are transcendental. In Section 8 we show that the Thurston iteration scheme that associates a sequence of holomorphic maps in the model space to a given topological map is well defined for our classes. We follow the framework given in [J] to these families to find conditions under which a topological map with finite post-singular set is combinatorially equivalent to a transcendental map. Note that because the degree of these maps is not finite, the sequence generated by the iteration process does not automatically remain in a compact subset of the parameter space of model maps. Therefore, in addition to requiring a bounded geometry condition, we need to require a compactness condition that ensures that, under the iteration scheme, the iterates converge in the associated Teichmüller space. We discuss this compactness condition in more detail in Section 13.

The first family we define is modeled on the family of entire maps Pe^Q, where P are Q are polynomials of degrees p and q. These maps have one finite asymptotic value, one asymptotic value at infinity and $p+q-1$ critical points. We call it the family of *topological maps of type* (p,q) and we denote it by $\mathcal{T}E_{p,q}$.

The second family is modeled on transcendental maps with two asymptotic values and no critical values. An example of such a map is $\tan z$. The topological class consists of universal covering maps from the plane to the sphere with two removed points. The removed points are a topological version of asymptotic values which we again call asymptotic values. We call this the space of topological transcendental maps with two asymptotic values and we denote it by $\mathcal{AV}2$. Note that $\mathcal{T}E_{0,1} \subset \mathcal{AV}2$.

We will prove that the holomorphic or meromorphic maps in the topological families are precisely the model maps. The model maps Pe^Q for $\mathcal{T}E_{p,q}$ are parameterized naturally by the coefficients of P and Q. The model maps for $\mathcal{AV}2$ are parameterized naturally by their asymptotic values (or simple functions of them).

The *compactness condition* says that the sequence of holomorphic or meromorphic maps generated by the *Thurston iteration scheme* remains inside a compact subset of the parameter space of the model maps.

We follow the framework given in [J] to prove that a map $f \in \mathcal{T}E_{p,q} \cup \mathcal{AV}2$ with finite post-singular set that satisfies the *bounded geometry* and *compactness* conditions described above is combinatorially equivalent to a holomorphic or meromorphic map.

Our first theorem is

Theorem 1.1. *A post-singularly finite map f in $\mathcal{T}E_{p,q} \cup \mathcal{AV}2$ is combinatorially equivalent to a post-singularly finite entire map of the form $E = Pe^Q$ or a post-singularly finite meromorphic map with two asymptotic values if and only if it has bounded geometry and satisfies the compactness condition. The realization is unique up to conjugation by an affine map of the plane.*

In some cases, bounded geometry actually implies compactness. This is the case for some $f \in \mathcal{T}E_{p,q}$ and for $f \in \mathcal{AV}2$.

This is the content of our main theorems.

Theorem 1.2. *A post-singularly finite map f in $\mathcal{T}E_{p,1}$, $p \geq 1$, with only one non-zero simple branch point c such that either c is periodic or c and $f(c)$ are both not periodic, is combinatorially equivalent to a unique post-singularly finite entire map of the form $\alpha z^p e^{\lambda z}$, where $\alpha = (-\lambda/p)^p e^{-\lambda(-p/\lambda)^p}$, if and only if it has bounded geometry.*

Theorem 1.3. *A post-singularly finite map f in $\mathcal{AV}2$ is combinatorially equivalent to a post-singularly finite transcendental meromorphic function g with constant Schwarzian derivative if and only if it has bounded geometry. The realization is unique up to conjugation by an affine map of the plane.*

Theorem 1.2 is the first one to use the Thurston iteration scheme to characterize a transcendental entire map with critical points and is thus completely new. Therefore, we give a detailed proof of this result .

Theorem 1.3 is the content of our paper [CJK]. The main difference in the proofs of the two theorems is the proof that the compactness condition holds. Because it is not very long, we include the compactness part of the proof of Theorem 1.3 for the convenience of the reader.

Our techniques involve adapting the Thurston iteration scheme to our situation. We work with a fixed normalization. The proof of Theorem 1.1 applies to an arbitrary post-singularly finite map in either $\mathcal{T}E_{p,q}$ or $\mathcal{AV}2$. It shows that bounded geometry together with the assumption of compactness implies the convergence of the iteration scheme to fixed point in the Teichmüller space that corresponds to an entire or meromorphic map of the same type (see section 11). Its proof involves an analysis of quadratic differentials associated to the functions in the iteration scheme. To prove Theorems 1.2 and 1.3, which apply to the special cases, we need a topological constraint (see section 12). We then prove that bounded geometry together with the topological constraint implies compactness.

To conclude this introduction, we make a remark about the second step in Thurston's iteration scheme in the infinite degree context. It is still an open problem to prove that compactness of the sequence in moduli space is equivalent to non-existence of an appropriately generalized "Thurston obstruction" for maps in $\mathcal{T}E_{p,q}$ or $\mathcal{AV}2$. In a recent paper [HSS], Hubbard, Schleicher, and Shishikura gave a partial answer in the case of topological exponential maps. They used the non-existence of a Levy cycle, a very special type of Thurston obstruction, to characterize when the sequence of iterates generated by a map in $\mathcal{T}E_{0,1}$ is contained in a compact subset of moduli space. A full answer to the question of finding an appropriate generalization of a Thurston obstruction is still open, even for this special case.

The paper is organized as follows. In §2, we review the covering properties of (p,q)-exponential maps $E = Pe^Q$. In §3, we define the family $\mathcal{T}E_{p,q}$ of (p,q)-topological exponential maps f. In §4, we review the properties of meromorphic maps with two asymptotic values. In §5, we describe the space of universal covering maps from the Eucildean plane to the sphere with two removed points. In §6, we define combinatorial equivalence between post-singularly finite maps in $\mathcal{T}E_{p,q}$ or $\mathcal{AV}2$ and prove there is a local quasiconformal map in every combinatorial equivalence class. In §7, we define the Teichmüller space T_f for a post-singularly finite (p,q)-topological exponential map f or a post-singularly finite map in $\mathcal{AV}2$. In §8, we introduce the induced map σ_f from the Teichmüller space T_f into itself; this is the crux of the Thurston iteration scheme. In §9, we define the concept of "bounded geometry" and in §10 we prove the necessity of the bounded geometry condition. In §11, we give the proof of sufficiency assuming compactness for any post-singularly finite map f in either $\mathcal{T}E_{p,q}$ or $\mathcal{AV}2$. In §12, we define a topological constraint for the maps in Theorem 1.2 and Theorem 1.3; this involves defining markings and the winding number of a homotopy class. We prove that the winding numbers and the homotopy classes are unchanged under iteration of the map σ_f. Furthermore, in §13 we prove that bounded geometry together with the topological constraint implies compactness. This completes the proofs of Theorem 1.2 and Theorem 1.3.

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2. The space $\mathcal{E}_{p,q}$ of (p,q)-Exponential Maps

We use the following notation: $\mathbb{C}$ is the complex plane, $\hat{\mathbb{C}}$ is the Riemann sphere and $\mathbb{C}^*$ is the complex plane punctured at the origin.

A (p,q)-*exponential map* is an entire function of the form $E = Pe^Q$ where P and Q are polynomials of degrees $p \geq 0$ and $q \geq 0$ respectively such that $p + q \geq 1$. We use the notation $\mathcal{E}_{p,q}$ for the set of (p,q)-exponential maps.

Note that if $P(z) = a_0 + a_1 z + \dots a_p z^p$, $Q(z) = b_0 + b_1 z + \dots b_q z^q$, $\widehat{P}(z) = e^{b_0} P(z)$ and $\widehat{Q}(z) = Q(z) - b_0$ then

$$P(z)e^{Q(z)} = \widehat{P}e^{\widehat{Q}(z)}.$$

To avoid this ambiguity we always assume $b_0 = 0$. If $q = 0$, then E is a polynomial of degree p. Otherwise, E is a transcendental entire function with an essential singularity at infinity.

The growth rate of an entire function f is defined as

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

where $M(r) = \sup_{|z|=r} |f(z)|$. It is easy to see that the growth rate of E is q.

Recall the following definitions:

Definition 2.1. *Given an entire or meromorphic function g, the point v is an* asymptotic value *of g if there is a path $\gamma(t)$ such that* $\lim_{t\to 1} \gamma(t) = \infty$ *and* $\lim_{t\to 1} g(\gamma(t)) = v$. *It is a* logarithmic singularity for the map g^{-1} *if there is a neighborhood U_v of v and a component V of $g^{-1}(U_v \setminus \{v\})$ such that the map $g : V \to U_v \setminus \{v\}$ is a holomorphic universal covering map. If an asymptotic value is isolated, it is a logarithmic singularity. (This will always be the case in this paper.) The domain V is called an* asymptotic tract for v. *A point may be an asymptotic value for more than one asymptotic tract. An asymptotic value may be an omitted value.*

Definition 2.2. *Given a holomorphic or meromorphic function g, the point v is an* algebraic singularity for the map g^{-1} *if there is a neighborhood U_v such that for every component V_i of $g^{-1}(U_v)$ the map $g : V_i \to U_v$ is a degree $d_i \geq 1$ branched covering map and $d_i > 1$ for some V_i. For these components, if $c_i \in V_i$ satisfies $g(c_i) = v$ then $g'(c_i) = 0$; that is c_i is a* critical point of g *for $i = 1, \dots, n$ and v is a* critical value.

Note that if a meromorphic function had exactly one asymptotic value and no critical values it would be a covering map $g : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{a\}$ where a is the asymptotic value. Such a map is a conformal homeomorphism that extends to a Möbius transformation. Therefore if a meromorphic map of degree greater than one has no critical values it must have at least two asymptotic values. By the big Picard theorem, no entire function can omit more than one value and no transcendental meromorphic function $g : \mathbb{C} \to \hat{\mathbb{C}}$ can omit more than two values.

The following theorem about functions E in $\mathcal{E}_{p,q}$ is a summary of Theorem 3.5 in [Z].

Proposition 2.3. *If $q \geq 1$, E has $2q$ distinct asymptotic tracts that are separated by $2q$ rays. Each tract maps to a punctured neighborhood of either zero or infinity and these are the only asymptotic values.*

Proof. From the growth rate of E we see that for $|z|$ large, the behavior of the exponential dominates. Since $Q(z) = b_q z^q +$ lower order terms, in a neighborhood of infinity there are $2q$ branches of $\Re Q = 0$ asymptotic to equally spaced rays. In the $2q$ sectors defined by these rays the signs of $\Re Q$ alternate. If $\gamma(t)$ is an asymptotic path such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\gamma(t)$ stays in one sector for all large t, then either $\lim_{t\to\infty} E(\gamma(t)) = 0$ or $\lim_{t\to\infty} E(\gamma(t)) = \infty$, as $\Re Q$ is negative or positive in the sector. It follows that there are exactly q sectors that are asymptotic tracts for 0 and q sectors that are asymptotic tracts for infinity. Because the complement of these tracts in a punctured neighborhood of infinity consists entirely of these rays, there can be no other asymptotic tracts. $\square$

Remark 2.4. *The directions dividing the asymptotic tracts are called* Julia rays *or* Julia directions *for E. If $\gamma(t)$ tends to infinity along a Julia ray, $E(\gamma(t))$ remains in a compact domain in the plane. It spirals infinitely often around the origin.*

Two (p,q)-exponential maps E_1 and E_2 are conformally equivalent if they are conjugate by a conformal automorphism M of the Riemann sphere $\hat{\mathbb{C}}$, that is, $E_1 = M \circ E_2 \circ M^{-1}$. The automorphism M must be a Möbius transformation and it must fix both 0 and ∞ so that it must be the affine stretch map $M(z) = az$, $a \neq 0$. We are interested in conformal equivalence classes of maps, so by abuse of notation, we treat conformally equivalent (p,q)-exponential maps E_1 and E_2 as the same.

The critical points of $E = Pe^Q$ are the roots of $P' + PQ' = 0$. Therefore, E has $p + q - 1$ critical points counted with multiplicity which we denote by

$$\Omega_E = \{c_1, \cdots, c_{p+q-1}\}.$$

Note that if $E(z) = 0$ then $P(z) = 0$. This in turn implies that if $c \in \Omega_E$ maps to 0, then c must also be a critical point of P. Since P has only $p - 1$ critical points counted with multiplicity, there must be at least q points (counted with multiplicity) in Ω which are not mapped to 0. Denote by

$$\Omega_{E,0} = \{c_1, \cdots, c_k\}, \quad k \leq p - 1,$$

the (possibly empty) subset of Ω_E consisting of critical points such that $E(c_i) = 0$. Denote its complement in Ω_E by

$$\Omega_{E,1} = \Omega_E \setminus \Omega_{E,0} = \{c_{k+1}, \cdots, c_{p+q-1}\}$$

and denote the set of non-zero critical values of E by

$$\mathcal{V} = E(\Omega_{E,1}) = \{v_1, \cdots, v_m\}.$$

If $\Omega_{E,0}$ is not empty, the set of critical values is

$$\mathcal{V} \cup \{0\}.$$

When $q = 0$, E is a polynomial. The point at infinity is a fixed critical point and hence a critical value. The *post-singular set* in this special case is the same as the *post-critical set*. It is defined as

$$P_E = \overline{\cup_{n\geq 1} E^n(\Omega_E)} \cup \{\infty\}.$$

From now on we will always assume $q > 0$ so that E is not a polynomial. Since $q \geq 1$, E always has one finite asymptotic value at the origin. It has another asymptotic value at infinity which we can think of as a fixed point of E so the full set of singular values $\hat{\mathcal{V}}$ of E always contains 0 and ∞. Note, however, that $E(\infty)$ is not defined. That is,

$$\hat{\mathcal{V}} = \mathcal{V} \cup \{0\} \cup \{\infty\}.$$

The *post-singular set* is defined as

$$P_E = \overline{\cup_{n\geq 0} E^n(v), v \in \mathcal{V} \cup \{\infty\}}.$$

Conjugating by an affine map $z \to az$ of the complex plane, we normalize so that $0, 1, \infty \in P_E$.

To avoid trivial cases here we will assume that $\#(P_E) \geq 4$. When $q = 1$ and $p = 0$, $\Omega_E = \emptyset$ and $\mathcal{E}_{0,1}$ consists of exponential maps $\alpha e^{\lambda z}$, $\alpha, \lambda \in \mathbb{C}^*$. Since the singular set contains only the two asymptotic values, 0 and ∞, the *post-singular set* in this special case can be written as

$$P_E = \overline{\cup_{n\geq 0} E^n(0)} \cup \{\infty\}.$$

Conjugating by an affine stretch $z \mapsto \alpha z$ of the complex plane, we normalize so that $E(0) = 1$. Note that after this normalization $0, 1, \infty \in P_E$ and the family takes the form $e^{\lambda z}$, $\lambda \in \mathbb{C}^*$.

When $q \geq 2$ and $p = 0$ or when $q \geq 1$ and $p \geq 1$, $\Omega_{E,1}$ is a non-empty set; that is $\hat{\mathcal{V}}$ contains at least one value other than 0 or ∞.

We normalize as follows so that we always have $0, 1, \infty \in P_E$:

If E does not fix 0, which is always true if $q \geq 2$ and $p = 0$, we conjugate by an affine stretch $z \to az$ so that $E(0) = 1$.

If $E(0) = 0$, there is a critical point c_{k+1} in $\Omega_{E,1}$ with $c_{k+1} \neq 0$ and $v_1 = E(c_{k+1}) \neq 0$. In this case we normalize so that $v_1 = 1$. The family $\mathcal{E}_{1,1}$ consists of functions of the form $\alpha z e^{\lambda z}$. After normalization they take the form

$$-\lambda e z e^{\lambda z}.$$

An important family we consider in this paper is $\mathcal{E}_{p,1}$, $p \geq 1$; each map in this family has only one non-zero simple critical point. After normalization, the functions take the form

$$E(z) = \alpha z^p e^{\lambda z}, \quad \alpha = \Big(-\frac{\lambda}{p}\Big)^p e^p.$$

This is the family for which we have the strongest new results.

3. Topological Exponential Maps of Type (p, q)

In this section we define the space of maps of the Euclidean plane, denoted here by $\mathbb{R}^2$, whose covering properties are modeled on the maps in the holomorphic family $\mathcal{E}_{p,q}$ with $p+q \geq 1$ discussed above. We call this the space of *topological exponential maps of type* (p, q) and denote it by $\mathcal{T}E_{p,q}$.

The maps in $f \in \mathcal{T}E_{p,q}$ are topological maps of $\mathbb{R}^2$ that are branched (with finite order) at $p+q-1$ points. In analogy with holomorphic maps we call these branch points *critical points* and their images *critical values.* If $q > 0$, the maps f also have one singular point a over which the map is not a covering that has the following property. There is a punctured neighborhood of the point $U \setminus \{a\}$ and a simply connected component V of $f^{-1}(U \setminus \{a\})$ such that $f : V \to U \setminus \{a\}$ is a universal covering. In analogy with holomorphic maps we call this an *asymptotic value of* f.

Remark 3.1. *In the definitions below, we assume that the maps are conjugated by an affine map so that the asymptotic value $a = 0$. This is only for convenience. In fact the classes we define are invariant under pre or post composition by a homeomorphism.*

In [Z], Zakeri gives a description of the covering properties for the family $\mathcal{E}_{p,q}$. Our definition of the covering properties for the family $\mathcal{T}E_{p,q}$ is modeled on his description.

If $q = 0$, then $\mathcal{T}E_{p,0}$ consists of all topological polynomials P of degree p: these are degree p branched coverings of the sphere such that $f^{-1}(\infty) = \{\infty\}$.

If $q = 1$ and $p = 0$, the space $\mathcal{T}E_{0,1}$ consists of universal covering maps $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$. These are discussed at length in [HSS], where they are called topological exponential maps.

The polynomials P and Q contribute differently to the covering properties of maps in $\mathcal{E}_{p,q}$. As we saw there, the degree of Q controls the growth and behavior at infinity. We therefore first define the space $\mathcal{T}E_{0,q}$ using maps e^Q as our model.

In the definitions below, we use the term *ray* to mean a simple topological curve starting from a finite point in $\mathbb{R}^2$ and extending to infinity.

Definition 3.2. *If $q \geq 2$ and $p = 0$, the space $\mathcal{T}E_{0,q}$ consists of infinite to one topological maps $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$ whose set of critical points, $\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$, consists of $q-1$ points counted with multiplicity. Let $\mathcal{V} = \{v_1, \cdots, v_m\} = f(\Omega_f) \subset \mathbb{R}^2 \setminus \{0\}$ be the set of critical values. For each map in $\mathcal{T}E_{0,q}$ we can choose a set of pairwise disjoint rays L_i in $\mathbb{R}^2 \setminus \{0\}$, $i = 1, \ldots, m$,* *starting at v_i, in such a way that the set $f^{-1}(L_i)$ consists of infinitely many rays starting at points in the pre-image set $f^{-1}(v_i)$. These are divided into two categories: if $x \in f^{-1}(v_i) \cap \Omega_f$, there are $d_x = \deg_x f$ rays meeting at x and these are called* critical rays*; if $x \in f^{-1}(v_i) \setminus \Omega_f$, there is only one ray emanating from x and it is called a* non-critical ray*.*

The existence of the rays L_i and $f^{-1}(L_i)$ implies that the maps in $\mathcal{T}E_{0,q}$ have covering properties that are like those of the maps e^Q. Set

$$W = \mathbb{R}^2 \setminus (\cup_{i=1}^m L_i \cup \{0\}).$$

The set of critical rays meeting at points in Ω_f divides $f^{-1}(W)$ into $q = 1 + \sum_{c \in \Omega_f}(d_c - 1)$ open unbounded connected components $W_1, \cdots, W_q$. For each $1 \leq i \leq q$, the components are simply connected and the map $f : W_i \to W$ is a universal covering. This means that the map restricted to each W_i is a topological model for the exponential map $z \mapsto e^z$. The local degree at the critical point c_i determines the number of W_i that meet a c_i.

We now define the space $\mathcal{T}E_{p,q}$ in full generality where we assume $p > 0$ and there is additional behavior modeled on the role of the new critical points of Pe^Q introduced by the non-constant polynomial P. (See Figure 1.)

Definition 3.3. *If $q \geq 1$ and $p \geq 1$, the space $\mathcal{T}E_{p,q}$ consists of infinite to one topological maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

i) *$f^{-1}(0)$ consists of p points counted with multiplicity.*
ii) *The set of critical points, $\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$ consists of $p + q - 1$ points counted with multiplicity.*
iii) *Let $\Omega_{f,0} = \Omega_f \cap f^{-1}(0)$ be the $k < p$ critical points that map to 0 and $\Omega_{f,1} = \Omega_f \setminus \Omega_{f,0}$ the $p+q-1-k$ critical points that do not. Note that $\Omega_{f,1}$ contains at least q points and the set of critical values $\mathcal{V} = \{v_1, \cdots, v_m\} = f(\Omega_{f,1})$ is contained in $\mathbb{R}^2 \setminus \{0\}$.*

    For each map in $\mathcal{T}E_{p,q}$ we can choose a set of pairwise disjoint rays L_i in $\mathbb{R}^2 \setminus \{0\}$, $i = 1, \ldots, m$, starting at v_i, in such a way that the set $f^{-1}(L_i)$ consists of infinitely many rays starting at points in the pre-image set $f^{-1}(v_i)$. These are divided into two categories: if $x \in f^{-1}(v_i) \cap \Omega_{f,1}$, there are $d_x = \deg_x f$ rays meeting at x and these are called critical rays*; if $x \in f^{-1}(v_i) \setminus \Omega_{f,1}$, there is only one ray emanating from x and this is called a* non-critical ray*.*

Again we see that the existence of the rays L_i and $f^{-1}(L_i)$ implies that the maps in $\mathcal{T}E_{p,q}$ have the covering properties that mimic the covering properties of the maps of the form $E = Pe^Q$. Set

$$W = \mathbb{R}^2 \setminus (\cup_{i=1}^m (L_i) \cup \{0\}).$$

The set $f^{-1}(W)$ has the following properties:

(1) The collection of all critical rays meeting at points in $\Omega_{f,1}$ divides $f^{-1}(W)$ into $l = p + q - k = 1 + \sum_{c \in \Omega_{f,1}}(d_c - 1)$ open unbounded connected components.

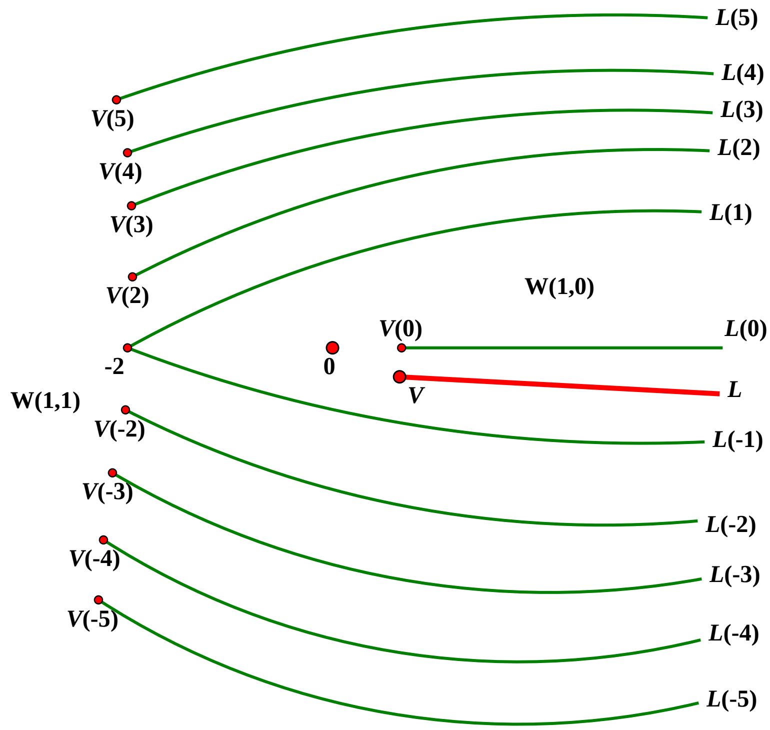

FIGURE 1. The function here is $f(z) = \alpha z^2 e^z$ with critical point -2, critical value V and fixed asymptotic value at 0. 0 is also a critical point and critical value. The non-critical pre-images of V are labeled by $V(k)$, $k \in \mathbb{Z}$. The line L is a ray emanating from the critical value V and its pre-images under f are labeled by $L(k)$.

(2) The set $f^{-1}(0) = \{a_i\}_{i=1}^{p-k}$ contains $p-k$ distinct values. Each a_i is contained in a distinct component of $f^{-1}(W)$; label these components $W_{i,0}$, $i = 1, \ldots p-k$. Then the restriction $f : W_{i,0} \setminus \{a_i\} \to W$ is an unbranched covering map of degree $d_i = deg_{a_i} f$ where $d_i > 1$ if $a_i \in \Omega_{f,0}$ and $d_i = 1$ otherwise.

(3) Label the remaining q connected components of $f^{-1}(W)$ by $W_{j,1}$, $j = 1, \ldots, q$. They are simply connected and the restriction $f : W_{j,1} \to W$ is a universal covering map.

From [Z, Section 3], we see that the (p,q)-exponential maps are in $\mathcal{TE}_{p,q}$ and that the converse follows from Lemma 3.1. We include a proof for the convenience of the reader.

Theorem 3.4. *Suppose $f \in \mathcal{TE}_{p,q}$ is analytic. Then $f = Pe^Q$ for two polynomials P and Q of degrees p and q. That is, an analytic topological exponential map of type (p,q) is a (p,q)-exponential map.*

Proof. If $q = 0$, then f is a polynomial P of degree p.

If $q \geq 1$, then f is an entire function with p roots, counted with multiplicity. Every such function can be expressed as

$$f(z) = P(z)e^{g(z)}$$

where P is a polynomial of degree p and g is some entire function (see [A, Section 2.3]).

Consider

$$f'(z) = (P(z)g'(z) + P'(z))e^{g(z)}.$$

It is also an entire function, and by assumption it has $p+q-1$ roots so that $Pg'+P'$ is a polynomial of degree $p+q-1$. It follows that g' is a polynomial of degree $q-1$ and $g = Q$ is a polynomial of degree q. □

Remark 3.5. *Since the functions in $\mathcal{T}E_{p,q}$ are defined by their topological properties, post or pre-composing with a homeomorphism results in a function that is again in $\mathcal{T}E_{p,q}$.*

Note that if $f \in \mathcal{T}E_{p,q}$, $q \neq 0$, the origin plays a special role: it is the only point with no or finitely many pre-images. Since $q \neq 0$, it is an asymptotic value so there are some rays R in components W_i such that $f(R)$ limits on 0. The point at infinity is not in the domain or range of f but it too plays a special role. There are rays L in W with pre-image rays $R = f^{-1}(L)$ in the components W_i. This means that infinity is also an asymptotic value for f. Below we treat these asymptotic values differently because the orbit of the origin is always defined but the orbit of infinity is not.

Below we define the *post-singular set* P_f of f. When $q = 0$, f is topologically conjugate to a polynomial and, as mentioned in the introduction, a full discussion of such maps is treated elsewhere. We therefore always assume $q \geq 1$.

Definition 3.6. *For $f \in \mathcal{T}E_{p,q}$, we define the* post-singular set *as follows:*

i) *When $q = 1$ and $p = 0$, the* post-singular set *is*

$$P_f = \overline{\cup_{n\geq 0} f^n(0)} \cup \{\infty\}.$$

ii) *When $q \geq 1$ and $p \geq 1$, the set of branch (critical) points is*

$$\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$$

and and the set of critical values is $\mathcal{V} = f(\Omega_f)$. Note that 0 *is not necessarily in $\mathcal{V}$. The* post-singular set *is*

$$P_f = \overline{\cup_{n\geq 0} f^n(\mathcal{V} \cup \{0\})} \cup \{\infty\}.$$

Since the conjugate of $f \in \mathcal{T}E_{p,q}$ by $z \mapsto az, a \in \mathbb{C}^*$ is also in $\mathcal{T}E_{p,q}$ and conjugate maps are conformally equivalent, we can always normalize so that $\{0, 1, \infty\} \in P_f$. If $q > 1$ or if $q = 1$ and $f(0) \neq 0$, we normalize so that $f(0) = 1 \in P_f$. If $f(0) = 0$, then, by the assumption $q \geq 1$, there is a branch point $c_{k+1} \neq 0$ such that $v_1 = f(c_{k+1}) \neq 0$. In this case we normalize so that $v_1 = 1 \in P_f$.

To avoid trivial cases we assume that $\#(P_f) \geq 4$.

It is clear that, in any case, P_f is forward invariant, that is,

$$f(P_f \setminus \{\infty\}) \cup \{\infty\} \subseteq P_f$$

or equivalently,

$$f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\} \supset P_f.$$

Note that since we assume $q \geq 1$, $f^{-1}(P_f \setminus \{\infty\}) \setminus (P_f \setminus \{\infty\})$ contains infinitely many points.

Definition 3.7. *We call* $f \in \mathcal{T}E_{p,q}$ post-singularly finite *if* $\#(P_f) < \infty$.

4. The Space $\mathcal{M}_2$

In this section we define the space of meromorphic functions $\mathcal{M}_2$. It is the model for the more general space of topological functions $\mathcal{AV}2$ that we define in the next section.

The space $\mathcal{M}_2$ consists of meromorphic functions whose only singular values are its asymptotic values. We will see later that in this situation these must be omitted values.

Definition 4.1. *The space* $\mathcal{M}_2$ *consists of meromorphic functions* $g : \mathbb{C} \to \hat{\mathbb{C}}$ *with exactly two asymptotic values and no critical values.*

4.1. **Examples.** Examples of functions in $\mathcal{M}_2$ are the exponential functions $\alpha e^{\beta z}$ and the tangent functions $\alpha \tan i\beta z = i\alpha \tanh \beta z$ where α, β are complex constants.

The asymptotic values for the exponential functions above are $\{0, \infty\}$; the half plane $\Re\beta z < 0$ is an asymptotic tract for 0 and the half plane $\Re\beta z > 0$ is an asymptotic tract for infinity. The asymptotic values for the tangent functions above are $\{\alpha i, -\alpha i\}$ and the asymptotic tract for αi is the half plane $\Im\beta z > 0$ while the asymptotic tract for $-\alpha i$ is the half plane $\Im\beta z < 0$.

4.2. **Nevanlinna's Theorem.** To find the form of the most general function in $\mathcal{M}_2$ we use a theorem of Nevanlinna [N, Chapter 11][1]

Theorem 4.2 (Nevanlinna)**.** *Every meromorphic function g with exactly p asymptotic values and no critical values has the property that its Schwarzian derivative is a polynomial of degree* $p-2$. *That is*

$$S(g) = \Big(\frac{g''}{g'}\Big)' - \frac{1}{2}\Big(\frac{g''}{g'}\Big)^2 = a_{p-2}z^{p-2} + \ldots a_1 z + a_0. \tag{1}$$

[1] In this chapter Nevanlinna constructs rational functions with a fixed number of branch points and then lets the order of branching go to infinity. By an earlier theorem the limit function is analytic and has asymptotic values and no critical points. The functions all have rational Schwarzian where the degree is bounded only by the number of points and so is bounded in the limit. Since there are no critical points the limit Schwarzian is a polynomial. A proof of the converse can be found in [H2, Chapter 10].

Conversely, for every polynomial $P(z)$ of degree $p-2$, the solution to the Schwarzian differential equation $S(g) = P(z)$ is a meromorphic function with p asymptotic values and no critical values.

Here we apply this theorem to the special case when $P(z)$ is a constant. It is easy to check that $S(\alpha e^{\beta z}) = -\frac{1}{2}\beta^2$ and $S(\alpha \tan \frac{i\beta}{2} z) = -\frac{1}{2}\beta^2$.

To find all functions in $\mathcal{M}_2$, let $\beta \in \mathbb{C}$ be constant and consider the Schwarzian differential equation

$$S(g) = -\beta^2/2 \tag{2}$$

and the related second order linear differential equation

$$w'' + \frac{1}{2}S(g)w = w'' - \frac{\beta^2}{4}w = 0. \tag{3}$$

It is straightforward to check that if w_1, w_2 are linearly independent solutions to equation (3), then $g_\beta = w_2/w_1$ is a solution to equation (2). The converse is also true: if g is a meromorphic function defined in a simply connected domain in $\mathbb{C}$, then two solutions w_1 and w_2 of (3) can be found, and furthermore, these are unique up to a common scale factor. Thus solutions to the differential equation (3) give all the solutions of the equation (2). (This classical result can be found, for example, in [H1, sec. 17.6]).

Normalizing so that $w_1(0) = 1, w_1'(0) = -\beta/2, w_2(0) = 1, w_2'(0) = \beta/2$ and solving equation (3), we have $w_1 = e^{-\frac{\beta}{2}z}, w_2 = e^{\frac{\beta}{2}z}$ as linearly independent solutions and $g_\beta(z) = e^{\beta z}$ and $g_{-\beta}(z) = e^{-\beta z}$ as linearly independent solutions to equation (2). An arbitrary solution to equation (2) then has the form

$$\frac{Aw_2 + Bw_1}{Cw_2 + Dw_1},\ A, B, C, D \in \hat{\mathbb{C}},\ AD - BC = 1 \tag{4}$$

and its asymptotic values are $\{A/C, B/D\}$.

Remark 4.3. *The asymptotic values are distinct and omitted.*

Remark 4.4. *If $B = C = 0, AD = 1, A = \sqrt{\alpha}$ we obtain the exponential family $\{\alpha e^{\beta z}\}$ with asymptotic values at 0 and ∞. If $A = -B = \sqrt{\frac{\alpha i}{2}},\ C = D = \sqrt{-\frac{i}{2\alpha}}$ we obtain the tangent family $\{\alpha \tan \frac{i\beta}{2} z\}$ whose asymptotic values $\{\pm\alpha i\}$ are symmetric with respect to the origin.*

Remark 4.5. *Note that in the solutions of $S(g) = -\beta^2/2$ what appears are e^β and $e^{-\beta}$, and not β (or β^2); this creates an ambiguity about which branch of the logarithm of e^β corresponds to a given solution of equation (2). In section 12 we address this ambiguity in our situation. We show that the topological map we start with determines a topological constraint which in turn, defines the appropriate branch of the logarithm for each of the iterates in our iteration scheme.*

Remark 4.6. *One of the basic features of the Schwarzian derivative is that it satisfies the following cocycle relation: if f, g are meromorphic functions then*

$$S(g \circ f)(z) = S(g)(f(z))f'(z)^2 + S(f)(z).$$

In particular, if T is a Möbius transformation, $S(T)(z) = 0$ and $S(T \circ g)(z) = S(g)(z)$ so that post-composing by T doesn't change the Schwarzian.

In our dynamical problems the point at infinity plays a special role and the dynamics are invariant under conjugation by an affine map. Thus, we may assume that all the solutions have one asymptotic value at 0 and that they take the value 1 at 0.

Since this is true for $g_\beta(z) = e^{\beta z}$, any solution with this normalization has the form[2]

$$g_{\alpha,\beta}(z) = \frac{\alpha g_\beta(z)}{(\alpha - \frac{1}{\alpha})g_\beta(z) + \frac{1}{\alpha}} \tag{5}$$

where α is an arbitrary value in $\mathbb{C}^*$. The second asymptotic value is $\lambda = \frac{\alpha}{\alpha - \frac{1}{\alpha}}$. It takes values in $\mathbb{C} \setminus \{0, 1\}$. The point at infinity is an essential singularity for all these functions.

The parameter space $\mathcal{P}$ for these functions is the two complex dimensional space

$$\mathcal{P} = \{\alpha, \beta \in \mathbb{C}^*\}.$$

The parameters define a natural complex structure for the space $\mathcal{M}_2$. The subspace of *entire* functions in $\mathcal{M}_2$ is the one dimensional subspace of $\mathcal{P}$ defined by fixing $\alpha = 1$ and varying β;

$$g_\beta(z) = e^{\beta z}.$$

The tangent family has symmetric asymptotic values. Renormalized, it forms another one dimensional subspace of $\mathcal{P}$. This is defined by fixing $\alpha = \sqrt{2}$ and varying β;

$$g_{\sqrt{2},\beta}(z) = 1 + \tanh \frac{\beta}{2} z = \frac{\sqrt{2} e^{\beta z}}{\frac{1}{\sqrt{2}} e^{\beta z} + \frac{1}{\sqrt{2}}}.$$

These functions have asymptotic values at $\{0, 2\}$ and $g_{\frac{1}{\sqrt{2}},\beta}(0) = 1$.

Definition 4.7. *For $g_{\alpha,\beta}(z) \in \mathcal{M}_2$, the set $\Omega = \{0, \lambda\}$ of asymptotic values is the set of singular values. The* post-singular set *P_g is defined by*

$$P_g = \overline{\bigcup_{n \geq 0} g^n(\Omega)} \cup \{\infty\}.$$

[2]Notice that $g_{\alpha,\beta}$ is obtained from g_β by a Möbius transformation with determinant 1. To see this, multiply the top and bottom of (4) by $e^{\frac{\beta}{2} z}$, require that one asymptotic value is 0 and make the determinant 1.

Note that we include the point at infinity separately in P_g because whether or not it is an asymptotic value, it is an essential singularity and its forward orbit is not defined. The asymptotic values are in P_g and, since 0 and λ are omitted and $g_{\alpha,\beta}(0) = 1 \in P_g$, $\#P_g \geq 3$.

5. The Space $\mathcal{AV}2$

We now want to consider the topological structure of functions in $\mathcal{M}_2$ and define $\mathcal{AV}2$ to be the set of maps with the same topology.

Definition 5.1. *Let X be a simply connected open surface and let $S^2 = \mathbb{R}^2 \cup \{\infty\}$ be the 2-sphere. For any pair of distinct points $a, b \in S^2$ let $f_{a,b} : X \to S^2 \setminus \{a, b\}$ be an unbranched covering map; that is, a universal covering map. We say the pair $(X, f_{a,b})$ is equivalent to the pair $(Y, f_{c,d})$ if and only if there is a homeomorphism $h : X \to Y$ such that $f_{c,d} \circ h = f_{a,b}$. The space of these pairs is denoted by $\mathcal{AV}2$.*

Let $(X, f_{a,b})$ be a representative of a map in $\mathcal{AV}2$. By abuse of notation, we will often suppress the dependence on the equivalence class and identify X with $\mathbb{R}^2 = S^2 \setminus \{\infty\}$ and refer to $f_{a,b}$ as an element of $\mathcal{AV}2$.

By definition $f_{a,b}$ is a local homeomorphism and satisfies the following conditions:

For $v = a$ or $v = b$, let $U_v \subset X$ be a neighborhood of v whose boundary is a simple closed curve that separates a from b and contains v in its interior. Since $f_{a,b}$ is a universal cover, v is an omitted value and

(1) $f_{a,b}^{-1}(U_v \setminus \{v\})$ is connected and simply connected.
(2) The restriction $f_{a,b} : f_{a,b}^{-1}(U_v \setminus \{v\}) \to U_v \setminus \{v\}$ is a regular covering of a punctured topological disk whose degree is infinite.
(3) $f^{-1}(\partial U_v)$ is an open curve extending to infinity in both directions.

In analogy with meromorphic functions with isolated singularities we say

Definition 5.2. *v is called a* logarithmic singularity *of $f_{a,b}^{-1}$ or, equivalently, an* asymptotic value *of $f_{a,b}$. The domain $V_v = f_{a,b}^{-1}(U_v \setminus \{v\})$ is called an* asymptotic tract *for v.*

Definition 5.3. $\Omega_f = \{a, b\}$ is the set of singular values of $f_{a,b}$.

Because it is compact, we can endow S^2 with the standard complex structure and identify it with $\hat{\mathbb{C}}$. By the classical uniformization theorem,[3] for any pair $(X, f_{a,b})$, there is a map $\pi : \mathbb{C} \to X$ such that $g_{a,b} = f_{a,b} \circ \pi$ is meromorphic. It is called *the meromorphic function associated to $f_{a,b}$*.

By Nevanlinna's theorem $S(g(z))$ is constant and moreover,

Proposition 5.4. *If $g(z) \in \mathcal{M}_2$ with $\Omega_g = \{a, b\}$ then $g(z) = g_{a,b}(z) \in \mathcal{AV}2$ and, conversely, if $g_{a,b} \in \mathcal{AV}2$ is meromorphic then $g_{a,b} \in \mathcal{M}_2$.*

[3]The proof of Lemma 6.2 shows that there is a quasiconformal map. By the measurable Riemann mapping theorem, [AB], it can be made meromorphic.

Proof. Any $g(z) \in \mathcal{M}_2$ is a universal cover $g : \mathbb{C} \to \hat{\mathbb{C}} \setminus \Omega_g$ and so belongs to $\mathcal{AV}2$. Conversely, if $g_{a,b} \in \mathcal{AV}2$, it is meromorphic and its only singular values are the omitted values $\{a, b\}$; it is thus in $\mathcal{M}_2$. □

We define the post-singular set for maps in $\mathcal{AV}2$ just as we did for functions in $\mathcal{M}_2$.

Definition 5.5. *For $f = f_{a,b} \in \mathcal{AV}2$, the* post-singular set P_f *is defined by*

$$P_f = \overline{\bigcup_{n\geq 0} f^n(\Omega_f)} \cup \{\infty\}$$

Note that under the identification of S^2 with the Riemann sphere and X with the complex plane, $S^2 \setminus X$ is the point at infinity and it has no forward orbit although it may be an asymptotic value. We therefore include it in P_f.

Conjugation of $f_{a,b}$ by an affine transformation T results in another map in $\mathcal{AV}2$. In what follows, therefore, we always assume X is the Euclidean plane $\mathbb{R}^2$, one asymptotic value is $a = 0$ and the second asymptotic value is $b = \lambda$ and we normalize so that $f(0) = 1$.

Definition 5.6. *We call $f \in \mathcal{AV}2$* post-singularly finite *if $\#(P_f) < \infty$.*

6. Combinatorial Equivalence

Definition 6.1. *Suppose f and g are a pair of post-singularly finite maps either in $\mathcal{T}E_{p,q}$ or in $\mathcal{AV}2$. We say that f and g are* combinatorially equivalent *if there are two homeomorphisms ϕ and ψ of the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ fixing 0 and ∞ such that $\phi \circ f = g \circ \psi$ on $\mathbb{R}^2$ and if, in addition, $\phi^{-1} \circ \psi$ is isotopic to the identity of S^2 rel P_f; that is, $\phi|P_f = \psi|P_f$.*

The commutative diagram for the above definition is

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\psi"] \arrow[d, "f"] & \mathbb{R}^2 \arrow[d, "g"] \\
\mathbb{R}^2 \arrow[r, "\phi"] & \mathbb{R}^2
\end{tikzcd}$$

The isotopy condition says that $P_g = \phi(P_f)$.

Consider $\mathbb{R}^2 \cup \{\infty\}$ equipped with the standard conformal structure as the Riemann sphere and let f be a map from $X \subset \mathbb{C}$ into $\hat{\mathbb{C}}$. We say f is *locally K-quasiconformal* for some $K > 1$ if for any $z \in \hat{\mathbb{C}} \setminus (\Omega_f \cup \{0, 1\})$ there is a neighborhood U of z such that $f : U \to f(U)$ is K-quasiconformal. Since the maps f we are working with are isotopies rel a finite set, the following lemma is standard. We only give a proof for $f \in \mathcal{T}E_{p,q}$. For $f \in \mathcal{AV}2$, the proof is similar.

Lemma 6.2. *Any post-singularly finite $f \in \mathcal{T}E_{p,q}$ (or $f \in \mathcal{AV}2$) is combinatorially equivalent to some locally K-quasiconformal map $g \in \mathcal{T}E_{p,q}$ (or $g \in \mathcal{AV}2$).*

Proof. Recall that Ω_f is the set of branch points of f in $\mathbb{C}$ and 0 is the only asymptotic value in $\mathbb{C}$. Consider the space $X = \mathbb{C} \setminus \Omega_f$. For every $p \in X$, let U_p be a small neighborhood about p such that $\phi_p = f|_{U_p} : U_p \to f(U_p) \subset \mathbb{C}$ is injective. Then $\alpha = \{(U_p, \phi_p)\}_{p \in X}$ defines an atlas on X with charts (U_p, ϕ_p). If $U_p \cap U_q \neq \emptyset$, then $\phi_p \circ \phi_q^{-1}(z) = z : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$. Thus all transition maps are conformal $(1-1$ and analytic) and the atlas α defines a Riemann surface structure on X which we again denote by α. Denote the Riemann surface by $S = (X, \alpha)$. From the uniformization theorem, S is conformally equivalent to the Riemann surface $\mathbb{C} \setminus A$ with the standard complex structure induced by $\mathbb{C}$, where A consists of $n = \#(\Omega_f) + 1$ points . The homeomorphism $h : \mathbb{C} \to \mathbb{C}$ with $h(0) = 0$ and $h : S = (X, \alpha) \to \mathbb{C} \setminus A$ is conformal so that $R = f \circ h^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic with critical points at $h(\Omega_f)$ and one asymptotic value at 0. Since the set P_f is finite, following the standard procedure in quasiconformal mapping theory, there is a K-quasiconfornal homeomorphism $k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that h is isotopic to k rel P_f. The map $g = R \circ k$ is a locally K-quasiconformal map in $\mathcal{T}E_{p,q}$ and combinatorially equivalent to f. This completes the proof of the lemma. $\square$

Thus without loss of generality, in the rest of the paper, for our purposes, we will assume that any post-singularly finite $f \in \mathcal{T}E_{p,q}$ (or $f \in \mathcal{AV}2$) is locally K-quasiconformal for some $K \geq 1$. The argument in Lemma 6.2 can be adapted to show that we may assume the maps ϕ and ψ in Definition 6.1 are quasiconformal. We will do so in the rest of the paper.

7. Teichmüller Space T_f

Recall that we denote $\mathbb{R}^2 \cup \{\infty\}$ equipped with the standard conformal structure by $\widehat{\mathbb{C}}$. Let $\mathcal{M} = \{\mu \in L^\infty(\widehat{\mathbb{C}}) \mid \|\mu\|_\infty < 1\}$ be the unit ball in the space of all measurable functions on the Riemann sphere. Each element $\mu \in \mathcal{M}$ is called a Beltrami coefficient. For each Beltrami coefficient μ, the Beltrami equation

$$w_{\overline{z}} = \mu w_z$$

has a unique quasiconformal solution w^μ which maps $\hat{\mathbb{C}}$ to itself fixing $0, 1, \infty$. Moreover, w^μ depends holomorphically on μ.

Let f be a post-singularly finite map in $\mathcal{T}E_{p,q}$ (or $\mathcal{AV}2$) with post-singular set P_f. The Teichmüller space $T(\hat{\mathbb{C}}, P_f)$ is defined as follows. Given Beltrami differentials $\mu, \nu \in \mathcal{M}$ we say that μ and ν are equivalent in $\mathcal{M}$, and denote this by $\mu \sim \nu$, if $(w^\nu)^{-1} \circ w^\mu$ is isotopic to the identity map of $\widehat{\mathbb{C}}$ rel P_f. The equivalence class of μ under $\sim$ is denoted by $[\mu]$. We set

$$T_f = T(\hat{\mathbb{C}}, P_f) = \mathcal{M}/\sim .$$

The classical Teichmüller space $Teich(\hat{\mathbb{C}} \setminus P_f) = Teich(X_0)$ of Riemann surfaces with basepoint $X_0 = \hat{\mathbb{C}} \setminus P_f$ consists of equivalence classes of pairs (g, X) where X is the Riemann sphere punctured at $n = \#P_f$ points and g is a quasiconformal map $g : X_0 \to X$; (g_1, X_1) is equivalent to (g_2, X_2) if there is a conformal map

$h : X_1 \to X_2$ such that $g_2^{-1} \circ h \circ g_1$ is isotopic to the identity. It is well known that $Teich(X_0)$ is a finite dimensional complex analytic space.

If μ_g is the Beltrami differential of g in the pair (g, X), we identify $[\mu_g]$ with $[\mu] \in T_f$. This implies that T_f has the same dimension as $Teich(X_0)$. Therefore, the Teichmüller distance d_T and the Kobayashi distance d_K on T_f coincide (see e.g. [GJW]).

8. Induced Holomorphic Map σ_f

For any post-singularly finite f in $\mathcal{T}E_{p,q}$ (or $\mathcal{AV}2$), there is an induced map $\sigma = \sigma_f$ from T_f into itself given by

$$\sigma([\mu]) = [f^*\mu],$$

where

$$f^*\mu(z) = \frac{\mu_f(z) + \mu_f((f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu_f(f(z))\theta(z)}, \quad \mu_f(z) = \frac{f_{\bar{z}}}{f_z}, \quad \theta(z) = \frac{\bar{f}_z}{f_z}. \tag{6}$$

It is a holomorphic map, so it contracts the the Kobayashi distance d_K. By the final paragraph in the last section, this means it is a contraction in the Teichmüller distance d_T. Thus we have that

Lemma 8.1. *For any two points τ and $\tilde{\tau}$ in T_f,*

$$d_T(\sigma(\tau), \sigma(\tilde{\tau})) \le d_T(\tau, \tilde{\tau}).$$

The next lemma follows directly from the definitions.

Lemma 8.2. *A post-singularly finite f in $\mathcal{T}E_{p,q}$ (or $\mathcal{AV}2$) is combinatorially equivalent to a (p,q)-exponential map $E = Pe^Q$ (or a meromorphic map in $\mathcal{M}2$) if and only if σ has a fixed point in T_f.*

Proof. Suppose σ has a fixed point $\tau = [\mu]$, that is, $\sigma(\tau) = [f^*\mu] = \tau = [\mu]$. This implies that w^μ and $w^{f^*\mu}$ are isotopy rel P_f. Using Formula (6) and Lemma 6.2, one can check that

$$E = w^\mu \circ f \circ (w^{f^*\mu})^{-1}$$

is holomorphic. This says that f is combinatorially equivalent to $E(z) = P(z)e^{Q(z)}$ or equivalently that it belongs to $\mathcal{AV}2$. This proves the "if" part.

For the "only if" part, suppose there are two quasiconformal homeomorphisms ϕ and ψ as in Definition 6.1 such that ϕ and ψ are isotopic rel P_f and $E = \phi \circ f \circ \psi^{-1} = Pe^Q$. Let μ be the Beltrami coefficient of ϕ. Then the last equality implies that $f^*\mu$ is the Beltrami coefficient of ψ. Thus w^μ and $w^{f^*\mu}$ are isotopic rel P_f. This implies that $\sigma(\tau) = \tau$ as required. $\square$

9. Bounded Geometry

For any $\tau_0 \in T_f$, let $\tau_n = \sigma^n(\tau_0)$, $n \geq 1$. The iteration sequence $\tau_n = [\mu_n]$ determines a sequence of finite subsets

$$P_{f,n} = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \cdots .$$

Since all w^{μ_n} fix $0, 1, \infty$, it follows that $0, 1, \infty \in P_{f,n}$. Note that $P_{f,n}$ depends only on τ_n and not μ_n by the Teichmüller equivalence relation.

Definition 9.1 (Spherical Version)**.** *We say f has* bounded geometry *if there is a constant $b > 0$ and a point $\tau_0 \in T_f$ such that*

$$d_{sp}(p_n, q_n) \geq b$$

for $p_n, q_n \in P_{f,n}$ and $n \geq 0$. Here

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}$$

is the spherical distance on $\hat{\mathbb{C}}$.

Note that $d_{sp}(z, \infty) = \frac{|z|}{\sqrt{1+|z|^2}}$. Away from infinity the spherical metric and Euclidean metric are equivalent. Precisely, for any bounded $S \subset \mathbb{C}$, there is a constant $C > 0$ which depends only on S such that

$$C^{-1} d_{sp}(x, y) \leq |x - y| \leq C d_{sp}(x, y) \quad \forall x, y \in S.$$

Consider the hyperbolic Riemann surface $R = \hat{\mathbb{C}} \setminus P_f$ equipped with the standard complex structure as the basepoint $\tau_0 = [0] \in T_f$. A point τ in T_f defines another complex structure τ on R. Denote by R_τ the hyperbolic Riemann surface R equipped with the complex structure τ.

A simple closed curve $\gamma \subset R$ is called *non-peripheral* if each component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of P_f. Recall that we are assuming that $\#P_f \geq 4$. Let γ be a non-peripheral simple closed curve in R. For any $\tau = [\mu] \in T_f$, let $l_\tau(\gamma)$ be the hyperbolic length of the unique closed geodesic homotopic to γ in R_τ. The bounded geometry property can be stated in terms of hyperbolic geometry as follows.

Definition 9.2 (Hyperbolic version)**.** *We say f has* bounded geometry *if there is a constant $a > 0$ and a point $\tau_0 \in T_f$ such that $l_{\tau_n}(\gamma) \geq a$ for all $n \geq 0$ and all non-peripheral simple closed curves γ in R.*

The above definitions of bounded geometry are equivalent because of the following lemma and the fact that we have normalized so that $0, 1, \infty$ always belong to P_f.

Lemma 9.3. *Consider the hyperbolic Riemann surface $\hat{\mathbb{C}} \setminus X$, where X is a finite subset of $\hat{\mathbb{C}}$ such that $0, 1, \infty \in X$, equipped with the standard complex structure. Let $a > 0$ be a constant. Every simple closed geodesic in $\hat{\mathbb{C}} \setminus X$ has hyperbolic length greater than a, if and only if spherical distance between any two distinct points in*

X is bounded below by a bound $b > 0$ which depends only on a and the number of points $m = \#(X)$.

We omit the proof and refer the interested reader to [CJK] for a detailed proof.

10. The Proof of Necessity

Our theorems have two parts: the necessity and sufficiency of the bounded geometry condition. The necessity is relatively easy and can be proved once for all cases together. We prove the following statement.

Theorem 10.1 (Necessity)**.** *If a post-singularly finite map $f \in \mathcal{T}E_{p,q}$ (or $\mathcal{AV}2$) is combinatorially equivalent to a (p,q)-exponential map $E = Pe^Q \in \mathcal{E}_{p,q}$ (or a meromorphic map $g \in \mathcal{M}2$), then f has bounded geometry.*

Proof. If f is combinatorially equivalent to $E = Pe^Q \in \mathcal{E}_{p,q}$ (or $g \in \mathcal{M}2$), then σ has a fixed point τ_0 so that $\tau_n = \tau_0$ for all n. The complex structure on $\hat{\mathbb{C}} \setminus P_f$ defined by τ_0 induces a hyperbolic metric on it. The shortest closed geodesic in this metric gives a lower bound on the lengths of all geodesics so that f satisfies the hyperbolic definition of bounded geometry. $\square$

11. Sufficiency under Compactness

The proof of the sufficiency of bounded geometry in our theorems is more complicated and needs some preparatory material. There are two parts: one is a compactness argument and the other is a fixed point argument. Once one has compactness, the proof of the fixed point argument is quite standard (see [J]) and works for any $f \in \mathcal{T}E_{p,q}$ and any $f \in \mathcal{AV}2$. This is the content of Theorem 1.1 whose proof we give in this section. We only give the details for $f \in \mathcal{T}E_{p,q}$. For $f \in \mathcal{AV}2$, the proof is similar but uses different notation (see [CJK]).

The normalized functions in $\mathcal{E}_{p,q}$ are determined by the $p+q+1$ coefficients of the polynomials P and Q. This identification defines an embedding into $\mathbb{C}^{p+q+1}$ and hence a topology on $\mathcal{E}_{p,q}$.

Given $f \in \mathcal{T}E_{p,q}$ and given any $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by σ. Let w^{μ_n} be the normalized quasiconformal map with Beltrami coefficient μ_n. Then

$$E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{E}_{p,q}$$

This follows because by Remark 3.5 it belongs to $\mathcal{T}E_{p,q}$, by construction it preserves μ_0 and hence is holomorphic so that by Theorem 3.4 it is in $\mathcal{E}_{p,q}$. This gives a sequence $\{E_n\}_{n=0}^{\infty}$ of maps in $\mathcal{E}_{p,q}$ and a sequence of subsets $P_{f,n} = w^{\mu_n}(P_f)$. Note that $P_{f,n}$ is not, in general, the post-singular set P_{E_n} of E_n.

The compactness condition. We say f satisfies the compactness condition if the sequence $\{E_n\}_{n=1}^{\infty}$ generated in the Thurston iteration scheme is contained in a compact subset of $\mathcal{E}_{p,q}$.

From a conceptual point of view, the compactness condition is very natural and simple. From a technical point of view, however, it is not at all obvious. We give a detailed proof showing how to get a fixed point assuming both bounded geometry and compactness.

Suppose f is a post-singularly finite topological exponential map in $\mathcal{T}E_{p,q}$. For any $\tau = [\mu] \in T_f$, let T_τ and T^*_τ be the tangent space and the cotangent space of T_f at τ respectively. Let w^μ be the corresponding normalized quasiconformal map fixing $0, 1, \infty$. Then T^*_τ coincides with the space $\mathcal{Q}_\mu$ of integrable meromorphic quadratic differentials $q = \phi(z)dz^2$. Integrablility means that the norm of q, defined by

$$||q|| = \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\overline{z}$$

is finite. This condition implies that the poles of q must occur at points of $w^\mu(P_f)$ and that these poles are simple.

Set $\tilde{\tau} = \sigma(\tau) = [\tilde{\mu}]$ and denote by w^μ and $w^{\tilde{\mu}}$ the corresponding normalized quasiconformal maps. We have the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus f^{-1}(P_f) & \xrightarrow{w^{\tilde{\mu}}} & \hat{\mathbb{C}} \setminus w^{\tilde{\mu}}(f^{-1}(P_f)) \\ \downarrow f & & \downarrow E_{\mu,\tilde{\mu}} \\ \hat{\mathbb{C}} \setminus P_f & \xrightarrow{w^\mu} & \hat{\mathbb{C}} \setminus w^\mu(P_f). \end{array}$$

Note that in the diagram, by abuse of notation, we write $f^{-1}(P_f)$ for $f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\}$. Since by definition $\tilde{\mu} = f^*\mu$, the map $E = E_{\mu,\tilde{\mu}} = w^\mu \circ f \circ (w^{\tilde{\mu}})^{-1}$ defined on $\mathbb{C}$ is analytic. Again by Remark 3.5 it belongs to $\mathcal{T}E_{p,q}$, by construction it preserves μ_0 and hence is holomorphic so that by Theorem 3.4 it is in $\mathcal{E}_{p,q}$. Therefore $E_{\mu,\tilde{\mu}} = P_{\tau,\tilde{\tau}} e^{Q_{\tau,\tilde{\tau}}}$ for a pair of polynomials $P = P_{\tau,\tilde{\tau}}$ and $Q = Q_{\tau,\tilde{\tau}}$ of respective degrees p and q.

Let $\sigma_* : T_\tau \to T_{\tilde{\tau}}$ and $\sigma^* : T^*_{\tilde{\tau}} \to T^*_\tau$ be the tangent and co-tangent maps of σ, respectively. Take a co-tangent vector $\tilde{q} = \tilde{\phi}(w)dw^2$ in $T^*_{\tilde{\tau}}$. Let $q = \sigma^*\tilde{q}$ be the corresponding co-tangent vector in T^*_τ. Then q is also the push-forward integrable quadratic differential of $\tilde{q}$ by E

$$q = E_*\tilde{q} = \phi(z)dz^2. \tag{7}$$

To see this, recall from section 3 that E, and a choice of curves L_i from the branch points, determine a finite set of domains W_i on which E is an unbranched covering to a domain homeomorphic to $\mathbb{C}^*$. Since E restricted to each W_i is either a topological model for e^z or z^k, we may divide each W_i into a collection of fundamental domains on which E is bijective. Therefore the coefficient $\phi(z)$ of q is given by the formula

$$\phi(z) = (\mathcal{L}\tilde{\phi})(z) = \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2} = \frac{1}{z^2} \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(\frac{P'(w)}{P(w)} + Q'(w))^2} \tag{8}$$

Here $\mathcal{L}$ is called a transfer operator in thermodynamical formalism. Following some standard calculations (see e.g. [J]) on transfer operators, we have,

$$||q|| < ||\tilde{q}||. \tag{9}$$

Remark 11.1. *The main point in the calculations is to note that E has infinite degree and q has finitely many poles. If there were a $\tilde{q}$ with $||q|| = ||\tilde{q}|| \neq 0$ and poles comprising a set Z, then the poles of q would be contained in the set $E(Z) \cup \mathcal{V}_E$, where $\mathcal{V}_E$ is the set of critical values of E. Thus, by formula (8),*

$$E^* q = \phi(E(w)) dw^2 = n\tilde{q}(w),$$

where n is the degree of E. Furthermore,

$$E^{-1}(E(Z) \cup \mathcal{V}_E) \subseteq Z \cup \Omega_E.$$

Since n is infinite, the last inclusion formula can not hold because the left hand side is infinite and the right hand side is finite.

An immediate corollary of inequality (9) is

Corollary 11.2. *For any two points τ and $\tilde{\tau}$ in T_f,*

$$d_T\Big(\sigma(\tau), \sigma(\tilde{\tau})\Big) < d_T(\tau, \tilde{\tau}).$$

In addition, inequality (9) also implies uniqueness.

Corollary 11.3. *If σ has a fixed point in T_f, then this fixed point must be unique. This is equivalent to saying that a post-singularly finite f in $\mathcal{T}E_{p,q}$ is combinatorially equivalent to at most one (p,q)-exponential map $E = Pe^Q$.*

We can now finish the proof of the sufficiency in Theorem 1.1.

Proof of Theorem 1.1. Suppose $f \in \mathcal{T}E_{p,q}$ has both bounded geometry and compactness. Suppose $\tau_0 = [\mu_0]$ satisfies the bounded geometry condition and μ_n is defined by $\sigma^n(\tau_0) = [\mu_n]$. Recall that the map defined by

$$E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \tag{10}$$

is a (p,q)-exponential map.

If $q = 0$, E_n is a polynomial and the theorem follows from the arguments given in [CJ] and [DH] so we assume now that $q > 0$.

Note that if $P_f = \{0, 1, \infty\}$, then f is a universal covering map of $\mathbb{C}^*$ and is therefore combinatorially equivalent to $e^{2\pi niz}$. The Teichmüller space in this case is a single point. Thus in the following argument, we assume that $\#(P_f) \geq 4$. Then, given our normalization conventions and the bounded geometry hypothesis we see that the functions E_n, $n = 0, 1, \ldots$ satisfy the following conditions:

1) $m = \#(w^{\mu_n}(P_f)) \geq 4$ is fixed.
2) $0, 1, \infty \in w^{\mu_n}(P_f)$.
3) $\Omega_{E_n} \cup \{0, 1, \infty\} \subseteq E_n^{-1}(w^{\mu_n}(P_f))$.
4) there is a $b > 0$ such that $d_{sp}(p_n, q_n) \geq b$ for any $p_n, q_n \in w^{\mu_n}(P_f)$.

As a consequence of the compactness, we have that in the sequence $\{E_n\}_{n=1}^{\infty}$, there is a subsequence $\{E_{n_i}\}_{i=1}^{\infty}$ converging to a map $E = Pe^Q \in \mathcal{E}_{p,q}$ where P and Q are polynomials of degrees p and q respectively.

To complete the proof we need to show that the sequence has a unique limit. To this end, note that any integrable quadratic differential $q_n \in T^*_{\tau_n}$ has, at worst, simple poles in the finite set $P_{f,n} = w^{\mu_n}(P_f)$. Since $T^*_{\tau_n}$ is a finite dimensional linear space, there is a quadratic differential $q_{n,max} \in T^*_{\tau_n}$ with $\|q_{n,max}\| = 1$ such that

$$0 \leq a_n = \sup_{||q_n||=1} \|(E_n)_* q_n\| = \|(E_n)_* q_{n,max}\| < 1.$$

Moreover, by the bounded geometry condition any simple poles of $\{q_{n,max}\}_{n=1}^{\infty}$ lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on $\hat{\mathbb{C}}$ with, at worst, simples poles at $m = \#(P_f)$ points.

Let

$$a_{\tau_0} = \sup_{n \geq 0} a_n.$$

Let $\{n_i\}$ be a sequence of integers such that the subsequence $a_{n_i} \to a_{\tau_0}$ as $i \to \infty$. Taking a further subsequence if necessary, we obtain a convergent sequence of sets $P_{n_i,\tau_0} = w^{\mu_{n_i}}(P_f)$ with limit set X. By bounded geometry, $\#(X) = \#(P_f)$ and $d_{sp}(x, y) \geq b$ for any $x, y \in X$. Thus we can find a subsequence $\{q_{n_i,max}\}$ converging to an integrable quadratic differential q of norm 1 whose only poles lie in X and are simple. Now by inequality (9), we have that

$$a_{\tau_0} = ||E_* q|| < 1.$$

Thus we have proved that there is an $0 < a_{\tau_0} < 1$, depending only on b and f, such that

$$\|\sigma_*\| = \|\sigma^*\| \leq a_{\tau_0}.$$

Let l_0 be a curve connecting τ_0 and τ_1 in T_f and set $l_n = \sigma_f^n(l_0)$ for $n \geq 1$. Then $l = \cup_{n=0}^{\infty} l_n$ is a curve in T_f connecting all the points $\{\tau_n\}_{n=0}^{\infty}$. For each point $\tilde{\tau}_0 \in l_0$, we have $a_{\tilde{\tau}_0} < 1$. Taking the maximum gives a uniform $a < 1$ for all points in l_0. Since σ is holomorphic, a is an upper bound for all points in l. Therefore,

$$d_T(\tau_{n+1}, \tau_n) \leq a\, d_T(\tau_n, \tau_{n-1})$$

for all $n \geq 1$. Hence, $\{\tau_n\}_{n=0}^{\infty}$ is a convergent sequence with a unique limit point τ_∞ in T_f and τ_∞ is a fixed point of σ. This together with Lemma 8.2 completes the proof of sufficiency in Theorem 1.1. □

In the case of rational maps (see [J]) the bounded geometry condition always guarantees that the compactness condition holds. The statements of Theorems 1.2 and 1.3 do not assume that the compactness condition holds for the families they treat. To complete the proofs of these theorems we need to show that the bounded geometry condition implies compactness for these families. From Definition 3.3 of the family $\mathcal{T}E_{p,q}$ and the subsequent discussion, it follows that the connected components W_i in that definition can be decomposed into fundamental domains; that is, subsets that map one to one onto W. We need to have some control of these fundamental domains to prove the compactness condition holds. To do this, in the next section we define some topological constraints on the families of Theorems 1.2 and 1.3. In Section 12.3 we show how the topological constraints, together with the bounded geometry condition, control the fundamental domains.

12. Topological Constraints.

In section 3 we defined two different normalizations for functions in $\mathcal{T}E_{p,q}$ that depend on whether or not 0 is a fixed point of the map. The topological constraints for post-singularly finite maps also follow this dichotomy.

12.1. **For $f \in \mathcal{T}E_{0,1}$ satisfying the hypotheses of Theorem 1.3.** A function in $f \in \mathcal{T}E_{0,1}$ is also $\mathcal{AV}2$ because it has the topological covering properties of the exponential map and as such has two topological asymptotic values, a finite one normalized to be 0 and one at infinity. Since it has no branch points its post-singular set is $P_f = \overline{\cup_{k\geq 0} f^k(0) \cup \{\infty\}}$. Recall that we are assuming that $\#P_f \geq 4$.

The hypotheses of Theorem 1.3 are that P_f is finite and that f has bounded geometry. Because P_f is finite the orbit of 0 is either periodic or pre-periodic. Since the asymptotic value 0 is omitted it cannot be part of a periodic orbit so the orbit of 0 must be pre-periodic. Let $c_k = f^k(0)$ for $k \geq 0$. By the pre-periodicity, there are a minimal integer $k_1 \geq 0$ and a minimal integer $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. This says that

$$\{c_{k_1+1}, \ldots, c_{k_1+l}\}$$

is a periodic orbit of period l. Let $k_2 = k_1+l$. Let γ be a continuous curve connecting c_{k_1} and c_{k_2} in $\mathbb{R}^2$ disjoint from P_f, except for its endpoints. Because

$$f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1},$$

the image curve $\delta = f(\gamma)$ is a closed curve.

12.2. **For $f \in \mathcal{T}E_{p,1}$ with a non-periodic critical point satisfying the hypotheses of Theorem 1.2.** Any such f has exactly one non-zero simple branch point which we denote by c; 0 is the only other branch point and it is fixed with multiplicity $p-1$. By our normalization, $f(c) = 1$. In this case

$$P_f = \cup_{k\geq 1} f^k(c) \cup \{0, \infty\}.$$

Again by the hypothesis of Theorem 1.2, P_f is finite. Set $c_k = f^k(c)$ for $k \geq 0$.

If c is periodic, we will see below that compactness follows directly from bounded geometry so we assume here that c is not periodic. Then, as above, there are minimal integers $k_1 \geq 0$ and $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. Again,

$$\{c_{k_1+1}, \ldots, c_{k_1+l}\}$$

is a periodic orbit of period l. Let $k_2 = k_1 + l$.

As above, let γ be a continuous curve connecting c_{k_1} and c_{k_2} in $\mathbb{R}^2$ disjoint from P_f, except for its endpoints. Since

$$f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1},$$

the image curve $\delta = f(\gamma)$ is a closed curve.

12.3. **Winding number.** In each of the above cases, 0 is either omitted or is its only pre-image. Since the curve γ avoids P_f it doesn't pass through any pre-image of 0. Therefore the *winding number* η of the closed curve $\delta = f(\gamma)$ about 0 is well defined. Since the winding number is independent of homotopy rel P_f we can replace γ by a curve that intersects as few fundamental domains as possible. Thus, the winding number essentially counts the number of fundamental domains between c_{k_1} and c_{k_2} and thus defines a "distance" between these fundamental domains. (See Figure 2).

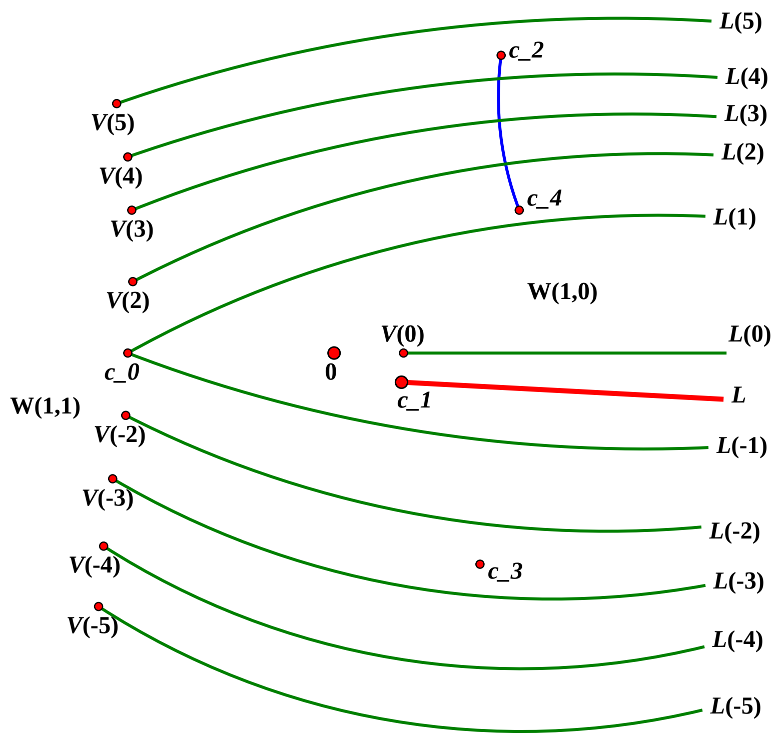

FIGURE 2. The figure shows $f(z) = \alpha z^2 e^{\lambda z}$ with critical point $c_0 = -2/\lambda$, critical value $c_1 = 1$ and fixed asymptotic value at 0. We assume that c_0 lands on the cycle $\{c_3, c_4\}$ so the points c_2 and c_4 are the pre-images of c_3. We show them separated by 3 fundamental domains.

The following lemma is a crucial point in the proof that the compactness condition holds for each family of functions satisfying the hypotheses of Theorem 1.2 and Theorem 1.3.

Lemma 12.1. *Suppose $f \in \mathcal{T}E_{p,q}$ or $\mathcal{AV}2$ with $4 \leq \#P_f < \infty$ and normalized so that $\{0, 1, \infty\} \in P_f$. Let δ be any closed curve in $\mathbb{C} \setminus P_f$ with winding number η about 0. If w^{μ_n} is a normalized quasiconformal map arising from the Thurston iteration procedure, set $\delta_n = w^{\mu_n}(\delta)$. Then δ_n has winding number η about 0. That is, the winding number is invariant under Thurston iteration.*

Proof. Given $\tau_0 = [\mu_0] \in T_f$, apply the Thurston map σ to obtain the sequence $\tau_n = \sigma^n(\tau_0) = [\mu_n]$. Let w^{μ_n} be the normalized quasiconformal map in its equivalence class with Beltrami coefficient μ_n. Then

$$E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$$

is holomorphic since it preserves the standard structure μ_0 and is holomorphic. See the following diagram.

$$\begin{array}{ccc} \hat{\mathbb{C}} & \stackrel{w^{\mu_{n+1}}}{\longrightarrow} & \hat{\mathbb{C}} \\ \downarrow f & & \downarrow E_n \\ \hat{\mathbb{C}} & \stackrel{w^{\mu_n}}{\longrightarrow} & \hat{\mathbb{C}}. \end{array}$$

For the given closed curve δ in $\mathbb{C} \setminus P_f$ whose winding number about 0 is η, set

$$\delta_n = w^{\mu_n}(\delta).$$

Note that since w^{μ_n} is a homeomorphism δ_n is a closed curve in $\mathbb{C} \setminus w^{\mu_n}(P_f)$; because w^{μ_n} fixes $0, 1, \infty$, these points belong to $w^{\mu_n}(P_f)$ and δ_n has a winding number η_n about 0. Now the winding number is a topological invariant so $\eta_n = \eta$ for all n. $\square$

We apply this lemma first to the functions $f \in \mathcal{T}E_{p,1}$, $p \geq 0$ with finite P_f and no periodic critical point. Using the notation of the previous two sections, if c is the non-zero critical point of f, set $c_k = f^k(c)$, $k \geq 0$, and let $c_{k,n} = w^{\mu_n}(c_k)$. Because c is preperiodic, there are minimal integers $k_1 \geq 1$ and k_2 such that

$$\{c_{k_1+1}, \ldots, c_{k_2}\}$$

is a periodic orbit.

Let γ be a continuous curve from c_{k_1} to c_{k_2} in $(\mathbb{R}^2 \setminus P_f) \cup \{c_{k_1}, c_{k_2}\}$. Then $\delta = f(\gamma) \in (\mathbb{R}^2 \setminus P_f) \cup \{c_{k_1+1}\}$. The curve

$$\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$$

is continuous and goes from $c_{k_1,n+1}$ to $c_{k_2,n+1}$ avoiding any other points in $w^{\mu_{n+1}}(P_f)$. Then

$$\delta_n = E_n(\gamma_{n+1}) = w^{\mu_n}(f((w^{\mu_{n+1}})^{-1}(\gamma_{n+1}))) = w^{\mu_n}(f(\gamma)) = w^{\mu_n}(\delta)$$

is a closed curve through the point $c_{k_1+1,n} = w^{\mu_n}(c_{k_1+1})$ that avoids all other points in $w^{\mu_n}(P_f)$. By the lemma, it has winding number η around 0. We can interpret η as the number of fundamental domains or "distance" between $c_{k_1,n+1}$ and $c_{k_2,n+1}$.

We also apply this lemma to functions $f \in \mathcal{AV}2$ with asymptotic values $\Omega_f = \{0, \lambda\}$, with $4 \leq \#P_f < \infty$ and normalized so that $f(0) = 1$. The application is a bit different from the one above.

The fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda\}) = \mathbb{Z}$. If we draw a curve α on $\hat{\mathbb{C}}$ from 0 to λ, then $f^{-1}(\hat{\mathbb{C}} \setminus \alpha)$ is topologically equivalent to the union of strips $\hat{\mathbb{C}} \setminus \{\cup_n \Im z = 2\pi n\}$. If we take a closed curve $\beta \in \hat{\mathbb{C}} \setminus \{0, \lambda\}$ separating 0 and λ, its winding number counts the number of strips it crosses. Since P_f is finite the orbits $\{c_k = f^k(0)\}_{k=0}^{\infty}$ and $\{c'_k = f^k(\lambda)\}_{k=0}^{\infty}$ are both finite, and thus, preperiodic. Note that neither can be periodic because the asymptotic values are omitted. Because 0 is preperiodic, there are minimal integers $k_1 \geq 1$ and k_2 such that

$$\{c_{k_1+1}, \ldots, c_{k_2}\}$$

is a periodic orbit.

Let γ be a continuous curve connecting c_{k_1} to c_{k_2} in $(\mathbb{R}^2 \setminus P_f) \cup \{c_{k_1}, c_{k_2}\}$. Because $f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1}$, the image curve $\delta = f(\gamma)$ is a closed curve in $(\mathbb{R}^2 \setminus P_f) \cup \{c_{k_1+1}\}$. We can choose γ once and for all such that δ separates 0 and λ; that is, so that δ is a non-trivial curve closed curve in $\mathbb{R}^2 \setminus P_f$ except at its endpoints. Let η be its winding number about 0.

The fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda\}) = \mathbb{Z}$ so the homotopy class $\eta = [\delta]$ in the fundamental group is an integer which essentially counts the number of fundamental domains (strips) between c_{k_1} and c_{k_2} and defines a "distance" between the fundamental domains. The integer η depends only on the choice of γ and since γ is fixed, so is η.

The iteration defines the map

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{AV}2$$

which is holomorphic since it preserves the standard structure μ_0. The continuous curve

$$\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$$

is simple and goes from $c_{k_1,n+1} = w^{\mu_{n+1}}(c_{k_1})$ to $c_{k_2,n+1} = w^{\mu_{n+1}}(c_{k_2})$. The image curve

$$\delta_n = g_n(\gamma_{n+1}) = w^{\mu_n}(f((w^{\mu_{n+1}})^{-1}(\gamma_{n+1}))) = w^{\mu_n}(f(\gamma)) = w^{\mu_n}(\delta)$$

is a closed curve through the point $c_{k_1+1,n} = w^{\mu_n}(c_{k_1+1})$.

From our normalization, it follows that

$$g_n(z) = g_{\alpha_n,\beta_n}(z) = \frac{\alpha_n e^{\beta_n z}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n z} + \frac{1}{\alpha_n}}. \tag{11}$$

and 0 is an omitted value for g_n. Since $\lambda_n = w^{\mu_n}(\lambda)$, it is also omitted for g_n and

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_{f,n} = w^{\mu_n}(P_f). \tag{12}$$

Applying Lemma 12.1 with the curve δ, we see that the curves δ_n are have winding number equal to η, the winding number of δ.

Moreover, since the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \mathbb{Z}$, we deduce that $\eta = [\delta_n] \in \pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \mathbb{Z}$. Thus the homotopy class of δ_n in the space $\hat{\mathbb{C}} \setminus \{0, \lambda_n\}$ is the same throughout the iteration. We can interpret the integer representing the homotopy class as the number of fundamental domains between c_{k_1} and c_{k_2}.

Note that when $f \in \mathcal{AV}2$ with $\lambda = \infty$, it is also in $\mathcal{T}E_{0,1}$. So the homotopy class defined in this section is the same as the winding number defined above.

13. Compactness.

The arguments that the invariance under the Thurston iteration scheme of the winding number and the homotopy class together with the bounded geometry condition imply compactness are different in the proofs of Theorem 1.2 and Theorem 1.3. We present these arguments in the two subsections below. Recall that

$$P_{f,n} = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \ldots.$$

13.1. **The proof of Theorem 1.2.** For a map in $\mathcal{T}E_{p,1}$, $f(0) = 0$, 0 is a branch point of multiplicity $p-1$ and f has exactly one non-zero branch point c with $f(c) = 1$. All the functions in the Thurston iteration have the form

$$E_n(z) = \alpha_n z^p e^{\lambda_n z}, \quad \alpha_n = e^p \Big(-\frac{\lambda_n}{p} \Big)^p.$$

Note that $E_n(0) = 0$ and 0 is a critical point of multiplicity $p-1$. It is also the asymptotic value and hence it has no other finite pre-images. Moreover, $E_n(z)$ has exactly one non-zero simple critical point

$$c_n = -\frac{p}{\lambda_n} = w^{\mu_n}(c)$$

and α_n is defined by the normalization condition $E_n(c_n) = 1$. Recall that we are always assuming that $\#P_f \geq 4$ which implies $f(1) \neq 1$ and this in turn implies $E_n(1) \neq 1$.

If c is periodic, then $c \in P_f$. It follows that $c_n (\neq 0, \infty) \in P_{f,n}$ and thus its spherical distance from either $0, 1$ or ∞ is bounded below. That is, there are two constants $0 < \kappa < K < \infty$ such that

$$\kappa \leq |\lambda_n| \leq K, \quad \forall n > 0.$$

Because $\mathcal{E}_{p,1}$ is a one parameter family parameterized by λ, this implies that the sequence $\{E_n\}_{n=1}^{\infty}$ is contained in a compact subset.

Now suppose c is not periodic. By the the hypotheses in Theorem 1.2, $f(c) = 1$ is also not periodic and therefore $k_1 \geq 1$.

We have

$$0, \quad 1 = E_n(c_n), \quad E_n(1) = e^p \Big(-\frac{\lambda_n}{p} \Big)^p e^{\lambda_n} \in P_{f,n}.$$

Let $c_{k,n} = w^{\mu_n}(c_k)$. Then $c_{k,n} \in P_{f,n}$ for all $k \geq 1$. Let $\gamma_n = w^{\mu_n}(\gamma)$ and $\delta_n = w^{\mu_n}(\delta)$.

When f has bounded geometry, since $E_n(1) \neq 0$, its spherical distance from 0 is bounded below. This implies that the sequence $\{|\lambda_n|\}$ is bounded below; that is, there is a constant $\kappa > 0$ such that

$$\kappa \leq |\lambda_n|, \quad \forall n > 0.$$

By our hypothesis, $c_{k_1,n+1} \neq c_{k_2,n+1}$ both belong to $P_{f,n+1}$ and bounded geometry implies there are two constants, which we again denote by κ, K with $0 < \kappa < K < \infty$, such that

$$\kappa \leq |c_{k_2,n+1}|, |c_{k_1,n+1}| \leq K \ \text{ and } \ \kappa \leq |c_{k_2,n+1} - c_{k_1,n+1}| \leq K, \quad \forall\, n \geq 1.$$

Now we prove that the sequence $\{|\lambda_n|\}$ is also bounded above. Recall that when we chose γ, we assumed it did not go through 0 and thus by the normalization, none of the γ_{n+1} go through 0 either. Therefore, for each n we can find a simply connected domain $D_{n+1} \supset \gamma_{n+1}$ that does not contain 0. Now we compute

$$\eta = \frac{1}{2\pi i}\oint_{\delta_n} \frac{1}{w} dw = \frac{1}{2\pi i}\int_{\gamma_{n+1}} \frac{E_n'(z)}{E_n(z)} dz = \frac{1}{2\pi i}\int_{\gamma_{n+1}} \Big(\frac{p}{z} + \lambda_n\Big) dz$$

so that

$$2\pi i\eta = \int_{\gamma_{n+1}} \frac{p}{z} dz + \int_{\gamma_{n+1}} \lambda_n dz = \int_{\gamma_{n+1}} \frac{p}{z} dz + \lambda_n(c_{k_2,n+1} - c_{k_1,n+1}).$$

Rewriting we have

$$\lambda_n(c_{k_2,n+1} - c_{k_1,n+1}) = 2\pi i\eta - \int_{\gamma_{n+1}} \frac{p}{z} dz.$$

This implies that

$$\kappa|\lambda_n| \leq |\lambda_n(c_{k_2,n+1} - c_{k_1,n+1})| \leq 2\pi\eta + \Big|\int_{\gamma_{n+1}} \frac{p}{z} dz\Big|.$$

Thus, if we can bound the integral on the right we will be done.

Notice that $\log z$ can be defined as an analytic function on the simply connected domain D_{n+1} containing γ_{n+1} that we chose above. We take $\log z = \log|z| + i\arg(z)$ as the principal branch, with $-\pi \leq \arg(z) < \pi$. We then estimate

$$\Big|\int_{\gamma_{n+1}} \frac{p}{z} dz\Big| = p|\log c_{k_2,n+1} - \log c_{k_1,n+1}|$$

$$\leq p(|\log|c_{k_2,n+1}| - \log|c_{k_1,n+1}|| + |\arg(c_{k_2,n+1}) - \arg(c_{k_1,n+1})|).$$

We have

$$|\arg(c_{k_2,n+1}) - \arg(c_{k_1,n+1})| < 2\pi(\eta + 1)$$

because of the topological constraint. By Lemma 12.1, the number of fundamental domains between the points is bounded by the topological invariant η and this is independent of n. (See Figure 2.) Then, including the width of the domains containing the points $c_{k_1,n+1}$ and $c_{k_2,n+1}$, we have

$$\Big| \int_{\gamma_{n+1}} \frac{p}{z} dz \Big| \leq p((\log K - \log \kappa) + 2\pi(\eta + 1)).$$

Finally we have

$$|\lambda_n| \leq p(\frac{2\pi\eta + (\log K - \log \kappa) + 2\pi(\eta+1)}{\kappa}).$$

which proves that $\{E_n(z)\}_{n=0}^{\infty}$ is contained in a compact subset in $\mathcal{E}_{p,1}$. This combined with Theorem 1.1 completes the proof of Theorem 1.2 for $f \in \mathcal{T}E_{p,1}$.

13.2. **Proof of Theorem 1.3.** By hypothesis f has bounded geometry and by the normalization of f, $\Omega_f = \{0, \lambda\}$, $f(0) = 1$ so that $\{0, 1, \lambda, \infty\} \subset P_f$. Moreover the iterates

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$$

belong to $\mathcal{M}_2$.

Recall that $P_{f,n} = w^{\mu_n}(P_f)$ and because w^{μ_n} fixes $\{0, 1, \infty\}$ for all $n \geq 0$, $\{0, 1, \infty\} \subset P_{f,n}$. By equation (11),

$$g_n(1) = w^{\mu_n}(f(1)) = \frac{\alpha_n e^{\beta_n}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n} + \frac{1}{\alpha_n}} \in P_{f,n}.$$

so that

$$\lambda_n = \frac{\alpha_n}{(\alpha_n - \frac{1}{\alpha_n})}$$

and

$$\{0, 1, \lambda_n, g_n(1), \infty\} \subseteq P_{f,n}.$$

The $\mathcal{M}2$ here is parameterized by the asymptotic value λ and the coefficient β. The compactness condition says that the parameter λ_n stays a bounded distance from 0 and 1 and that β_n is bounded away from 0 and ∞.

In the case that $\lambda = \infty$, f is in $\mathcal{T}E_{0,1}$ so that all the functions in the Thurston iteration have the form $g_n(z) = e^{\beta_n z}$. From our normalization, we have

$$0,\ 1 = g_n(0),\ g_n(1) = e^{\beta_n} \in P_{f,n+1}.$$

As usual, we assume $\#(P_f) \geq 4$ because if $\#(P_f) = 3$, then $f(1) = 1$ so that $g_n(1) = 1$ for all $n \geq 0$ and therefore $g_n(z) = e^{2\pi m_n z}$. The homotopy class of δ_n is determined by η, the winding number about the origin in the complex analytic sense. Thus $m_n = \eta$ for all n and $g_n = e^{2\pi i \eta z}$, which is fixed under Thurston iteration so trivially lies in a compact subset in $\mathcal{E}_{0,1} \subset \mathcal{M}2$.

Now still assuming that $\lambda = \infty$, suppose that $\#(P_f) \geq 4$ so that $g_n(1) \neq 1$. When f has bounded geometry, the spherical distance between 1 and $g_n(1)$ is bounded away from zero. That is, there is a constant $\kappa > 0$ such that

$$\kappa \leq |\beta_n|, \ \ \forall n \geq 0.$$

Now we prove that the sequence $\{|\beta_n|\}$ is also bounded above. Again we compute

$$\eta = \frac{1}{2\pi i}\oint_{\delta_n}\frac{1}{w}dw = \frac{1}{2\pi i}\int_{\gamma_{n+1}}\frac{g_n'(z)}{g_n(z)}dz = \frac{1}{2\pi i}\int_{\gamma_{n+1}}\beta_n dz.$$

The integral therefore depends only on the endpoints and we have

$$\eta = \frac{1}{2\pi i}\int_{\gamma_{n+1}}\beta_n dz = \frac{\beta_n}{2\pi i}(c_{k_2,n+1} - c_{k_1,n+1}).$$

Since 0 is omitted, it cannot be periodic. Therefore, both $c_{k_2,n+1} \neq c_{k_1,n+1} \in P_{f,n+1}$; by bounded geometry therefore, there is a positive constant which we again denote by κ such that

$$|c_{k_2,n+1} - c_{k_1,n+1}| \geq \kappa.$$

This gives us the estimate

$$|\beta_n| \leq \frac{2\pi\eta}{|c_{k_2,n+1} - c_{k_1,n+1}|} \leq \frac{2\pi\eta}{\kappa}$$

which proves that $\{g_n(z)\}_{n=0}^{\infty}$ is contained in a compact family in $\mathcal{E}_{0,1} \subset \mathcal{M}2$.

Finally, let us prove compactness of the iterates when $\lambda \neq \infty$. In this case, since

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_{f,n+1}$$

has a definite spherical distance from 0, 1 and ∞, bounded geometry implies there are two constants $0 < k < K < \infty$ such that

$$k \leq |\alpha_n|, \ \ |\alpha_n - 1| \text{ and } \ |\alpha_n| \leq K, \quad \forall\, n \geq 0.$$

In this case, we also know that $g_n(1) \neq 1$. Since $g_n(1) \in P_{f,n+1}$, bounded geometry implies that the constant k can be chosen such that

$$k \leq |\beta_n|, \ \ \forall n > 0.$$

Again we use the topological constraint of Lemma 12.3 to prove that $\{|\beta_n|\}$ is also bounded from above. Let

$$M_n(z) = \frac{\alpha_n z}{(\alpha_n - \frac{1}{\alpha_n})z + \frac{1}{\alpha_n}}$$

so that $g_n(z) = M_n(e^{\beta_n z})$. The map M_n is a homeomorphism that fixes 0. Thus, setting $\tilde{\delta}_n = M_n^{-1}(\delta_n)$, the winding number about 0 of $\tilde{\delta}_n$ is equal to the the winding number of δ_n about 0. By Lemma 12.3 this is an integer η, independent of n.

Remark 13.1. *We can consider M_n as a map from $\hat{\mathbb{C}} \setminus \{0,\infty\} \to \hat{\mathbb{C}} \setminus \{0,\lambda_n\}$ so so it induces an isomorphism from the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0,\infty\})$ to the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0,\lambda_n\})$. Then, because these fundamental groups are isomorphic to the integers, the homotopy classes of δ_n and $\tilde{\delta}_n$ are represented by the same integer.*

Note that $\tilde{\delta}_n$ is the image of γ_{n+1} under $\widetilde{g}_n(z) = e^{\beta_n z}$. Since $\widetilde{\delta_n}$ is a closed curve in $\hat{\mathbb{C}} \setminus \{0, \infty\}$, η is the winding number of $\tilde{\delta}_n$ about the origin in the complex analytic sense, and we can compute

$$\eta = \frac{1}{2\pi i} \oint_{\widetilde{\delta_n}} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{\tilde{g}_n'(z)}{\widetilde{g}_n(z)} dz = \frac{\beta_n}{2\pi i} (c_{k_2,n+1} - c_{k_1,n+1}).$$

As above, $c_{k_2,n+1}, c_{k_1,n+1} \in P_{f,n+1}$, and by bounded geometry there is a constant $k > 0$ such that

$$|c_{k_2,n+1} - c_{k_1,n+1}| \geq k,$$

so that

$$|\beta_n| \leq \frac{2\pi\eta}{|c_{k_2,n+1} - c_{k_1,n+1}|} \leq \frac{2\pi\eta}{k}.$$

This inequality proves that $\{g_n(z) = g_{\alpha_n,\beta_n}(z)\}$ forms a compact subset in $\mathcal{M}2$.

Thus, we have shown that in all cases bounded geometry implies the sequence $\{g_n\}$ is in a compact subset in $\mathcal{M}2$. This combined with Theorem 1.1 completes the proof of Theorem 1.3.

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