SLICES OF PARAMETER SPACE FOR MEROMORPHIC MAPS WITH TWO ASYMPTOTIC VALUES

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Abstract. This paper is part of a program to understand the parameter spaces of dynamical systems generated by meromorphic functions with finitely many singular values. We give a full description of the parameter space for a specific family based on the exponential function that has precisely two finite asymptotic values and one attracting fixed point. It represents a step beyond the previous work in [GK] on degree 2 rational functions with analogous constraints: two critical values and an attracting fixed point. What is interesting and promising for pushing the general program even further, is that, despite the presence of the essential singularity, our new functions exhibit a dynamic structure as similar as one could hope to the rational case, and that the philosophy of the techniques used in the rational case could be adapted.

1. Introduction

A general principle in complex dynamics is that singular values control the dynamical behavior. There is now a long history of isolating interesting families of functions whose singular values can be parameterized in a way that allows one to understand how the dynamics varies across the family. In practice, one constrains the number of singular values and the behavior of one or more of them–for example, by demanding that the orbit of one tend to an attracting fixed point.

This paper is a step along the way to a general theory for meromorphic functions with finitely many singular values. We adapt a technique developed by Douady, Hubbard and their students to study spaces of cubic polynomials, and used in [GK] for rational maps of degree 2, in which the parameter space is modeled on the dynamic space of a fixed map in the family. We will be looking at a family of meromorphic functions that are close enough to rational maps of degree 2 that there should be (and is) a direct similarity between the behaviors. To put this in context, it helps to review some of the history.

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The study of the parameter space for families of complex dynamical systems began with the family of quadratic polynomials. They have one critical value whose behavior determines the dynamics and it is this behavior that is captured by the Mandelbrot set and its complement. The next step was to study families with two free critical values — cubic polynomials and rational maps of degree two. Moving out of the realm of rational functions and into that of dynamics of transcendental functions, we see more substantial differences between entire and meromorphic functions than between polynomials and rationals. Rational maps define finite coverings of the plane, but transcendental maps define infinite coverings. Moreover, while the poles of rational maps are no different from regular points, the poles of meromorphic functions add a new flavor to the dynamics. It turns out that there are more similarities between the parameter space of rational maps of degree 2 and that of the tangent family $\lambda \tan z$ than between quadratic polynomials and the exponential family. See e.g. [DFJ, DK, FG, KK, RG, Sch]. Is this similarity just good fortune, or is it suggestive of a more general pattern of relationships with rational maps?

Thanks to invariance under Möbius transformations, in order to study rational maps of degree 2, we can restrict our attention to maps of the form $(z + b + 1/z)/\rho$ where $b \in \mathbb{C}$ and $\rho \in \mathbb{C}^*$. This family is often called Rat_2 in the literature. These functions fix infinity where the derivative (multiplier) is $\rho \neq 0$ and have *two free* critical values, $(b \pm 2)/\rho$, rather than one as in the quadratic polynomial case. Constraining ρ to lie in the punctured unit disk, $\mathbb{D}^*$, makes infinity an attracting fixed point for all values of b. In the paper [GK], a structure theorem is proved for this family that is as close as one could hope to the earlier examples:

Theorem (Structure Theorem for Rat_2)**.** *Fix $\rho \in \mathbb{D}^*$, and consider the family $(z + b + 1/z)/\rho$ where $b \in \mathbb{C}$. The b plane is divided into three components by a bifurcation locus: two copies of the Mandelbrot set that meet at the origin and are symmetric about it, and a "shift locus". For b in either copy of the Mandelbrot set, one or the other critical value is attracted to infinity and the other is not. In the shift locus, both critical values are attracted to infinity.*

This paper looks at the family of meromorphic functions whose members "look like degree 2 rationals": they have two finite omitted asymptotic values λ and μ and an attracting fixed point (in this case, at the origin) with multiplier ρ:

$$f_{\lambda,\rho}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}} \text{ where } \frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho},\ \rho \in \mathbb{D}^*. \tag{1}$$

We use $\mathcal{F}_2$ to denote this family. Our main result is a structure theorem for the slice of the parameter space defined by a fixed $\rho \in \mathbb{D}^*$, and those λ's for which that ρ has a corresponding μ, namely λ not equal to 0 or $\rho/2$. It is a direct analogue of the Rat_2 theorem:

Theorem (Main Structure Theorem)**.** *For each $\rho \in \mathbb{D}^*$, the parameter slice, $\lambda \in \mathbb{C} \setminus \{0, \rho/2\}$ divides into three distinct regions: two copies of connected and full sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ in which only one of the asymptotic values μ or λ is attracted to the origin and a "shift locus" $\mathcal{S}$ in which both asymptotic values are attracted to the origin. The shift locus $\mathcal{S}$ is conformally equivalent to a punctured annulus. The puncture is at the origin. The other puncture of the parameter plane, $\rho/2$, is on the boundary of the shift locus.*

We are able to give a description of the sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$. Like the Mandelbrot set in Rat_2, $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ contain hyperbolic components in which one or the other of the asymptotic values tends to a non-zero attracting cycle. Within each component the functions are quasiconformally conjugate. Components like this were first studied in [KK] where they occur in the parameter plane of the tangent family $\lambda \tan z$. There, and in other later work, (see [KK, FK, CK]), it was proved that each component is a universal cover of $\mathbb{D}^*$; based on the computer pictures, these components were called *shell components*. Thus, unlike the Mandelbrot set, the hyperbolic components do not contain a "center" where the periodic cycle contains the critical value and has multiplier zero. Instead, they contain a distinguished boundary point with the property that as the parameter approaches this point, the limit of the multiplier of the periodic cycle attracting the asymptotic value is zero. It is thus called a "virtual center".

Like the characterization of centers of the components of the Mandelbrot set in terms of the sequence of inverse branches that keep the critical value fixed, a virtual center λ^* can also be characterized by the property that there is some n such that $f_{\lambda^*}^{n-1}(\lambda^*) = \infty$ or $f_{\lambda^*}^{n-1}(\mu(\lambda^*, \rho)) = \infty$; the point is thus also called a "virtual cycle parameter of order n". In this paper we give a complete combinatorial description of the virtual cycle parameters:

Theorem (Combinatorial Structure Theorem)**.** *The virtual cycle parameters $\lambda_{\mathbf{k}_n}$ of order n can all be labelled by sequences $\mathbf{k}_n = k_{n-1} \dots k_1$, where $k_i \in \mathbb{Z}$, in such a way that each of the parameters $\lambda_{\mathbf{k}_n}$ is an accumulation point in $\mathbb{C}$ of a sequence of parameters $\lambda_{\mathbf{k}_{n+1}}$ of order $n+1$ and related to $\mathbf{k}_n$; that is, $\mathbf{k}_{n+1} = k_{n-1} \dots k_1 k_{0,j}$, $j \in \mathbb{Z}$. This combinatorial description of the virtual cycle parameters determines combinatorial descriptions of the sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$.*

In [CJK], we proved a "transversality theorem" for functions in the tangent family. Combining the techniques in the proof of this theorem with the results here, we prove

Theorem (Common Boundary Theorem)**.** *Every virtual cycle parameter is both a boundary point of a shell component and a boundary point of the shift* *locus. Furthermore, the dynamics of the family* $\{f_\lambda\}$ *is transversal at each of these virtual cycle parameters (see Definition 5 and Remark 5.1). And even further, the set of all virtual cycle parameters is dense in the common boundary of the shift locus* $\mathcal{S}$ *and the sets* $\mathcal{M}_\lambda \cup \mathcal{M}_\mu$.

The paper is broken down into two parts. Part 1 provides the background we need and some of the basic facts about the dynamical systems for functions in $\mathcal{F}_2$. Part 2 contains the main results of the paper.

We begin by quickly reviewing the basic definitions and facts we need about the dynamics of meromorphic functions and a theorem of Nevanlinna's, theorem 2.1, that characterizes the functions we work with in terms of their Schwarzian derivatives. We next take a detailed look at $\mathcal{F}_2$. In particular, in 3.3 we show that there is a dichotomy in the dynamics in this family analogous to that for quadratic polynomials: either the Julia set is a Cantor set or there is a connected "filled Julia set" analogous to the filled Julia set of a quadratic polynomial.

Part 2 begins with the description of the sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ from the Main Structure Theorem and gives the definitions of virtual cycle parameters and virtual centers. The combinatorial description of the virtual cycle parameters, the Combinatorial Structure Theorem, is given in section 4.2. Pictures of the parameter plane follow and the rest of the paper contains the the proof of the Common Boundary Theorem, which leads to a detailed discussion of the shift locus and the rest of Main Structure Theorem.

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Part 1. **Background**

2. Basic Dynamics

Here we give the basic definitions, concepts and notations we will use. We refer the reader to standard sources on meromorphic dynamics for details and proofs. See e.g. [Berg, BF, BKL1, BKL2, BKL3, BKL4, DK, KK].

We denote the complex plane by $\mathbb{C}$, the Riemann sphere by $\widehat{\mathbb{C}}$ and the unit disk by $\mathbb{D}$. We denote the punctured plane by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the punctured disk by $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$.

Given a family of meromorphic functions, $\{f_\lambda(z)\}$, we look at the orbits of points formed by iterating the function $f(z) = f_\lambda(z)$. If $f^k(z) = \infty$ for some $k > 0$, z is called a pre-pole of order k — a pole is a pre-pole of order 1. For meromorphic functions, the poles and pre-poles have finite orbits that end at infinity. The *Fatou set or Stable set,* F_f, consists of those points at which the iterates $\{f_\lambda^n\}_{n=0}^\infty$ are well-defined and form a normal family in a neighborhood of each of them. The Julia set J_f is the complement of the Fatou set and contains all the poles and pre-poles.

A point z such that $f^n(z) = z$ is called *periodic*. The minimal such $n > 0$ is called the period. Periodic points are classified by their multipliers, $\nu(z) = (f^n)'(z)$ where n is the period: they are repelling if $|\nu(z)| > 1$, attracting if $0 < |\nu(z)| < 1$, super-attracting if $\nu = 0$ and neutral otherwise. A neutral periodic point is *parabolic* if $\nu(z) = e^{2\pi i p/q}$ for some rational p/q. The Julia set is the closure of the repelling periodic points. For meromorphic f, it is also the closure of the pre-poles, (see e.g. [BKL1]).

If D is a component of the Fatou set, either $f^n(D) \subseteq f^m(D)$ for some integers n, m or $f^n(D) \cap f^m(D) = \emptyset$ for all pairs of integers $m \neq n$. In the first case D is called *eventually periodic* and in the latter case it is called *wandering*. The periodic cycles of stable domains are classified as follows:

- Attracting or super attracting if the periodic cycle of domains contains an attracting or superattracting cycle in its interior.
- Parabolic if there is a parabolic periodic cycle on its boundary.
- Rotation if $f^n : D \to D$ is holomorphically conjugate to a rotation map. Rotation domains are either simply connected or topological annuli. These are called *Siegel disks and Herman rings* respectively.
- Essentially parabolic, or Baker, if there is a point $z_\infty \in \partial D$ such that $f^n(z_\infty)$ is not well defined and for every $z \in D$, $\lim_{k\to\infty} f^{nk}(z) = z_\infty$.

A point a is a *singular value* of f if f is not a regular covering map over a.

- a is a *critical value* if for some z, $f'(z) = 0$ and $f(z) = a$.
- a is an *asymptotic value* if there is a path $\gamma(t)$, called an *asymptotic path*, such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a$.
- The *set of singular values* S_f consists of the closure of the critical values and the asymptotic values. The *post-singular set is*

$$P_f = \overline{\cup_{a\in S_f} \cup_{n=0}^{\infty} f^n(a) \cup \{\infty\}}.$$

  For notational simplicity, if a pre-pole s of order p is a singular value, $\cup_{n=0}^{p} f^n(s)$ is a finite set with $f^p(s) = \infty$.

A map f is *hyperbolic* if $J_f \cap P_f = \emptyset$.

A standard result in dynamics is that each attracting, super-attracting, parabolic or Baker cycle of domains contains a singular value. Moreover, unless the cycle is superattracting, the orbit of the singular value is infinite and accumulates on the cycle, or the orbit of z_∞ associated with the Baker domain. The boundary of each rotation domain is contained in the post singular set. (See e.g. [Mil], chap 8-11 or [Berg], Sect.4.3.)

In this paper we confine our attention to the family $\mathbb{F}_2$ of meromorphic functions that have no critical values and exactly two finite simple asymptotic values. Because the asymptotic values are isolated, they are *logarithmic*; that is, if U is a neighborhood of an asymptotic value a containing no other singular values and $U^* = U \setminus \{a\}$, then $f^{-1}(U^*)$ contains an unbounded, simply connected component $\mathcal{A}_a$ called the *asymptotic tract* of a, such that $f : \mathcal{A}_a \to U^*$ is a universal covering map. The number of different asymptotic tracts for a is called its *multiplicity*. The asymptotic value a is called *simple* if it has only one asymptotic tract. Every map we consider in this paper has two simple asymptotic values and there are two distinct asymptotic tracts.

2.1. **Nevanlinna's theorem.** Recall that the Schwarzian derivative is defined by

$$S(f) = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2.$$

It satisfies the cocycle relation

$$S(f \circ g) = S(f)(g')^2 + S(g).$$

Since the Schwarzian derivative of a Möbius transformations is zero, solutions to the Schwarzian differential equation $S(f)(z) = P(z)$ are unique up to post-composition by a Möbius transformation. See [Hil] and [Nev1] for proofs.

Nevanlinna's theorem characterizes transcendental functions with finitely many singular values and finitely many critical values in terms of thier Schwarzian derivatives.

Theorem 2.1 (Nevanlinna,[Nev1], Chap XI, [Hil])**.** *Every meromorphic function g with $p < \infty$ asymptotic values and $q < \infty$ critical points has the property that its Schwarzian derivative is a rational function of degree $p + q - 2$. If $q = 0$, the Schwarzian derivative is a polynomial $P(z)$. In the opposite direction, for every polynomial function $P(z)$ of degree $p - 2$, the solution to the Schwarzian differential equation $S(g) = P(z)$ is a meromorphic function with exactly p asymptotic values and no critical points. The only essential singularity is at infinity.*

A summary of the proof is given in [DK1] where the behavior of the function in a neighborhood of infinity is described. There are p equally spaced asymptotic tracts separated by Julia directions along which the poles tend asymptotically to infinity. An immediate corollary of the theorem is

Corollary 2.2. *If f is a meromorphic functions with p finite simple asymptotic values and no critical values and h is a homeomorphism of the complex plane $\mathbb{C}$ such that $g = h^{-1} \circ f \circ h$ is holomorphic (meromorphic), then $S(g)$ is a polynomial of degree p.*

In [DK1], this corollary is used to prove that if f has polynomial Schwarzian derivative and all its asymptotic values are finite, then f cannot have a Baker domain.

Our focus in this paper is on parameter spaces of meromorphic functions with two finite simple asymptotic values and no critical values. By the above theorem, such functions are characterized by the property that they have a constant Schwarzian derivative .

It is easy to compute that $S(e^{2kz}) = -2k^2$ and therefore that the most general solution to the equation $S(f) = -2k^2$ is

$$f(z) = \frac{ae^{kz} + be^{-kz}}{ce^{kz} + de^{-kz}},\ ad - bc \neq 0, \tag{2}$$

and its asymptotic values are $\{a/c, b/d\}$. Note that both of them are omitted values. According to theorem 2.1, the converse is true too. Moreover, by corollary 2.2, the solution to is unique up to post composition by an affine map. Precomposition by an affine map multiplies the constant k by the scaling factor.

3. Functions with two asymptotic values

Functions of the form (2) have a single essential singularity at infinity and their dynamics are invariant under affine conjugation. If one of the asymptotic values is equal to infinity, c or $d = 0$ and the family is the well-studied exponential family. See e.g. [DFJ, RG]. The dynamics are quite different if both asymptotic values are finite and here we restrict ourselves to this situation.

Because we assume the asymptotic values are finite, neither c nor d can be zero. We choose a representative of an equivalence class, where the equivalence relation is defined by affine conjugation, such that $k = 1$ and $f(0) = 0$; this implies $b = -a$. If the asymptotic values are λ and μ we have

$$f_{\lambda,\mu}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}}$$

where $\lambda, \mu \in \mathbb{C}^*$. If $f'(0) = \rho \in \mathbb{C}^*$ we have the relation

$$\frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho}. \tag{3}$$

We denote this family by $\mathbb{F}_2$. We still have the freedom to conjugate by the affine map $z \to -z$ so we see that the maps $f_{\lambda,\mu}(z)$ and $f_{\lambda',\mu'}(z) = -f_{\lambda,\mu}(-z)$ have the same dynamics. That is,

$$f_{\lambda',\mu'}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda'} - \frac{e^{-z}}{\mu'}} = \frac{e^z - e^{-z}}{\frac{e^z}{-\mu} - \frac{e^{-z}}{-\lambda}},$$

where

$$\frac{1}{\lambda'} - \frac{1}{\mu'} = \frac{2}{\rho} = -\frac{1}{\mu} + \frac{1}{\lambda}. \tag{4}$$

Set $f_{\lambda,\mu} \sim f_{\lambda',\mu'}$ if $\lambda' = -\mu, \mu' = -\lambda$ and use this equivalence relation to define the space of pairs of functions:

$$\widehat{\mathbb{F}}_2 = \left\{ f_{\lambda,\rho}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}} \;\middle|\; \rho \in \mathbb{C}^*, \;\; \lambda \in \mathbb{C}^* \setminus \{\rho/2\}, \;\; \frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho} \right\} \Big/ \sim .$$

Note that each pair of complex numbers (λ, ρ) uniquely determines a pair of functions so that we also use $\widehat{\mathbb{F}}_2$ to denote the moduli space of $\mathbb{F}_2$.

Because of the ambiguity left by the normalization, it is difficult to study $\widehat{\mathbb{F}}_2$ directly. This situation is similar to the space Rat_2 of rational functions of degree 2 with a fixed point at infinity discussed in the introduction. The affine conjugation $z \to -z$ identifies maps with the same dynamics and sends the parameter b to $-b$. Thus the (b, ρ) space is a 2-fold covering map of the space of functions. To understand the role of the parameters, however, it is easier to work in this covering space. This can be done by *marking* the singular points and choosing a "preferred" point. In [GK], the preferred point was taken as $R(+1)$. The conjugation $z \to -z$ interchanged the marking and corresponded to the involution $b \to -b$ in the lifted parameter space of functions with marked critical values. See e.g. [M1, GK] for more details.

We proceed in the same way here. To mark the asymptotic values, we choose λ as the "preferred" value and μ as the "non-preferred" value. That is, $\lambda = \lim_{t\to\infty} f_{\lambda,\rho}(\gamma(t))$ where $\Re\gamma(t) \to +\infty$. We call the space with marked asymptotic values $\mathbb{F}_2$. Again the marked space is a 2-fold cover of the space of functions. Note that if $\lambda = \infty, \mu = -\rho/2$ and if $\mu = \infty, \lambda = \rho/2$.

Because the stable dynamics of functions in $\mathbb{F}_2$ are controlled by the behavior of the orbits of the asymptotic values, it will be convenient to choose a one dimensional "slice" in this covering space of $\mathbb{F}_2$ in such a way that at each point in the slice, the orbit of one asymptotic value has fixed dynamics. One way to do this is to require that both asymptotic values have similar behavior. For example, if $\mu = -\lambda$, so that $\lambda = \rho$, the slice obtained is the tangent family $f_\rho(z) = \rho \tanh z = i\rho \tan(iz)$. The properties of this slice have been investigated in [KK, CJK].

3.1. **The space $\mathcal{F}_2$.** In this paper, we begin with the 2 dimensional subfamily $\mathcal{F}_2 \subset \mathbb{F}_2$ where ρ is in the punctured unit disk $\mathbb{D}^*$. This means that the origin is an attracting fixed point so the orbit of at least one of the asymptotic values converges to zero. It may be either the preferred asymptotic value λ or not. We can parameterize this subspace as

$$\mathcal{F}_2 = \{f_{\lambda,\rho}\} = \big(\mathbb{C} \setminus \{0, \rho/2\}\big) \times \mathbb{D}^*$$

Each $\rho \in \mathbb{D}^*$ defines a one dimensional slice we denote by $\mathcal{F}_{2,\rho}$. This is a "dynamically natural slice" in the sense of [FK] because one asymptotic value is always attracted to the origin where the multiplier is fixed and the other is free. We choose the asymptotic value λ as parameter for our slice; either it, or $\mu(\lambda)$ (determined by equation (4)) is the free asymptotic value. Note that because of equation (4), when $\lambda = \rho/2$, $\mu = \infty$ and the function is in the exponential family, not our family. Also, if $\lambda = 0$, the function reduces to the constant 0. The points of the slice are denoted by λ, f_λ or $f_{\lambda,\rho}$ if we want to emphasize the dependence on ρ; if the context is clear, for readablity we use f. We will prove these slices all have the same structure.

Simple calculations show

$$f_{\lambda,\mu,\rho}(-z) = f_{\mu,\lambda,-\rho}(z) \text{ and } f_{\lambda,\mu,\rho}(-z) = f_{-\mu,-\lambda,\rho}(z)$$

so that interchanging the asymptotic values λ and μ changes multiplier from ρ to $-\rho$; interchanging and negating the asymptotic values changes the marking.

3.2. **Fatou components for $f_\lambda \in \mathcal{F}_2$.** For any $f_\lambda \in \mathcal{F}_2$, the origin is an attracting fixed point with multiplier ρ. Denote its attracting basin (which is non-empty) by A_λ.

Proposition 3.1. *The attracting basin A_λ is completely invariant.*

Proof. Since the origin is fixed, It is sufficient to prove that its immediate basin of attraction $I_\lambda \subset A_\lambda$ is backward invariant.

On a neighborhood $N \subset I_\lambda$ of the origin, we can define a uniformizing map $\phi_\lambda(z)$ such that $\phi_\lambda(0) = 0$, $\phi'_\lambda(0) = 1$ and $\phi_\lambda \circ f_\lambda = \rho\phi_\lambda$. It extends by analytic continuation to the whole immediate attractive basin I_λ. Denote by O_λ the largest neighborhood of the origin on which ϕ_λ is injective. One (or both) of the asymptotic values must be on the boundary of O_λ. Assume for argument's sake that $\mu \in \partial O_\lambda$. Choose a path γ joining 0 to μ in O_λ. If g is any inverse branch of f_λ, then $g(O_\lambda)$ contains a path joining $g^{-1}(0)$ to infinity that passes through the asymptotic tract $\mathcal{A}_\mu$ of μ. Thus all these paths are contained in the same component of $f_\lambda^{-1}(O_\lambda)$. Therefore this component contains all the pre-images of 0, and since one branch fixes 0, this component is I_λ. It follows that I_λ is backward invariant and $I_\lambda = A_\lambda$. □

We also have

Proposition 3.2. *Functions in $\mathcal{F}_2$ do not have Herman rings.*

Proof. For the tangent family, the structure of the asymptotic tracts at infinity is used in [KK] to prove that no map can have a Herman ring; the argument there used the symmetry of the asymptotic values. In our family, we again have two asymptotic values but they are no longer symmetric. Thus the argument there needs some modification.

Suppose an $f \in \mathcal{F}_2$ has a Herman ring R_0 with period $p \geq 1$ and for $i = 1, \cdots p-1$, let $R_i = f^i(R_0)$. Choose $z \in R_0$ and set $\gamma_0 = \overline{\{f^{pn}(z)\}_{n=0}^{\infty}}$. Then γ_0 is a topological circle. Its images, $\gamma_i = f^i(\gamma_0)$, $i = 1, \cdots p-1$ lie in

R_i. Let I_0 be the inner boundary of R_0; that is, it is contained in the bounded component of the complement of γ_0. It belongs to the Julia set.

Since the prepoles are dense in the Julia set, there is a prepole $s \in I_0$ such that $f^m(s) = \infty$. Then $f^m(R_0)$ is unbounded and $f^m(R_0) = R_i$ for some $i = 0, \dots, p-1$; thus if N_∞ is a neighborhood of infinity, $V = R_i \cap N_\infty$ is a non-empty subset of the Fatou set. The structure of solutions of $S_f = const$ in [Hil] shows that V must intersect an asymptotic tract of one of the asymptotic values. It follows that f is infinite to one from R_i to R_{i+1} and so cannot be part of a Herman ring cycle.

□

Proposition 3.1 implies the following trichotomy for $\mathcal{F}_2$:

- A_λ contains both asymptotic values: this is called the *shift locus* and denoted $\mathcal{S}$.
- A_λ contains only the preferred asymptotic value λ: in this case the other asymptotic value μ is not attracted to the origin and we call the set of such λ's $\mathcal{M}_\mu$. We denote the subset where μ is attracted to an attracting periodic cycle by $\mathcal{M}_\mu^0$.
- A_λ contains only the non-preferred asymptotic value μ: in this case the other asymptotic value λ is not attracted to the origin and we call the set of these λ's $\mathcal{M}_\lambda$. We denote the subset where λ is attracted to an attracting periodic cycle by $\mathcal{M}_\lambda^0$.

The maps in $\mathcal{S}$, $\mathcal{M}_\lambda^0$ and $\mathcal{M}_\mu^0$ are hyperbolic because the orbits of their asymptotic values accumulate on attracting cycles. The connected components of these three subsets of parameter space are thus called *hyperbolic components.*

As with the space Rat_2, there is an inversion of the space $\mathcal{F}_2$ that interchanges the regions $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ and leaves $\mathcal{S}$ invariant.

Let C_0 be the circle in the λ plane centered at the parameter singularity $\rho/2$ with radius $|\rho/2|$ and let D be the disk it bounds, punctured at the singularity $\rho/2$. The inversion

$$I(\lambda) = -\mu = \frac{\lambda}{2\lambda/\rho - 1}$$

leaves C_0 invariant and interchanges λ and $-\mu$.

Proposition 3.3. *If $f_\lambda^n(\lambda) \not\to 0$ as $n \to \infty$, then $f_{I(\lambda)}^n(I(\lambda)) \to 0$. That is, the inversion interchanges the regions $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ in the plane where only one of the asymptotic values goes to zero.*

Proof. Suppose $f_\lambda^n(\lambda) \not\to 0$ so that $f_\lambda^n(\mu) \to 0$. Since $I(\lambda) = -\mu$ and $I(\mu) = -\lambda$, we can write

$$f_{-\mu}(-\mu) = \frac{e^{-2\mu} - 1}{\frac{e^{-2\mu}}{-\mu} - \frac{1}{-\lambda}} = \frac{e^{2\mu} - 1}{\frac{e^{2\mu}}{\lambda} - \frac{1}{\mu}} = f_\lambda(\mu)$$

and

$$f_{-\mu}(-\lambda) = \frac{e^{-2\lambda} - 1}{\frac{e^{-2\lambda}}{-\mu} - \frac{1}{-\lambda}} = \frac{e^{2\lambda} - 1}{\frac{e^{2\lambda}}{\lambda} - \frac{1}{\mu}} = f_\lambda(\lambda).$$

□

It follows that the inversion also preserves the region $\mathcal{S}$ where both asymptotic values go to zero.

When ρ is real, we can say more.

Proposition 3.4. *Suppose ρ is real and $\lambda \in C_0$. Then both $f_\lambda^n(\lambda) \to 0$ and $f_\lambda^n(\mu) \to 0$.*

Proof. Set $\lambda_1 = I(\lambda)$. Then $-\mu = I(\lambda) = \bar{\lambda} = \lambda_1$. Thus,

$$\overline{f_\lambda(\lambda)} = f_{\bar{\lambda}}(\bar{\lambda}) = f_{\lambda_1}(\lambda_1);$$

so if $f_\lambda^n(\lambda) \to 0$, $f_{\lambda_1}^n(\lambda_1) \to 0$.

We can rewrite this as

$$f_\lambda(\mu) = f_\lambda(-\bar{\lambda}) = -\overline{f_\lambda(\lambda)}.$$

Therefore, either both asymptotic values iterate to zero or neither does. Since the origin is an attracting fixed point with multiplier ρ, at least one must and so they both do.

□

This proposition says when ρ is real, the region where both asymptotic values are attracted to zero contains the invariant circle of the inversion.

Notice that the point $\lambda = \rho$ is on the circle C_0. At that point we have $\mu = -\lambda$, $f_\lambda = \lambda \tanh z$ and $I(\lambda) = \lambda$ so that it is a branch point of the double covering defined by the marking. Moreover, because of the symmetry both asymptotic values are attracted to zero.

If $\lambda \in \mathcal{M}_\mu^0$ or $\mathcal{M}_\lambda^0$, f_λ has an attracting periodic cycle different from the origin. This cycle has an attractive basin which we denote by K_λ and $A_\lambda = \widehat{\mathbb{C}} \setminus \overline{K_\lambda}$. Thus ∂K_λ is the Julia set and $\overline{K_\lambda}$ is the "filled Julia set". Both of them are unbounded sets in $\mathbb{C}$.

3.3. The set $\overline{K_\lambda}$.

In [KK], it is proved that for $\rho \in \mathbb{D}^*$, the Julia set J_λ of the function $T_\rho(z) = \rho \tanh(z)$ is a Cantor set. Moreover, it is homeomorphic to a space consisting of finite and infinite sequences on an alphabet isomorphic to the natural numbers and infinity. The finite sequences end with infinity. The homeomorphism conjugates T_ρ to the shift map on this alphabet. See [DK, Mo] for details.

At this point in this paper we can prove:

Proposition 3.5. *If Ω is the hyperbolic component of the λ plane containing $f_{\lambda_0} = \rho \tanh z$ and $\lambda \in \Omega$, then the Julia set of f_λ is a Cantor set. If $\lambda \in \mathcal{M}_\lambda$ or $\mathcal{M}_\mu$ then $\overline{K_\lambda}$ is full.*

Remark 3.1. *In theorem 6.7 we will prove that the shift locus $\mathcal{S}$ is connected so that $\Omega = \mathcal{S}$. It will then follow that we have a dichotomy similar to that for quadratic polynomials.*

Proof. If $\lambda_0 = \rho$, by symmetry, both λ_0 and $\mu_0 = -\lambda_0$ are in A_λ. By the results in [KK], the Julia set of f_{λ_0} is a Cantor set. Suppose $\lambda \in \Omega$, and let $\lambda(t)$, with $\lambda(0) = \lambda_0$ and $\lambda(1) = \lambda$, be a path in Ω. By standard arguments using quasiconformal surgery, see e.g. [McMSul, BF] and sections 6.2.4 and 6.2.3, we can construct quasiconformal homeomorphisms $g(t)$ conjugating f_{λ_0} to $f_{\lambda(t)}$ that preserve the dynamics. Since the maps are hyperbolic, the Julia sets of $f_{\lambda(t)}$ are quasiconformally equivalent and thus also topologically equivalent.

Suppose now that $\lambda \in \mathcal{M}_\lambda$ so that λ is not in A_λ. The same argument works for $\lambda \in \mathcal{M}_\mu$, interchanging the roles of λ and μ. Take a generic small r, such that $\partial D_r(0)$ does not contain a point in the forward orbit of μ. Then by definition, $A_\lambda = \cup_{n\geq 1} f^{-n}(D_r(0))$ and $f^{-n}(D_r(0)) \subset f^{-(n+1)}(D_r(0))$. Note that because $\lambda \notin A_\lambda$, $f : f^{-(n+1)}(D_r(0)) \to f^{-(n)}(D_r(0)) \setminus \{\mu\}$ is a covering and so $f^{-1}(D_r(0))$ is simply connected. Therefore A_λ simply connected, which implies that it is complement $\overline{K_\lambda}$ is full.

$\square$

Note that the argument above adapts easily to show that if f_λ has a non-zero attracting or parabolic fixed point the attracting basin of this fixed point is unbounded and completely invariant. Other standard arguments, [Mil], show that if f_λ has a neutral fixed point with a Siegel multiplier, its boundary must be contained in the post singular set. Thus there are two completely invariant domains in the Fatou set separated by the Julia set. An example of this is shown in figure 1 where $\rho = 2/3$ and $\lambda = 2+2i$. The yellow is the basin of 0 and the blue is the basin of the fixed point $2.25818 + 2.12632i$. The proof of the Main Structure Theorem uses another example of a function with two attractive fixed points and its dynamic space is shown in figure 7.

Part 2. **Properties of the Hyperbolic Components of the λ-plane.**

4. Shell components: Properties of $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$

In this section, we work only with hyperbolic components in $\mathcal{M}_\lambda^0$. By propositions 3.3 and 3.4, the discussion for $\mathcal{M}_\mu^0$ is essentially the same. By definition, all the maps in $\mathcal{M}_\lambda^0$ are hyperbolic; $\mathcal{M}_\lambda^0$ consists of components in which standard arguments (see e.g. [BF]) show any two functions corresponding to parameters in the component are quasiconformally conjugate. Following [FK] we call these components *Shell Components*. In that paper, more general functions were considered and the properties of the shell components were described. Here we summarize what we need from that description. We begin with some definitions.

FIGURE 1. The dynamic plane of f_λ with $\rho = 2/3$ and $\lambda = 2+2i$. The fixed points are stars and the black dot is a pole.

4.1. **Virtual Cycle Parameters and Virtual Centers.** Let Ω be a hyperbolic component in $\mathcal{M}_\lambda$ and let $\lambda \in \Omega$. Both λ and μ are attracted by attracting cycles of f_λ, and since $\lambda \in \mathcal{M}_\lambda$, μ is attracted to the origin and λ is attracted to a different cycle of order $n \geq 1$. Since all the f_λ, $\lambda \in \Omega$ are quasiconformally conjugate, all the functions in Ω have non-zero attracting cycles of the same period, say n. We say Ω *has period* n and where appropriate, denote it by Ω_n.

We need the following definitions:

Definition 1. *If $\lambda \in \mathcal{F}_2$ and there exists an integer $n > 1$ such that either $f_\lambda^{n-1}(\lambda) = \infty$ or $f_\lambda^{n-1}(\mu) = \infty$, then λ is called a* virtual cycle parameter. *In the first case set $a_1 = \lambda$ and in the second case set $a_1 = \mu$. Next set $a_{i+1} = f_\lambda(a_i)$ where i is taken modulo n so that $a_0 = \infty$. We call the set $\mathbf{a} = \{a_1, a_2, \ldots, a_{n-1}, a_0\}$ a* virtual cycle*, or if we want to emphasize the period,* virtual cycle of period n.

This definition is justified by the following. Assume for argument's sake that we are in the first case. Let $\gamma(t)$ be an asymptotic path for $\lambda = a_1$, that is, $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = a_1$, and let g be the composition of branches of the inverse of f such that $g(\infty) = a_1$. Then

$$\lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} g(\gamma(t)) = \lambda = a_1,$$

so that in this limiting sense, the points form a cycle.

Definition 2. *Let Ω_n be a shell component of period n and let*

$$\mathbf{a}_\lambda = \{a_0, a_1, \ldots, a_{n-2}, a_{n-1}\}$$

be the attracting cycle of period n that attracts λ or μ. Suppose that as $k \to \infty$, $\lambda_k \to \lambda^* \in \partial\Omega_n$ and the multiplier $\nu_{\lambda_k} = \nu(\mathbf{a}_{\lambda_k}) = \Pi_{i=0}^{n-1} f'(a_i(\lambda_k)) \to 0$. Then λ^* is called a virtual center *of Ω_n.*

Remark 4.1. *Note that if $n = 1$ and $a_0(\lambda_k)$ is the fixed point, the definition implies that $f'(a_0(\lambda_k)) \to 0$. This in turn implies that either λ tends to ∞ or λ tends to the parameter singularity $\rho/2$ so that μ tends to ∞. These "would be" virtual centers do not belong to the parameter space but they share many properties with proper virtual centers including transversality (see definition 5).*

Since the attracting basin of the cycle $\mathbf{a}_\lambda$ must contain an asymptotic value, we will assume throughout the paper that the points in the cycle are labeled so that λ or μ and a_1 are in the same component of the immediate basin.

In the next theorem we collect the results in [FK] about shell components for fairly general families of functions. The proof of Parts [b] and [c] are based on an estimate of the growth of the orbits of the singular values given in lemma 2.2 of [RS], and on proposition 6.8 of [FK]. Part [d] combines theorem 6.10 of [FK] and Corollary A of [CK].

Theorem 4.1 (Properties of Shell Components of $\mathcal{F}_2$)**.** *Let Ω be a shell component in $\mathcal{F}_2$. Then*

(a) *The map $\nu_\lambda : \Omega \to \mathbb{D}^*$ is a universal covering map. It extends continuously to $\partial\Omega$ and $\partial\Omega$ is piecewise analytic; Ω is simply connected and ν_λ is infinite to one.*

(b) *There is a unique virtual center on $\partial\Omega$. If the period of the component is 1 and Ω is a shell component of $\mathcal{M}_\lambda$, the component is unbounded and the virtual center is at infinity; if, however, Ω is a shell component of $\mathcal{M}_\mu$ of period 1, then it is bounded and the virtual center is at the finite point $\rho/2$ which is a parameter singularity. This is the only difference between $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$.*

(c) *If $\lambda_k \in \Omega$ of period greater than 1 is a sequence tending to the virtual center λ^* and $a_0(\lambda_k)$ is the periodic point of the cycle $\mathbf{a}(\lambda_k)$ in the component containing the asymptotic tract and $a_1(\lambda_k) = f_{\lambda_k}(a_0(\lambda_k))$, then as $k \to \infty$, $a_0(\lambda_k) \to \infty$ and $a_1(\lambda_k) \to \lambda^*$.*

(d) *Every virtual center of a shell component is a virtual cycle parameter and every virtual cycle parameter is a virtual center.*

As a corollary we have

Corollary 4.2. *If λ^* is a virtual center of a shell component of period n, there are virtual centers of period $n + 1$ accumulating on it.*

Proof. The poles of f_λ are given by

$$p_k(\lambda) = \frac{1}{2}\mathrm{Log}(\frac{\rho - 2\lambda}{\rho}) + ik\pi, \tag{5}$$

where Log is the branch of the logarithm with imaginary part in $[-\pi, \pi)$. They and all their preimages are holomorphic functions of λ.

Let V be a neighborhood of λ^* in the parameter plane that does not contain any poles of f_λ^k for all $1 \le k < n-1$. Such a neighborhood exists because the poles of f_λ^k form a discrete set. The holomorphic function $h(\lambda) = f_\lambda^{n-1}(\lambda)$ maps V to a neighborhood W of infinity and $h(\lambda^*) = \infty$. Since infinity is an essential singularity, for $\lambda \in V$ and large enough $|k|$, W contains infinitely many poles $p_k(\lambda)$ of the functions $f_\lambda(z)$; moreover, for each λ, as $k \to \infty$, $p_k(\lambda)$ converge to infinity. The zeroes of the functions $h_k(\lambda) = h(\lambda) - p_k(\lambda)$ are virtual centers of components of period $n+1$. We want to show there is a sequence of these zeroes in V converging to λ^*.

The functions $\widehat{h}(\lambda) = 1/h(\lambda)$ and $\widehat{p}_k(\lambda) = 1/p_k(\lambda)$ take values in a neighborhood of the origin. Since the $p_k(\lambda)$ converge to infinity as $|k| \to \infty$ uniformly on $\overline{V}$ as long as V is small enough, we can find N large enough so that if $k > N$ and $\lambda \in \partial V$, then $|\widehat{p}_k(\lambda)| < |\widehat{h}(\lambda)|$. By Rouché's theorem, we conclude $\widehat{h}$ and $\widehat{h} - \widehat{p}_k$ have the same number of zeroes in V; $\widehat{h}$ has a zero at λ^* and thus each $\widehat{h} - \widehat{p}_k$ has a zero $\lambda_k \in V$; it follows that $h(\lambda_k) = p_k(\lambda_k)$ so that λ_k is a virtual center of period $n+1$. □

4.2. **Combinatorics.** Theorem 4.1 allows us to assign a label to each of the shell components of $\mathcal{M}_\lambda$ in terms of its virtual center. To label the virtual centers we need to know that the indices of the poles are well defined. In section 6.4 we will prove lemma 6.11 that says that we can find a simply connected domain Σ, containing $\mathcal{M}_\lambda$ and not containing $\mathcal{M}_\mu$, in which, after an initial choice, as above, of a basepoint and a branch of the logarithm, the poles and inverse branches of f_λ can be labelled consistently. The discussion here will assume that lemma.

Pick a basepoint that is not in $\mathcal{M}_\lambda$, for example, the symmetric point $\lambda_0 = -\mu_0 = \rho$. It has poles $p_k(\lambda_0)$ defined by the principal branch of the logarithm. With the poles $p_k(\lambda_0)$ defined by equation (5), denote the branch of $f_{\lambda_0}^{-1}$ that maps ∞ to $p_k(\lambda_0)$ by $g_{\lambda,k}(z) = g_{\lambda_0,k}(z)$. With the choice of a fixed base point and logarithm branch, the inverse branches are well defined since the set Σ (to be defined in section 6.4) is simply connected.

We use these branches to define labels for the prepoles of all orders, and thus for labels of the virtual cycle parameters. Because of part [d] of theorem 4.1 each virtual cycle parameter is a virtual center of a shell component so the label of the virtual center defines a label for the shell component.

The formula for $p_k(\lambda)$ shows that the poles are injective functions of λ in Σ. Let

$$\mathcal{V}_1 = \{\lambda_k^* \in \Sigma \,|\, g_{\lambda^*,k}(\infty) = \lambda^*\}.$$

That is, $\mathcal{V}_1$ is the set of λ_k^* such that $f_{\lambda_k^*}(\lambda_k^*) = \infty$. It is the the set of virtual cycle parameters of order 1 and hence virtual centers of shell components Ω_2 of period 2. We assign the label k to each point in $\mathcal{V}_1$ and the same label to the component for which it is the virtual center.

The prepoles $p_{k_1k_2}(\lambda) = g_{\lambda,k_2}(p_{k_1}(\lambda))$ are defined for all $\lambda \in \Sigma \setminus \mathcal{V}_1$. Since they are holomorphic functions of λ with non-vanishing derivative, each $g_{\lambda,k}$ is an injective function of both λ and z. Next, we inductively define the sets of virtual cycle parameters of order $n-1$ with labelled points by

$$\mathcal{V}_{n-1} = \{\lambda^*_{k_{n-1}\ldots k_1} \in \Sigma \setminus \cup_{i=1}^{n-2}\mathcal{V}_i \,|\, g_{\lambda^*_{k_{n-1}\ldots k_1}, k_{n-1}\ldots k_1}(\infty) = \lambda^*_{k_{n-1}\ldots k_1}\}.$$

The prepoles $p_{k_{n-1}\ldots k_1}(\lambda)$ are defined for all $\lambda \notin \mathcal{V}_{n-1}$ and, as above, move injectively.

We now assign the label $k_{n-1}\ldots k_1$ to the shell component of order n for which $\lambda^*_{k_{n-1}\ldots k_1}$ is the virtual center.

Definition 3. *We call the label $\mathbf{k}_n = k_n k_{n-1}\ldots k_1$ assigned to each prepole and each virtual cycle parameter its* itinerary*.*

We can also use the labelling of the inverse branches to assign an itinerary to each attractive cycle.

Definition 4. *For simplicity we suppress the dependence on λ and assume the shell component is in $\mathcal{M}_\lambda$. Suppose $f^n(a_0) = a_0$ for $n \geq 1$, where, by our numbering convention in part [c] of theorem 4.1, a_0 is in the asymptotic tract of λ and $a_j = f(a_{j-1})$, $j = 1,\ldots n$. Then for some k_j, $a_{j-1} = g_{k_j}(a_j)$. In fact there is a unique sequence $\{k_1,\ldots,k_n\}$ such that*

$$a_0 = g_{k_n} \circ \ldots g_{k_2} \circ g_{k_1}(a_0).$$

We say the cycle $\mathbf{a}$ has itinerary *$\mathbf{k}_n = k_n k_{n-1}\ldots k_2 k_1$.*

Proposition 4.3. *Let Ω_n be a shell component and suppose for $\lambda_0 \in \Omega_n$, the cycle $\mathbf{a}(\lambda_0)$ has itinerary $k_n k_{n-1}\ldots k_1$. Then for every $\lambda \in \Omega_n$, the itinerary of $\mathbf{a}(\lambda)$ is of the form $k_{0,j}k_{n-1}\ldots k_1$ for some $j \in \mathbb{Z}$.*

Proof. If the component of the basin $\mathbf{a}(\lambda)$ containing $a_j(\lambda)$ is denoted by $D_j(\lambda)$, then for $j = 1,\ldots n-1$, $f_\lambda : D_j(\lambda) \to D_{j+1}(\lambda)$ is one to one. Inside Ω_n, the points of the periodic cycle move holomorphically and are related by the inverse branches $g_{\lambda,k_j} : D_{j+1}(\lambda) \to D_j(\lambda)$. The branch is the same for all $\lambda \in \Omega_n$ since it is simply connected; in it the g_{λ,k_j} are quasiconformally conjugate and the $a_j(\lambda)$ move holomorphically. At the last step in the cycle, however, the map $f_\lambda : D_0(\lambda) \to D_1(\lambda)$ is infinite to one and so $a_1(\lambda)$ has infinitely many inverses, $a_{0,j}(\lambda) \in D_0(\lambda)$. They are all in the asymptotic tract of λ but only one of them can belong to the cycle. Thus although the inverse branch $g_j = g_{0,j}$ is well defined for each λ, the branch that defines the cycle changes as λ moves in Ω_n. □

Above we assigned a label, or itinerary to the virtual center of each shell component. We now address the questions of the uniqueness of these labels and their relation to the itineraries of their attracting cycles. As we stated above, this is based lemma 6.11, which will be proved later, that the inverse branches are defined as single valued functions of λ.

Proposition 4.4. *Every shell component $\Omega_n \in \mathcal{M}_\lambda$ and $\Omega'_n \in \mathcal{M}_\mu$, $n > 1$, has a unique label defined by the itinerary of its virtual center λ^*, a pre-pole of order $n-1$ where n is the minimal such integer.*

Because the shell components of period 1 have virtual centers that do not belong to the parameter space, we cannot label them in this way. There are only two such points, $\rho/2$ and ∞ and hence only two such components with no label. In order to have a label for every component, we arbitrarily assign the label ∞ to these components.

Proof. The boundary of each shell component Ω_n contains one and only one virtual center λ^* [1] and the label $\mathbf{k}_{n-1} = k_{n-1}k_{n-2}\dots k_1$ of the virtual center is its itinerary. Let V be a neighborhood of λ^* and let $W = \Omega_n \cap V$. By proposition 4.3, the itineraries of the points in W agree except for their first entry. By proposition 6.8 of [FK], as $\lambda \in W$ tends to the virtual center, the point $a_0(\lambda) = g_j(a_1(\lambda))$ of the cycle tends to infinity and the point $a_1(\lambda)$ tends to the virtual cycle parameter λ^*, a pre-pole of order $n-1$ with itinerary, $\mathbf{k}_{n-1} = k_{n-1}k_{n-2}\dots k_1$.

Note that because $\lambda \in \mathcal{M}_\lambda$ or $\mathcal{M}_\mu$, the cycle $\mathbf{a}(\lambda)$ attracts only one of the asymptotic asymptotic values. Therefore unlike the tangent family, where both asymptotic values can be attracted by a single cycle of double the period, $n-1$ is minimal. □

The proof of Corollary 4.2 also implies that

Proposition 4.5. *Let λ^* be the virtual center of a shell component Ω_n and let $\Omega_{n+1,i}$, be a sequence of components whose virtual centers λ^*_i converge to λ^* as i goes to infinity. If the itinerary of λ^* is given by $\mathbf{k}_{n-1} = k_{n-1}k_{n-2}\dots k_1$, the itineraries of the λ^*_i are given by $\mathbf{k}_{n,i} = k_{n-1}k_{n-2}\dots k_1k_{0,i}$.*

Remark 4.2. *There is an interesting duality here. As we approach the virtual center from inside a shell component of order n, we are taking a limit of cycle itineraries; the first entry in the itinerary (corresponding to the last inverse branch applied) disappears. Thus an itinerary with n entries becomes one with*

[1] It will follow from the Common Boundary Theorem that it is on the boundary of only one shell component. This is different from the tangent family where pairs of shell components share virtual centers. See e.g. [CJK]

$n-1$ entries. However, if we consider the labels of the shell components of order $n+1$ approaching the shell component of order n, it is the last entry (corresponding to the first inverse branch applied) that disappears in the limit.

Proof. As above, let V be a small neighborhood of λ^*. We may assume it contains no virtual center of order less than $n-1$. The functions g_{λ,k_j} that define the virtual cycle $a_j(\lambda^*)$, $j=1,\ldots n-1$ are defined in $V\cap\Omega_n$ where they track the attracting cycle. They also extend to all of $V\setminus\{\lambda^*\}$ by analytic continuation. Also for $\lambda\in V\cap\Omega_n$, the functions $a_{0,i}(\lambda)=g_{\lambda,k_{0,i}}(a_1(\lambda))$ are defined for all i but for only one i does it belong to the attracting cycle. All of these functions extend to $V\setminus\{\lambda^*\}$.

Now let W be a neighborhood of infinity and let $G(\lambda,z)$ be a map from $V\times W$ to $\mathbb{C}$ defined by $g_{\lambda,k_{n-1}}\circ\ldots\circ g_{\lambda,k_1}(z)$. By corollary 4.2 the neighborhood W contains the virtual centers λ_i^* of a sequence of shell components $\Omega_{n+1,i}$ with limit λ^*. These are poles p_i^* of $f_{\lambda_i^*}^{n-1}$ so we can find inverse branches of f_λ, which we denote by $g_{\lambda,k_{0,i}}$, such that $p_i^*=g_{\lambda_i^*,k_{0,i}}(\infty)$. It then follows that the itineraries of the λ_i^* are $\mathbf{k}_{n,i}=k_{n-1}k_{n-2}\ldots k_1k_{0,i}$ as claimed. ☐

Thus the combinatorics of the prepoles enable us to label each shell component $\Omega_n\in\mathcal{M}_\lambda$ and $\Omega'_n\in\mathcal{M}_\mu$ by the itinerary of its virtual center. If $\mathbf{k}_{n-1}=k_{n-1}k_{n-2}\ldots k_1$ is the itinerary of the virtual center of a shell component of period n, and we want to emphasize it, we write $\Omega_{\mathbf{k}_{n-1}}$ or $\Omega'_{\mathbf{k}_{n-1}}$.

The above discussion, modulo the proof of lemma 6.11, gives us a proof of the Combinatorial Structure Theorem:

Theorem (Combinatorial Structure Theorem)**.** *The virtual cycle parameters $\lambda_{\mathbf{k}_n}$ of order n can be labelled by sequences $\mathbf{k}_n=k_nk_{n-1}\ldots k_1$, where $k_i\in\mathbb{Z}$, in such a way that each of the parameters $\lambda_{\mathbf{k}_n}$ is an accumulation point in $\mathbb{C}$ of a sequence of parameters $\lambda_{\mathbf{k}_{n+1}}$ of order $n+1$, where $\mathbf{k}_{n+1}=k_nk_{n-1}\ldots k_1k_{0,j}$, $j\in\mathbb{Z}$. This combinatorial description of the virtual cycle parameters determines combinatorial descriptions of the sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$.*

4.3. **Parameter space pictures.** Figure 2 shows a picture of the λ parameter plane for $\rho=2/3$. The green region is $\mathcal{S}$ where both λ and μ are attracted to the origin. The unbounded multicolored region on the right in figure 2 is $\mathcal{M}_\lambda$ and the small bounded multicolored region inside the green region is $\mathcal{M}_\mu$. Since this figure is drawn to scale, in $\mathcal{M}_\mu$ the colors other than yellow are not visible. To make the structure of this region visible and show it is similar to $\mathcal{M}_\lambda$'s, in figure 3 we place a blown up neighborhood of $\mathcal{M}_\mu$ near $\mathcal{M}_\lambda$.

The shell components are colored according to their period: yellow is period 1, cyan is period 2, red is period 3 and so on. Periods higher than 10 are colored black. Note that there is only one unbounded domain, the yellow period 1 domain on the right, Ω_1; its virtual center is the point at infinity. The virtual center of the period 1 component of $\mathcal{M}_\mu$ is the leftmost point. It is the singular point $\rho/2$ of the parameter space. There is a cusp boundary

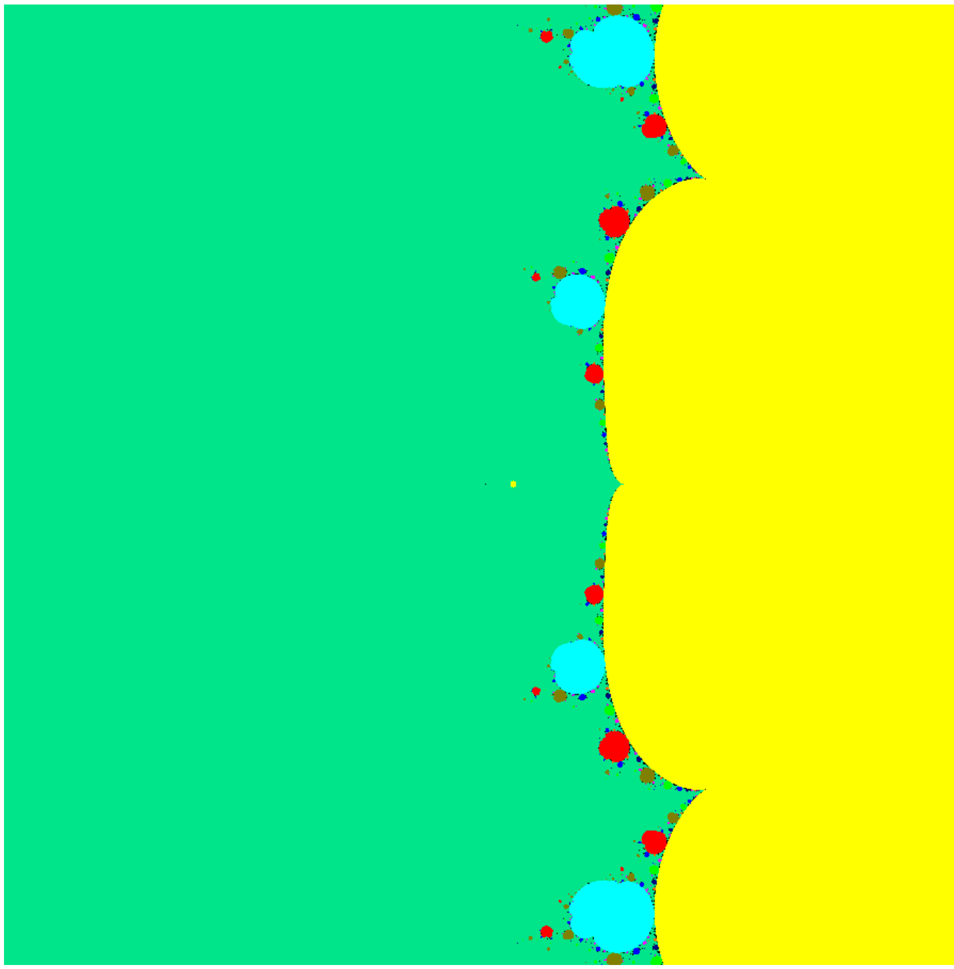

FIGURE 2. The λ plane divided into the shift locus and shell components. The green region represents the shift locus $\mathcal{S}$. The regions $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ are colored by the period of the component: period 1 is yellow , period 2 is cyan, period 3 is red, etc. The coloring is not visible for $\mathcal{M}_\mu$ because it is so small.

point of Ω_1 on the real axis where the multiplier of the cycle attracted to λ is $+1$. There are cyan period 2 components appearing as "buds" off of the yellow component Ω_1 where the multiplier of the cycle attracted to λ is $e^{(2m+1)\pi i}$; each of these has a virtual center with itinerary $\mathbf{k}_1 = m$.

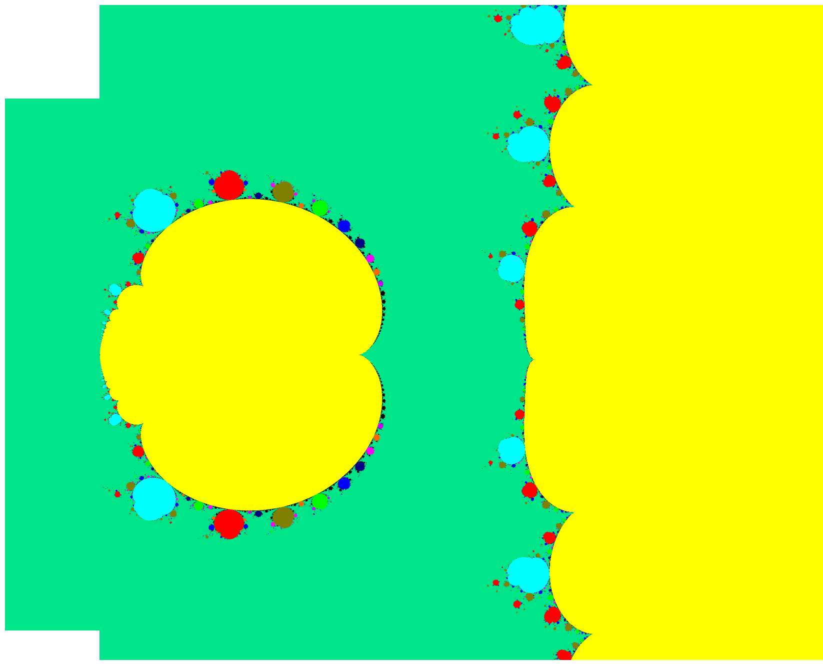

FIGURE 3. Blow up of $\mathcal{M}_\mu$ placed near $\mathcal{M}_\lambda$ for comparison. The coloring scheme is the same as in figure 2 and is now visible in the blown up $\mathcal{M}_\mu$.

In figure 4, we see a period 2 component Ω_2 budding off Ω_1. Although M_λ and M_μ look disconnected in the figure, as we will prove, they are not. Here we have only computed shell components for periods up to 10. To make a figure where $\mathcal{S}$ and $\mathcal{M}_\lambda$ look connected would require much more computation and many more colors to show components with much higher periods. What we do see, however, is a period 3, red component that is NOT a bud component of the period 1 component. In fact, there are infinitely many such converging to the virtual center of Ω_2 marked as v. We postpone a full discussion of the finer structure of the shell components to future work.

5. Boundaries of Hyperbolic Components and Virtual Cycle Parameters

In this section we show that each virtual center is a boundary point of both $\mathcal{S}$ and either $\mathcal{M}_\lambda$ or $\mathcal{M}_\mu$. To do this we need to use the concept of transversality.

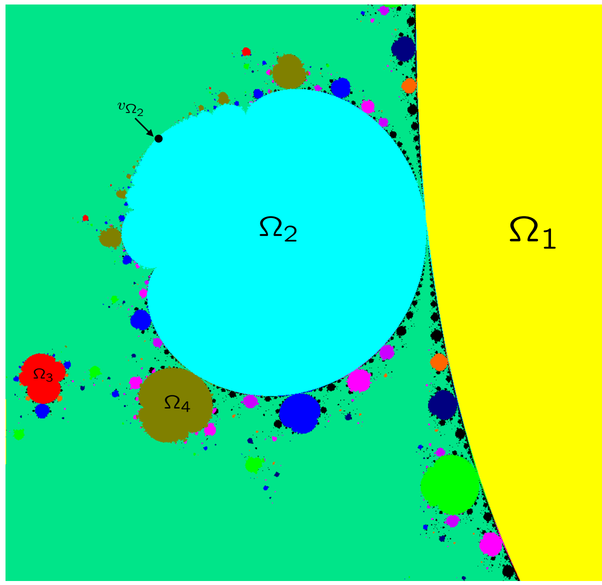

FIGURE 4. Blow-up of the λ plane near $\mathcal{M}_\lambda$ with the periods labelled.

Definition 5 (Transversality, [CJK]). *Suppose λ^* is a virtual center parameter. Let $p^*(\lambda)$ be the holomorphic prepole function such that $p^*(\lambda^*) = f_{\lambda^*}^{n-2}(\lambda^*)$. Define the holomorphic function,*

$$c_n(\lambda) = f_\lambda^{n-2}(\lambda) - p^*(\lambda).$$

We say f_λ is transversal at λ^* *or satisfies a* transversality condition *at λ^* if $c_n'(\lambda^*) \neq 0$.*

Theorem 5.1 (Common Boundary Theorem). *Every virtual cycle parameter is a boundary point of both a shell component and the shift locus. Furthermore, the family $\{f_\lambda\}$ is transversal at these parameters.*

Remark 5.1. *The transversality property translates to the dynamic planes of the functions f_λ as follows:*
If f_λ is transversal at λ^*, and if $\lambda(t)$ is any smooth path passing through λ^* at t^* such that $\lambda'(t^*) \neq 0$, then the dynamics of $f_{\lambda(t)}$ bifurcates at t^*. In particular, as $\lambda(t)$ moves from a shell component into the shift locus through the common boundary point, an asymptotic value, say $\lambda(t)$, moves from the attracting basin of an attractive cycle of $f_{\lambda(t)}$, through the pre-pole λ^* of the virtual cycle, and into the attracting basin of zero for $f_{\lambda(t)}$. Moreover, if ϵ is small enough so that $\lambda(t)$ does not contain any other virtual center when

$|t - t^*| < \epsilon$, then t^* is the only point in the interval $|t - t^*| < \epsilon$ where the dynamics of $f_{\lambda(t)}$ bifurcate. This is illustrated in figures 5 and 6.

In addition, transversality of f_λ at λ^* implies that the holomorphic functions defining the poles $p_k(\lambda)$ and pre-poles $p_{\mathbf{k}_n}(\lambda)$ satisfy $p'_k(\lambda^*) \neq 0$ and $p'_{\mathbf{k}_n}(\lambda^*) \neq 0$.

Proof of the Common Boundary Theorem. Let λ^* be a virtual cycle parameter. It follows from part (c) of theorem 4.1, that λ^* is on the boundary of a shell component. Suppose this component, Ω_n, is in $\mathcal{M}_\lambda$ so that μ^* is in A_{λ^*}, the attracting basin of 0, and $f_{\lambda^*}^{n-1}(\lambda^*) = \infty$. Let U be a small neighborhood of λ^* such that for $\lambda \in U$, $\mu(\lambda)$ is in A_λ and $a_{n-1}(\lambda) = f_\lambda^{n-1}(\lambda)$ is a holomorphic function on U with $a_{n-1}(\lambda^*) = \infty$. Since infinity is always a boundary point of the basin A_λ, the open mapping theorem implies that there is a $\lambda_U \in U$ with $\lambda_U \in A_{\lambda(U)}$. This says $\lambda_U \in \mathcal{S}$ and thus λ^* is a boundary point of $\mathcal{S}$.

In [CJK], we proved a transversality theorem for maps in the tangent family, $\lambda \tan z$ with $\lambda = it$, $t \in \mathbb{R}$. There $\lambda(t)$ is in the imaginary axis, and the proof shows that the function $c_n(\lambda(t))$ has no critical point at t^*. It involves the use of holomorphic motions and some ideas adapted from [LSS]. That proof can be adapted here by replacing the imaginary axis with a path $\lambda(t)$ in Ω_n defined by the condition that the multiplier of the attracting cycle $\mathbf{a}(\lambda)$ has argument equal to $2\pi i n$, for some n. Then the arguments there can be applied and show that as $t \to t^*$ in Ω_n, $c'_n(\lambda(t^*) \neq 0$, and the dynamics bifurcates smoothly. We refer the interested reader to that paper for the details. □

An immediate corollary of the Common Boundary Theorem is

Corollary 5.2. *Given an itinerary, $\mathbf{k}_{n-1} = k_{n-1}k_{n-2}\ldots k_1$, there is exactly one component in each of $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ with that itinerary label.*

Proof. Let $\mathbf{k}_{n-1}$ be a given itinerary. The prepoles $p_{\mathbf{k}_{n-1}}(\lambda)$ of order $n-1$ form a discrete set in dynamic space since they are solutions of $f_\lambda^{n-1}(z) = \infty$ and there is only one with itinerary $\mathbf{k}_{n-1}$. They are on the boundary of A_λ. The virtual centers form a discrete set in parameter space since they are solutions of $f_\lambda^{n-1}(\lambda) = \infty$.

We can find a sequence $\lambda_j \in \mathcal{S}$, tending to $\partial\mathcal{S}$ as j goes to infinity, such that $|f_{\lambda_j}^{n-1}(\lambda_j) - p_{\mathbf{k}_{n-1}}(\lambda)|$ goes to zero as j goes to infinity. It follows that $\lim_{j\to\infty} \lambda_j$ is a virtual cycle parameter λ^* with itinerary $\mathbf{k}_{n-1}$. By theorem 5, in a small neighborhood of λ^*, there is no other virtual center with itinerary $\mathbf{k}_{n-1}$ so that the component Ω_n with λ^* as virtual center is the only one in $\mathcal{M}_\lambda$ with this itinerary.

We obtain a different component Ω'_n if we choose a sequence λ'_j such that $f_{\lambda_j}^{n-1}(\mu_n)$ approaches the prepole with this itinerary, but that is the only other possibility. In this case, Ω'_n is in $\mathcal{M}_\mu$. □

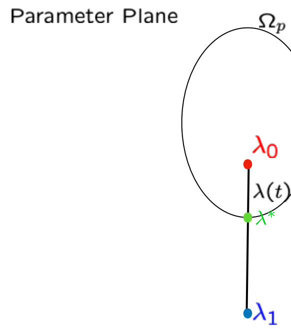

FIGURE 5. Transversality in the parameter plane

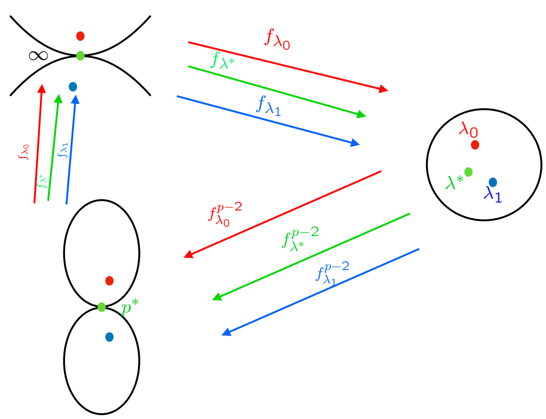

FIGURE 6. Transversality in the dynamic plane

5.1. **The Bifurcation Locus.** Denote the set of virtual center parameters by $\mathcal{B}_{cv}$. By theorem 4.1, each such parameter is on the boundary of a unique shell component and in section 4.2 we used these parameters to enumerate the shell components. Here we will prove that these parameters are dense in the boundary of the shift locus. To do so we need two definitions.

Definition 6 (Holomorphic family)**.** *A* holomorphic family *of meromorphic maps over a complex manifold X is a map $\mathcal{F} : X \times \mathbb{C} \to \widehat{\mathbb{C}}$, such that $\mathcal{F}(x, z) =: f_x(z)$ is meromorphic for all $x \in X$ and $x \mapsto f_x(z)$ is holomorphic for all $z \in \mathbb{C}$.*

Definition 7. *The* J-stable set *of the family $\mathcal{F}_2$, denoted by $\mathcal{J} = \mathcal{J}_\rho$, is the set*

$$\{\lambda \mid f_\lambda^n(\lambda) \text{ and } f_\lambda^n(\mu) \text{ form normal families}\}.$$

Its complement is called the bifurcation locus.

In [KK] it is proved that

Proposition 5.3 ([KK])**.** *If $\lambda_0 \in \mathcal{J}$, then the number of attracting cycles of f_{λ_0} is locally constant in a neighborhood of λ_0; in particular, $\mathcal{J}$ is open.*

We need the following generation of Montel's theorem.

Theorem 5.4 (See theorem 3.3.6 in [Bea])**.** *Let D be a domain, and suppose that the functions ϕ_1, ϕ_2 and ϕ_3 are analytic in D, and are such that the closures of the domains $\phi_j(D)$ are mutually disjoint. If $\mathcal{F}$ is a family of functions, each analytic in D, and such that for every z in D, and every f in $\mathcal{F}$, $f(z) \neq \phi_j(z)$, $j = 1, 2, 3$, then $\mathcal{F}$ is normal in D.*

If the iterates under f_λ of both λ and μ converge to attracting cycles, then $\lambda \in \mathcal{J}$. Thus $\mathcal{J}$ contains all the shell components and the shift locus. The set of virtual center parameters $\mathcal{B}_{cv}$ is clearly not contained in $\mathcal{J}$. By theorem 4.1 and theorem 5.1 the points in $\mathcal{B}_{cv}$ are on the boundaries of both a shell component and the shift locus. In addtion, we have

Theorem 5.5. *The boundary of $\mathcal{J}$ is contained in the closure of $\mathcal{B}_{cv}$, that is, $\partial\mathcal{J} \subset \overline{\mathcal{B}_{cv}}$.*

Proof. Since 0 is an attracting fixed point, at least one of the families $\{f_\lambda^n(\lambda)\}$ and $\{f_\lambda^n(\mu)\}$ converges to 0; that is, for each λ_0, one of them is always normal and, by proposition 5.3, in a neighborhood of λ_0 it is the same family that is normal. Suppose $\lambda_0 \in \partial\mathcal{J}$; without loss of generality, we may assume that $\{f_\lambda^n(\lambda)\}$ is not normal at λ_0.

Let U be any neighborhood of λ_0. The poles of f_λ,

$$p_k(\lambda) = \frac{1}{2}\log(\frac{\rho - 2\lambda}{\rho}) + ik\pi,\ k \in \mathbb{Z},$$

form a holomorphic family in U. If $f_{\lambda,\mu}^n(\lambda) \neq p_k(\lambda)$ for any k or $\lambda \in U$, then theorem 5.4 implies $f_\lambda^n(\lambda)$ is normal in U. This contradicts the hypothesis that $\lambda_0 \in \partial\mathcal{J}$. □

The bifurcation locus contains parabolic cusps and Misiurewicz points.

6. Topological structure of the Shift Locus

In this section we will show that the shift locus is homeomorphic to an annulus punctured at one point. This puncture corresponds to the point $\lambda = 0$ where f_λ is not defined.

Before we discuss this proof we need a lemma.

Lemma 6.1. *Suppose V is Riemann surface homeomorphic to a disk from which a (possibly empty) collection of finitely or countably many pairwise disjoint disks have been removed. Let λ and μ be two distinct points in V. Then there is a Riemann surface W, homeomorphic to a disk minus a countable collection of pairwise disjoint disks, and an infinite degree holomorphic covering map $h : W \to V \setminus \{\lambda, \mu\}$.*

Proof. There exists an embedding $e : V \to \widehat{\mathbb{C}}$ such that $e(\lambda) = 0$ and $e(\mu) = \infty$. Consider the exponential map

$$Exp(z) = e^z : \mathbb{C} \to \mathbb{C}$$

and set $W = Exp^{-1}(e(V))$. Each component U of $\widehat{\mathbb{C}} \setminus e(V)$ is simply connected and does not contain either 0 or ∞. Therefore $Exp^{-1}(U)$ is the union of infinitely many simply connected open sets so that W is an open set with infinitely many holes. Thus $h = e^{-1} \circ Exp : W \to V$ is the required map. □

Remark 6.1. *We inductively apply this lemma to construct a family of surfaces and infinite degree covering maps. The direct limit of this process defines a map that is used in a key step of the proof of the main structure theorem.*

As in lemma 6.1, let V_0 be a topological disk and let $\{U_j\}$ be a (possibly empty) collection of finitely or countably many pairwise disjoint disks in V_0. Set $U_0 = V_0 \setminus \cup_{j \in \mathbb{Z}} U_j$ and fix two points, λ_0 and μ_0 in U_0. Applying the lemma, we can find a Riemann surface $U_1 = V_1 \setminus \cup_{(j_1,j) \in \mathbb{Z}^2} U_{j_1 j}$, where V_1 is a topological disk and the $U_{j_1,j}$ are pairwise disjoint topological disks in V_1, and an infinite degree holomorphic covering map $h_1 : U_1 \to U_0 \setminus \{\lambda_0, \mu_0\}$.

Iterating this process, we choose points $\lambda_{n-1}, \mu_{n-1} \in U_{n-1}$ and obtain Riemann surfaces $U_n = V_n \setminus \cup_{(j_n, \cdots j_0) \in \mathbb{Z}^{n+1}} U_{j_n \cdots j_0}$, where V_n is a topological disk and the $U_{j_n \cdots j_0}$ are pairwise disjoint topological disks in V_n, and holomorphic covering maps of infinite degree

$$h_n : U_n \to U_{n-1} \setminus \{\lambda_{n-1}, \mu_{n-1}\}.$$

To carry out the proof on the structure of $\mathcal{S}$, recall the normalized uniformizing map ϕ_λ defined in the proof of proposition 3.1 that conjugates f_λ to a linear map near the origin. We divide the discussion into two parts depending on which of the asymptotic values is on the boundary of O_λ, the domain on which ϕ_λ is injective:

- Let $\mathcal{S}_\lambda = \{\lambda \in \mathcal{S} | \mu \in \partial O_\lambda\}$.
- Let $\mathcal{S}_\mu = \{\lambda \in \mathcal{S} | \lambda \in \partial O_\lambda\}$.

These sets have a common boundary, $\mathcal{S}_* = \mathcal{S}_\lambda \cap \mathcal{S}_\mu$, or equivalently,

$$\mathcal{S}_* = \{\lambda \in \mathcal{S}_\lambda | \lambda \in \partial O_\lambda\} = \{\lambda \in \mathcal{S}_\mu | \mu \in \partial O_\lambda\}.$$

In section 3.2 we defined the map $I(\lambda)$ which is the inversion in the circle C_0 defined by $|z - \rho/2| = |\rho/2|$. Using this map we have,

Proposition 6.2. *The common boundary set $\mathcal{S}_*$ is invariant under $I(\lambda)$.*

Proof. Note that the affine map $z \to -z$ conjugates f_λ to $f_{I(\lambda)}$. Therefore if $\phi_\lambda(z)$ is the uniformizing map for f_λ, then the uniformizing map $\phi_{I(\lambda)}$ for $f_{I(\lambda)} = f_{\lambda_1,\mu_1}$ is $\phi_\lambda(-z)$. Thus,

$$\phi_{I(\lambda)}(\lambda_1) = \phi_\lambda(\mu) \text{ and } \phi_{I(\lambda)}(\mu_1) = \phi_\lambda(\lambda).$$

It follows that $I(\lambda)$ interchanges $\mathcal{S}_\lambda$ and $\mathcal{S}_\mu$ and fixes $\mathcal{S}_*$.

□

Note that the point $\lambda = \rho$ is in $\mathcal{S}_*$.

We saw above, in proposition 3.4, that if ρ is real, the invariant circle C_0 of the inversion $I(\lambda)$ is in $\mathcal{S}$. If ρ is real, we can say more.

Proposition 6.3. *If ρ is real, then $\mathcal{S}_* = C_0$.*

Proof. Let $\sigma(z) = -\overline{z}$. Then if ρ is real, it is easy to check that for any z, $f_\lambda \circ \sigma(z) = \sigma \circ f_\lambda(z)$. Therefore

$$f_\lambda^n(\mu) = f_\lambda^n(-\overline{\lambda}) = -\overline{f_\lambda^n(\lambda)}$$

and by proposition 3.4, they both converge to 0.

To show $\mathcal{S}_* = C_0$, we need to show that $|\phi_\lambda(\lambda)| = |\phi_\lambda(\mu)|$ where ϕ_λ is the uniformizing map defined above such that $\phi_\lambda(f_\lambda(z)) = \rho\phi_\lambda(z)$. We claim, in fact, that $\phi_\lambda(\mu) = -\overline{\phi_\lambda(\lambda)}$.

Let $\phi = \sigma \circ \phi_\lambda \circ \sigma(z)$. We claim that $\phi_\lambda = \phi$ since

$$\phi(f_\lambda(z)) = \sigma\circ\phi_\lambda\circ\sigma(f_\lambda(z)) = \sigma\circ\phi_\lambda(f_\lambda(\sigma(z))) = \sigma(\rho\phi_\lambda(\sigma(z))) = \rho\sigma\circ\phi_\lambda\circ\sigma(z) = \rho\phi(z).$$

Then $\phi_\lambda(\mu) = \phi_\lambda(\sigma(z)) = \sigma\phi_\lambda(\lambda)) = -\phi_\lambda(-\overline{\lambda})$ as claimed. □

The following theorem says that the interior $\mathcal{S}_\lambda^0$ of $\mathcal{S}_\lambda$ is a topological annulus. It follows that it is connected.

Theorem 6.4. *There is a homeomorphism $E : \mathcal{S}_\lambda^0 \to \mathbb{A}$ where $\mathbb{A}$ is a topological annulus. The inverse map E^{-1} extends continuously to all points except one of one of the boundary components of $\mathbb{A}$.*

Remark 6.2. *The proof of this theorem is based on a lemma in which we explicitly construct a homeomorphism E from $\mathcal{S}_\lambda^0$ to an annulus. The construction depends on the choice of a particular "model map" in the period* 1 *component Ω_1.*

The proof of the lemma is based on a technique that originally appeared in the unpublished thesis of Wittner, [Wit]. *The technique, called "critical* *point surgery", is used to model pieces of the cubic connected locus on the dynamical plane of a quadratic polynomial. In* [GK], *it was adapted to describe slices of Rat_2, the parameter space of rational maps of degree two with an attracting fixed point. Like the Rat_2 case, we have two singular values, but unlike that case, our singular values are asymptotic values and our maps are infinite degree and have an essential singularity. We choose as our model an f_λ with λ in the period* 1 *shell component of $\mathcal{M}_\lambda$. This is the unbounded yellow component in figure 2 and is denoted by Ω_1. As we saw at the end of section 3.2, the attracting basin of the fixed point of f_λ is simply connected and completely invariant. Our model space will be the annulus formed by removing a dynamically defined disk from this basin.*

Before we give the proof in detail, we give an outline. Below, we assume, as we have been doing, that ρ is fixed and all the functions f_λ belong to $\mathcal{F}_2$.

(1) Since the multiplier map is a universal cover of a shell component to the punctured unit disk, we can find a λ_0 in $\Omega_1 \subset \mathcal{M}_\lambda$ such that the multiplier at the fixed point q_0 of f_{λ_0} equals the fixed value ρ. This choice is convenient because the map f_{λ_0} is quasiconformally conjugate to a map $\sigma \tan z$ whose Julia set, by [KK], is a quasiconformal image of the real line. In fact, if we take ρ real, σ is real, the Julia set of f_{λ_0} is a line parallel to the imaginary axis and the attracting basin of f_{λ_0} is a simply connected, completely invariant half plane containing the asymptotic value λ_0. Following the notation in section 3.2 we denote the basin of f_{λ_0} by K_0.

(2) We make the model space by removing from K_0 a closed dynamically defined topological disk Δ which contains the fixed point q_0 in its interior and λ_0 on its boundary. We define the map E from $\mathcal{S}^0_\lambda$ to $K_0 \setminus \Delta$ as follows: to each $\lambda \in \mathcal{S}^0_\lambda$, we construct a map ξ_λ from a subset of the attracting basin A_λ of 0 containing both asymptotic values into the attracting basin K_0 of f_{λ_0} such that $\xi_\lambda(0) = q_0$ and $\xi_\lambda(\mu) = \lambda_0$; we set $E(\lambda) = \xi_\lambda(\lambda)$. We then prove that E is injective.

(3) To show E is a homeomorphism, we construct an inverse.
   - We want to assign a map $f_\lambda \in \mathcal{S}^0_\lambda$ to each point p in $K_0 \setminus \Delta$. The point p should correspond to the asymptotic value λ of f_λ. Given p, we use induction to construct the stable region of a map with two asymptotic values at λ_0 and p. At the n^{th} step we obtain a domain U_n, homeomorphic to a disk minus an infinite collection of open disks, and a holomorphic map $Q_n : U_n \to U_n$ with omitted values λ_0 and p. Taking the direct limit of the pairs (U_n, Q_n) we obtain a pair (U_∞, Q_∞) where $Q_\infty : U_\infty \to U_\infty$ is a holomorphic covering map with the desired topology; that is, an infinite degree covering map with two asymptotic values.

- We construct a conformal embedding $e : U_\infty \to \mathbb{C}$ such that $e \circ U_\infty = f_\lambda \circ e$ for a unique $\lambda \in \mathcal{S}_\lambda^0$ such that $\xi_\lambda(\lambda) = p$. The construction depends on some Teichmüller theory. We give a brief summary of what we need before the construction.
- We extend this inverse map to the points of $\partial\Delta \setminus \{\lambda_0\}$ whose image, by construction, is $\mathcal{S}_*$. Note that the map is not defined for $p = \lambda_0$; its image must be a parameter singularity in $\overline{\mathcal{S}_*}$.

The proof of theorem 6.4 is contained in the next subsections.

6.1. **The Model Space.** Every point $\lambda \in \Omega_1$ corresponds to a function f_λ with a non-zero attracting fixed point denoted by q_λ; its attractive basin is denoted by K_λ. By propositions 3.1 and 3.5, it is simply connected and completely invariant. In fact, f_λ is quasiconformally conjugate to $t \tan z$ for some real $t \geq 1$ whose Julia set is the real line (see [DK]), so its Julia set is the quasiconformal image of a line (see figure 7 where $\rho = 2/3$). In figure 7, the cyan colored region is K_λ and the yellow region is the basin of 0, A_λ. The black dots are poles on the boundary of K_λ and are in the Julia set. Denote the closure of K_λ by $\overline{K_\lambda}$. It is the analogue of the filled Julia set for a quadratic map.

Because the multiplier map ν is a universal covering from Ω_1 to $\mathbb{D}^*$, we can find a sequence of points $\lambda_j \in \Omega_1$, $j \in \mathbb{Z}$ such that $\nu(\lambda_j) = f'_{\lambda_j}(q(\lambda_j)) = \rho$. We choose one, denote it by λ_0, and set $q_0 = q(\lambda_0)$. We set $Q(z) = f_{\lambda_0}(z)$ and let K_0 denote the attracting basin of q_0.

In figure 8 set $\overline{K}$ is depicted for ρ real and λ_0 taken as the real solution to $\nu(\lambda_0) = f'_{\lambda_0}(q(\lambda_0)) = \rho$. Since the multipliers of both attracting fixed points, 0 and q_0, are the same, there is a real $t = t(\rho)$ such that $Q(z)$ and $t \tanh z$ are not only quasiconformally conjugate but affine conjugate and the Julia set of $Q(z)$ is a vertical line.

There is a local uniformizing map, which we denote by ϕ_0, $\phi_0 : K_0 \to \mathbb{C}$, normalized so that ϕ_0 maps q_0 to 0, $\phi_0'(q_0) = 1$ and ϕ_0 conjugates Q to $\zeta \to \rho\zeta$ in a neighborhood of q_0. We can extend ϕ_0 to all of K_0 by analytic continuation. Note that $\phi_0(z) = 0$ if and only if $Q^n(z) = q_0$ for some n.

Let $r = |\phi_0(\lambda_0)|$ and let $\gamma^* = \phi_0^{-1}(re^{i\theta})$, $\theta \in \mathbb{R}$. It is a simple closed curve; let Δ be the closed topological disk in K_0 bounded by γ^*. Then ϕ_0 is injective on Δ and λ_0 is on $\partial\Delta$.

Lemma 6.5. *There is an injective holomorphic map $E : \mathcal{S}_\lambda^0 \to K_0 \setminus \Delta$. Set $w = E(\lambda)$; E satisfies:*

(i) For each $\lambda \in \mathcal{S}_\lambda$ such that $f_\lambda^n(\lambda) = 0$ for some n, E maps it to a preimage of q_0; that is, if $w = E(\lambda)$, then $Q^n(w) = q_0$.

(ii) For each $\lambda \in \mathcal{S}_\lambda$ such that $f_\lambda^n(\lambda) = f_\lambda^m(\mu)$ for some m, n, E maps it to a point in the grand orbit of λ_0; that is, if $w = E(\lambda)$, then $Q^n(w) = Q^m(\lambda_0)$.

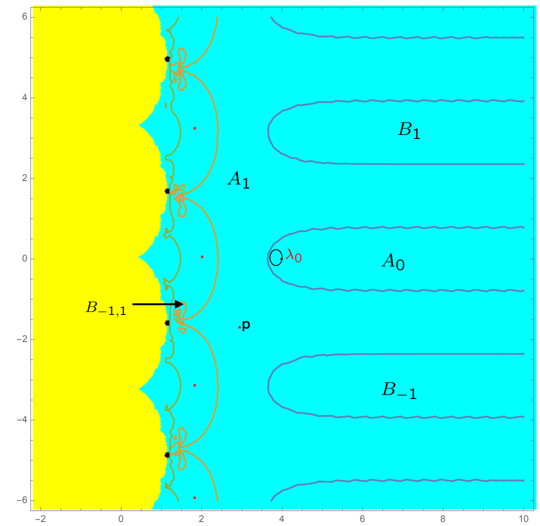

FIGURE 7. The "filled Julia set" of $Q(z)$. The black dots are poles.

(iii) As λ tends to the boundary $\mathcal{S}_*$ of $\mathcal{S}_\lambda$, $w = E(\lambda)$ tends to $\partial\Delta \setminus \{\lambda_0\}$.

Proof. The map E is defined as follows. Given $\lambda \in \mathcal{S}^0_\lambda$, we defined a conformal homeomorphism ϕ_λ from the neighborhood O_λ in the attracting basin A_λ conjugating f_λ to $\zeta \mapsto \rho\zeta$ and normalized so that $\phi_\lambda(0) = 0$ and $\phi'_\lambda(0) = 1$. Now we renormalize, and use the same notation, so that $\phi_\lambda(\mu) = \phi_0(\lambda_0)$. For each λ, define a map

$$\xi_\lambda = \phi_0^{-1} \circ \phi_\lambda : O_\lambda \to \Delta.$$

From the definitions of ϕ_λ and ϕ_0, it follows that $\xi_\lambda(0) = q_0$ and $\xi_\lambda(\mu) = \lambda_0$. Since $f'_{\lambda_0}(0) = Q'(q_0) = \rho$, the map ξ_λ is a conformal homeomorphism from O_λ to Δ and it conjugates f_λ to Q. Moreover, since K_0 is simply connected, the map ξ_λ has a unique analytic continuation to $\lambda \in K_0$.

Now we are ready to define the map E from $\mathcal{S}^0_\lambda \to K_0 \setminus \Delta$ as

$$E(\lambda) = \xi_\lambda(\lambda).$$

By construction E satisfies properties (i) and (ii).

Suppose $\xi_{\lambda'}(\lambda') = \xi_\lambda(\lambda)$; then the map $\xi_{\lambda',\lambda} = \xi_{\lambda'}^{-1}\xi_\lambda = \phi_{\lambda'}^{-1}\phi_\lambda$ restricted to a neighborhood of the origin in the basin A_λ is a holomorphic homeomorphism whose image is a neighborhood of the origin in the basin $A_{\lambda'}$. It extends by analytic continuation to a holomorphic conjugacy between the maps f_λ and $f_{\lambda'}$ on their respective stable sets A_λ and $A_{\lambda'}$. Since f_λ and $f_{\lambda'}$ are hyperbolic, their Julia sets are holomorphically removable; that is, the map $\xi_{\lambda',\lambda}$ can be extended holomorphically over the Julia sets as a conformal homeomorphism, and hence an affine map of the complex plane $\mathbb{C}$, which we still denote as $\xi_{\lambda',\lambda}$. As we saw in section 3, there are two choices for λ' but if we require that λ' is the preferred asymptotic value so that $\lambda' \in \mathcal{S}_\lambda$ then $\xi_{\lambda',\lambda}$ is the identity.

Property (iii) follows since as λ tends to the boundary $\mathcal{S}_*$ of $\mathcal{S}_\lambda$, the asymptotic value λ tends toward the leaf of the dynamically defined level curve containing μ in the dynamic plane of f_λ; thus $E(\lambda)$ tends to a point on the corresponding level curve, $\partial\Delta \setminus \{\lambda_0\}$ in K_0. □

Note that the map extends continuously to the preimages of the removed point λ_0 on $\partial\Delta$ so they must be parameter singularities in $\mathcal{S}_*$. There are only two such, 0 and $\rho/2$, and the latter is a virtual center on the boundary of $\mathcal{M}_\mu$. Because $\mathcal{S}_*$ is on the common boundary of $\mathcal{S}_\lambda$ and $\mathcal{S}_\mu$, any point in a neighborhood of this singularity is a point in $\mathcal{S}$ or $\mathcal{S}_*$ so the singularity cannot be a boundary point of $\mathcal{M}_\mu$. Therefore there is only one preimage of λ_0 and it is 0.

Remark 6.3. *The map ξ_λ ties together the attractive basin of the origin in the dynamical space of f_λ and the attractive basin of q_0 in the dynamical space of Q with the parameter space of f_λ.*

6.2. **Construction of an inverse for E.**

6.2.1. *Dynamic decomposition of K_0.* To define inverse branches R_j of Q on the K_0, let l^* be the gradient curve joining $Q(\lambda_0)$ to λ_0 in Δ and let $l \in Q^{-1}(l^*)$ be the curve joining λ_0 to infinity. Remove the line l from K_0 and define an inverse branch on its complement by the condition $R_0(q_0) = q_0$. Label the other branches as $R_j(q_0) = q_0 + \pi i j = q_j$. This is equivalent to choosing a principal branch for the logarithm. Having made this choice, we can extend the R_j analytically to all of K_0. Denote the preimages of q_0 under Q^{-1} by q_j, enumerated so that q_0 is fixed, and denote the inverse branch of Q that sends q_0 to q_j by R_j. Denote the upper and lower sides of the line l by l^+ and l^- and let $l_j = R_j(l^-)$, $l_{j+1} = R_j(l^+) = R_{j+1}(l^-)$. Then R_0 is a homeomorphism between the open region bounded by the lines l_0, l_1 and l onto $K_0 \setminus l$ and R_j, $j \neq 0$ is a homeomorphism from the open region between l_j and $l_j + 1$ onto $K_0 \setminus l$.

If ρ and λ_0 are real, this choice for the logarithm agrees with the labeling of the poles and inverse branches in section 4.2 where $R_0 = g_{\lambda_0,0}$, the branch of $Q^{-1} = f_{\lambda_0}^{-1}$ that fixes the origin. This is the labeling in figure 8. If ρ and/or λ_0 isn't real, and a different branch of the logarithm is chosen, there could be

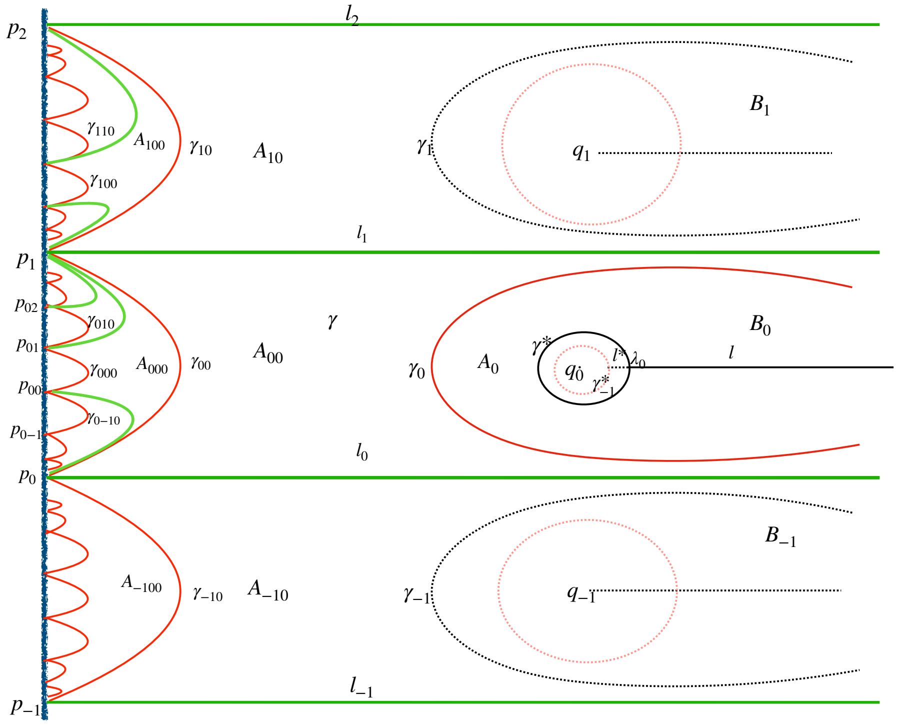

FIGURE 8. Domains and curves in the construction.

a shift by some k in the labelling. It would be the same shift throughout the rest of the paper so would not change the essence of the argument.

Recall that γ^* is the boundary of Δ in K_0 and it contains λ_0. Then, because λ_0 is an omitted value of Q, the curves $\{\gamma_j\} = R_j(\gamma^*)$, $j \in \mathbb{Z}$, are a countable collection of bi-infinite disjoint curves whose infinite ends approach infinity asymptotic to the lines l_j and l_{j+1}. Thus $R_j(\Delta)$ is an unbounded domain, with boundary γ_j, that contains q_j. Note that $R_0(\Delta)$ contains the removed line l. We label the complementary components of the γ_j as follows (see figure 8):

- $A_0 = R_0(\Delta)$ is the component of the complement of γ_0 containing the fixed point q_0 and the point λ_0.
- B_j, $j \in \mathbb{Z}\setminus\{0\}$ are the components of the complement of γ_j containing the non-fixed preimages q_j of q_0 and
- C_0 is the common complementary component in K_0 of A_0 and all the B_j.

To define the second preimages of q_0 and γ^* we need two indices. Thus $q_{j_2 j_1} = R_{j_2} R_{j_1}(q_0)$ and $\gamma_{j_2 j_1} = R_{j_2} R_{j_1}(\gamma^*)$ where $j_1, j_2, \in \mathbb{Z}$. They divide K_0 into domains as follows (see figure 8):

- Since A_0 is simply connected and contains one asymptotic value, $Q : A_1 = Q^{-1}(A_0) \to A_0 \setminus \{\lambda_0\}$ is a universal covering. Set $A_{j0} = R_j(A_0)$; it is bounded by l_j, l_{j+1} and γ_{j0}. Each $\{\gamma_{j0}\}$ joins the pole $R_j(\infty)$ to the pole $R_{j+1}(\infty)$; these two poles are different but adjacent because the infinite ends of γ_0 are on opposite sides of the line l defining the principal branch.
- Since B_{j_1} is simply connected and contains no asymptotic value for any $j_1 \neq 0$, each component of $Q^{-1}(\gamma_{j_1})$ is homeomorphic to γ_{j_1}. The curves $\gamma_{j_2 j_1}$ bound domains containing the preimages $q_{j_2 j_1}$. Label these domains $B_{j_2 j_1} = R_{j_2}(B_{j_1})$.
- There are domains $C_{j0} = R_j(C_0)$.

Inductively we have curves

$$\gamma_{j_n \dots j_1} = R_{j_n}(\gamma_{j_{n-1}} \dots j_1).$$

and the regions they define as follows (see figure 8);

- $A_n = R_0(A_{n-1})$; it contains q_0 and λ_0. It also contains all preimages of q_0 up to order $n-1$ but not those of order n. It is bounded by a curve $Q^{-n}(\gamma_0)$ that is a union of open arcs with endpoints at adjacent prepoles of order n. These are the red curves without labels closest to the vertical line in figure 8.
- $B_{j_n j_{n-1} \dots j_1} = R_{j_n}(B_{j_{n-1} \dots j_1})$; it contains the preimage $q_{j_n j_{n-1} \dots j_1}$ of q_0. These are not shown in the figure. They are bounded by a single curve with a boundary point at a prepole of order $n-1$.
- $C_{j_n j_{n-1} \dots, 0} = R_{j_n}(C_{j_{n-1} \dots 0})$.

6.2.2. *Inductive construction of the pair (U_∞, Q_∞).* See figure 9.

- Pick $p \in K_0 \setminus \Delta$. In figure 9, p is in A_{00}. Following the outline above, part (3), we construct a map with the asymptotic values p and λ_0 as follows. Let $\widehat{p}_j = R_j(p)$; in the figure the $\widehat{p}_j$ are in A_{j00}. Let N be the smallest integer such that p is in $A_N \cup B_{j_N, j_{N-1} \dots, j_1}$. The boundaries of the sets in this union are the level sets $\phi_0^{-1}(\rho^{-N} r e^{i\theta})$ where, as above, $r = |\phi_0(\lambda_0)|$. For every small $\epsilon > 0$, one component of the level set $\phi_0^{-1}((\rho^{-N} r + \epsilon) e^{i\theta})$ is an analytic curve, except at the prepoles of order $N-1$. It bounds a simply connected domain containing A_N; it thus contains the points p, q_0, λ_0 and the curves $\gamma_{j_N, \dots j_1}$, but none of preimages $\widehat{p}_j$ of p. Fix ϵ, and denote the resulting domain by U. Its boundary is denoted by the dotted black curve in figure 9. Since it is contained in the attracting basin of q_0, $Q(U) \subset U$. Moreover, since U

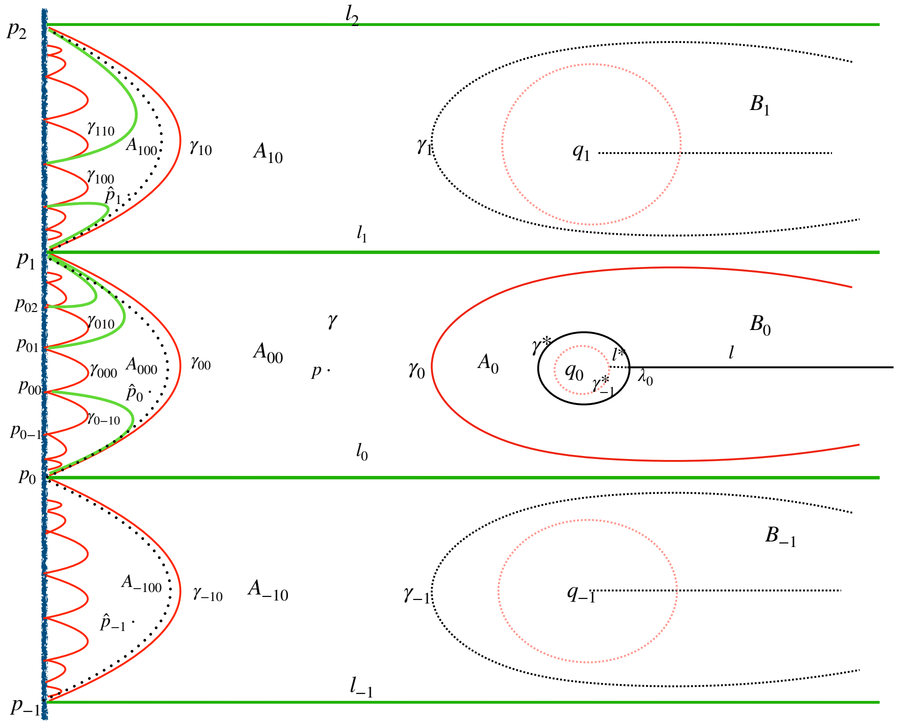

Figure 9. The point p is in A_{00} and the set U is the region to the right of the dotted curves.

does not contain any of the points $\widehat{p}_j$, $p \notin Q(U)$. Set $\widetilde{U} = U \setminus \{\lambda_0, p\}$.

- Lemma 6.1 implies there is a holomorphic unramified covering map $\Pi_1 : U_1 \to \widetilde{U}$ where U_1 is a Riemann surface that is topologically a disk minus a countable set of topological disks.
- Note that $Q : U \to Q(U)$ is a holomorphic universal covering map with omitted value λ_0. Set $U_1' = \Pi_1^{-1}(Q(U))$; since $\lambda_0 \in Q(U)$ and $p \notin Q(U)$, it is a topological disk so that $\Pi_1 : U_1' \to Q(U)$ is also a holomorphic unramified covering map that omits the value λ_0.

$$\begin{tikzcd} U_1 \arrow[r, dotted, "Q_1 = i_1 \circ \Pi_1"] \arrow[d, "\Pi_1"'] & U_1' \arrow[r, hook] \arrow[d, "\Pi_1"] & U_1 \\ U \arrow[r, "Q"] \arrow[ur, dotted, "i_1"] & Q(U) \arrow[r, hook] & U \end{tikzcd}$$

- Since both Π_1 and Q are regular coverings whose domains are simply connected, we can lift to obtain a conformal map $i_1 : U \to U_1'$ such that $\Pi_1 \circ i_1 = Q$. The choice of inverse branch will affect i_1 but the argument works for any choice. We now define $Q_1 : U_1 \to U_1'$ by $Q_1 = i_1 \circ \Pi_1$. It is an unramified regular infinite to one holomorphic endomorphism and omits the values $i_1(\lambda_0)$ and $i_1(p)$. Moreover,

$$Q_1 \circ i_1 = i_1 \circ \Pi_1 \circ i_1 = i_1 \circ Q;$$

that is, i_1 conjugates Q and Q_1, therefore Q_1 fixes $i_1(q_0)$.

We may, without loss of generality, assume $i_1(\lambda_0) = \lambda_0$, $i_1(p) = p$ and $i_1(q_0) = q_0$.

- Now set $Q_0 = Q$, $U_0 = U$ and $U_0' = Q(U)$. We proceed by induction: we assume that for $1 \leq j \leq n-1$ we have
  (1) Domains U_j, homeomorphic to an open disk from which infinitely many open disks been removed, and infinite to one unramified covering maps $\Pi_j : U_j \to U_{j-1}$ with two asymptotic values.
  (2) Holomorphic endomorphisms, $Q_j : U_j \to U_j' \subset U_j$, that are infinite to one, unramified, have one fixed point and two asymptotic values.
  (3) Conformal maps $i_j : U_{j-1} \to U_j'$ satisfying
  $$Q_j \circ i_j = i_j \circ Q_{j-1}.$$

  For the inductive step, we use Remark 6.1 to obtain the holomorphic unramified covering map $\Pi_n : U_n \to U_{n-1}$ where U_n is homeomorphic to $U \setminus \{U_{j_{n-1}\cdots j_1 j},\ (j_{n-1}, \ldots, j_1, j) \in \mathbb{Z}^n\}$. As in the first step we set $U_n' = \Pi_n^{-1}(Q_{n-1}(U_{n-1}))$. Both

  $$Q_{n-1} : U_n' \to Q_{n-1}(U_{n-1}) \text{ and } \Pi_n : U_n \to Q_{n-1}(U_{n-1})$$

  are unramified coverings with asymptotic values λ_0 and p. Lemma 6.1 implies there are infinitely many choices for a holomorphic isomorphism

  $$i_n : U_{n-1} \to U_n' \text{ satisfying } \Pi_n \circ i_n = Q_{n-1} \text{ on } U_{n-1}.$$

  Making one such choice (the choice doesn't matter) we define $Q_n : U_n \to U_n$ by $Q_n = i_n \circ \Pi_n$ so that

  $$Q_n \circ i_n = i_n \circ \Pi_n \circ i_n = i_n \circ Q_{n-1}.$$

  Therefore i_n conjugates Q_n to Q_{n-1} and the induction hypotheses are satisfied, completing the inductive step.
- The direct limit U_∞ of the system (U_n, i_n) is the quotient

  $$\cup_n U_n / \sim$$

  where the equivalence relation is defined by the identifications, $z \sim i_n(z)$, and the equivalence class is denoted by $[z]$. The Riemann surface U_∞ has infinite type. There is an infinite unramified holomorphic covering map Q_∞ defined by

$$Q_\infty([z]) = [Q_n(z)], \; z \in U_n$$

that has two omitted values $[\lambda_0]$ and $[p]$. It also fixes $[q_0]$ and since the maps Π_n and i_n are holomorphic we have $Q'_\infty([q_0]) = \rho$.

Topologically U_∞ is the complement in $\mathbb{C}$ of a Cantor set C isomorphic to the space of infinite sequences $\Sigma_\infty = s_1, \ldots s_{n-1}, s_n \ldots, s_j \in \mathbb{Z}$ together with the finite sequences $\Sigma_{n+1} = s_1, \ldots, s_n, \infty$ of length $n+1$. The map Q_∞ is conjugate to the shift map on C. See [Mo].

The final step of the proof is to show there is a conformal embedding $e : U_\infty \to \mathbb{C}$ such that

$$e \circ Q_\infty = f_\lambda \circ e$$

for some $\lambda \in \mathcal{S}^0_\lambda$ with $\xi_\lambda(\lambda) = \lambda_0$. To do this, we first give a brief informal review the results we need from Teichmüller theory and the theory of mapping classes of tori and punctured tori. We refer the reader to [Bir] for a full discussion and [GK] for a discussion analogous to what we need here.

6.2.3. *Teichmüller theory.* Fix $\lambda \in \mathcal{S}^0_\lambda$ and set $f = f_\lambda$.

Definition 8. *Let $QC(f)$ be the set of quasiconformal maps $h : \mathbb{C} \to \mathbb{C}$ such that $g = h \circ f \circ h^{-1}$ is meromorphic. Since g is a meromorphic infinite to one unbranched cover of the plane with two omitted values, by the corollary to Nevanlinna's theorem, corollary 2.2 , it is affine conjugate to a map $f_{\lambda'} \in \mathcal{F}_2$ and we choose the conjugacy so that λ' is the preferred asymptotic value.*

We define the Teichmüller equivalence relation on $QC(f)$ as follows: elements h_1, h_2 of $QC(f)$ are equivalent if there is an affine map a and an isotopy from h_1 to $a \circ h_2$ through elements of $QC(f)$. The quotient space of $QC(f)$ by this equivalence relation is called the Teichmüller space $Teich(f)$ with basepoint f.

Let $QC_0(f)$ denote the elements of $QC(f)$ that conjugate f to itself and $QC^*_0(f)$ those that preserve the marking of the asymptotic values.

Definition 9. *The* mapping class group, *$MCG(f)$, is the quotient of $QC_0(f)$ by the Teichmüller equivalence relation and the* pure mapping class group, *$MCG^*(f)$, is the quotient of $QC^*_0(f)$ by the Teichmüller equivalence relation. The* moduli space *and* pure moduli space *are defined as the quotients* $\mathrm{M}(f) = Teich(f)/MCG(f)$ *and* $\mathrm{M}^*(f) = Teich(f)/MCG^*(f)$.

Remark 6.4. *Because we are working in a dynamically natural slice of $\mathcal{F}_2$ defined by the conditions that* 0 *is fixed and has multiplier a fixed ρ, we restrict our considerations here to the slice $Teich(f, \rho) \subset Teich(f)$ of equivalence classes of quasiconfomal maps h such that $h \circ f \circ h^{-1}$ has a fixed point with multiplier ρ. The mapping class group and pure mapping class group act on $Teich(f, \rho)$. The λ parameter plane is identified with the pure moduli space*

$\mathrm{M}^*(f, \rho)$. *For readability below, since we always assume we are in this slice, we drop the* ρ *from the notation.*

Since f_λ is hyperbolic, its Teichmüller space $Teich(f_\lambda)$ contains $\mathcal{S}^0_\lambda$.

Because the quasiconformal maps conjugate the dynamics, and the dynamics are controlled by the orbits of the asymptotic values, the space $Teich(f_\lambda)$ is related to the Teichmüller space of a twice punctured torus defined by the dynamics. We explain this here.

Definition 10. *The points* z_1, z_2 *are* grand orbit equivalent *if there are integers* $m, n \geq 0$ *such that* $f^m_\lambda(z_1) = f^n_\lambda(z_2)$. *They are* small orbit equivalent *if for some* $n > 0$, $f^n_\lambda(z_1) = f^n_\lambda(z_2)$. *Denote the grand orbit equivalence classes by* $[z]$.

Now $\phi_\lambda(z) = 0$ if and only if z is grand orbit equivalent to 0. Let $\widehat{A_\lambda}$ denote the complement of the grand orbit of 0 in A_λ. We have

Lemma 6.6. *The restriction of* ϕ_λ *to* $\widehat{A_\lambda}$ *is a well defined map from each small equivalence class to a point in* $\mathbb{C}^*$.

Proof. If z_1, z_2 are small orbit equivalent there is some integer N such that for all $n \geq N$, $f^n_\lambda(z_1) = f^n_\lambda(z_2)$. Moreover, for all large n, $f^n_\lambda(z) \in O_\lambda$ and, since ϕ_λ is injective on O_λ, the lemma follows. □

Let Γ_ρ be the group generated by $z \mapsto \rho z$ in $\mathbb{C}^*$. The projection $\tau_\rho : \mathbb{C}^* \to \mathbb{C}^*/\Gamma_\rho = T_\rho$ is a holomorphic covering map onto a torus T_ρ. Following common usage, we say that its modulus is ρ. Set $T = T_\rho$ since ρ is fixed in this discussion.

Define the composition of ϕ_λ with τ_ρ by

$$\Phi_\lambda : \widehat{A_\lambda} \xrightarrow{\phi_\lambda} \mathbb{C}^* \xrightarrow{\tau_\rho} T.$$

By lemma 6.6, we see that Φ_λ identifies $\widehat{A_\lambda}$ in the dynamical space of f_λ with the torus T because each grand orbit in $\widehat{A_\lambda}$ maps to a unique point on T. Notice that T depends only on ρ and not on λ. Let γ^* be the level curve through the asymptotic value μ in $\widehat{A_\lambda}$, and β its projection on T.

Since $\lambda \in \mathcal{S}^0_\lambda$, the orbit of μ accumulates on 0 so it cannot be in the grand orbit of 0. It is possible that λ is in the grand orbit of zero, or that for some m, n, $f^n_\lambda(\lambda) = f^m_\lambda(\mu)$. This can happen only on a discrete set and we assume here that is does not happen for the λ we chose.

There are two special points on T, the points $\lambda^* = \Phi_\lambda(\lambda)$ and $\mu^* = \Phi_\lambda(\mu)$. We mark them so that λ^* is the preferred point. Let $T^2_\lambda = T \setminus \{\lambda^*, \mu^*\}$. Let $A^*_\lambda = \Phi^{-1}_\lambda(T^2_\lambda)$; then $A^*_\lambda \subset A_\lambda$ is the complement of the grand orbits of 0 and the asymptotic values. It is easy to see that $\Phi_\lambda : A^*_\lambda \to T^2_\lambda$ is a covering projection.

The Teichmüller space $Teich(T^2_\lambda)$ is defined as the set of equivalence classes of quasiconformal maps, $[H]$, defined on T^2_λ, where, as above, the equivalence is through isotopy. The pure mapping class group $MCG_*(T^2)$ and pure moduli space $\mathrm{M}^*(T^2)$ based at T^2_λ are defined as for $Teich(f)$: the pure mapping class group consists of equivalence classes $[H]$ that map T^2_λ to itself preserving the marking and the pure moduli space is formed by identifying points congruent under the pure mapping class group. Thus the map Φ_λ induces a map $\Psi : Teich(f) \to Teich(T^2)$. By standard arguments, see e.g. [McMSul], Ψ is a covering map so there is an injection on fundamental groups which translates to an injection of pure mapping class groups:

$$\Psi_* : MCG_*(f) \to MCG_*(T^2).$$

Since a quasiconformal map $H \in Teich(T^2_\lambda)$ is not necessarily the projection by Φ_λ of an h defined on A^*_λ, we need to characterize those that are. To do this, we need to understand the image $\Psi_*(MCG_*(f)) \subset MCG_*(T^2)$.

First of all, to remain in the slice, we require that $\omega(H(T^2_\lambda))$, the torus obtained by applying the "forgetful map" ω that fills in the punctures, is conformally equivalent to T and preserves the isotopy class of β.

Suppose $\tilde{\alpha}$ is a curve in A^*_λ with initial point μ and endpoint λ and $[h] \in MCG_*(f)$. Then $h(\tilde{\alpha})$ has the same property. The map $H = \Phi_\lambda \circ h \circ \Phi_\lambda^{-1}$ determines a point in $MCG_*(T^2)$ that maps the curve α^* on T^2_λ joining μ^* to λ^* to a curve $H(\alpha^*)$ with the same endpoints.

Every curve α' on T^2 that joins μ^* to λ^* has lifts $\Phi_\lambda^{-1}(\alpha')$ whose initial point is at a preimage of μ^*; let $\tilde{\alpha}'$ be the lift at the asymptotic value μ. The endpoint of $\tilde{\alpha}$ is in the grand orbit of λ, but it isn't necessarily at λ. Therefore, in order to construct maps in $Teich(f)$ from maps in $Teich(T^2)$, which we do below, we need to know that we can find those curves α whose lift to μ lands at λ. Let α^* be such a curve on T^2.

That we can always find these curves is proved in [Bir] where there is a full treatment of mapping class groups of surfaces. For a more detailed discussion analogous to the situation here see [GK].

In figure 10 we show how the region A_λ is divided into fundamental domains that project to T for two different values of λ. In both, μ is on γ^*, the boundary of O_λ, drawn in blue. The orange curves are the first pullbacks of γ^* by f_λ. The domain bounded by γ^* and one of the orange curves defines a fundamental domain for Γ_ρ. In the left figure, λ is in that fundamental domain. The green curves are the next pullback, and on the right figure, the red curve is the third pullback and λ is in a fundamental domain between the second and third pullbacks.

6.2.4. *Construction of the embedding e.* We now construct the conformal embedding $e : U_\infty \to \mathbb{C}$ such that

$$e \circ Q_\infty = f_\lambda \circ e$$

for some $\lambda \in \mathcal{S}^0_\lambda$ with $\xi_\lambda(\mu) = \lambda_0$.

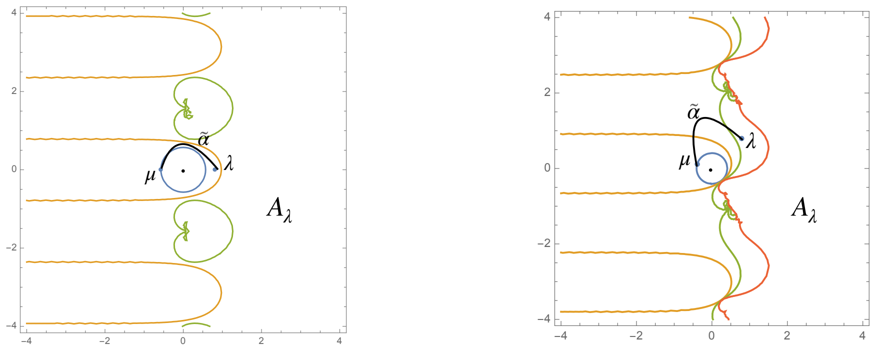

FIGURE 10. Two examples where the lifted curve is the one we need.

Delete the grand orbits of $[q_0], [\lambda_0]$ and $[p]$ from U_∞ to obtain a domain U_∞^*. As we did above for A_λ, we form the projection by the grand orbit equivalence

$$\Phi_\infty : U_\infty^* \to T_\infty^2 = T \setminus \{\Phi_\infty(p), \Phi_\infty(\lambda_0)\}$$

where again, T is a torus of modulus ρ.

As above, there is some α_∞ that is a curve on T_∞^2 with initial point $\Phi_\infty(\lambda_0)$ and endpoint $\Phi_\infty(p)$ whose lift to Q_∞ at λ_0 is a curve $\tilde{\alpha}_\infty$ joining λ_0 to p.

Let $H : T_\lambda^2 \to T_\infty^2$ be an orientation preserving homeomorphism that preserves the labeling of the punctures and satisfies $H(\alpha_*) = \alpha_\infty$. Then it lifts to a topological conjugacy h between $f_\lambda|A_\lambda$ and Q_∞.

$$\begin{tikzcd}
U_\infty \arrow[d, "Q_\infty"] & A_\lambda \arrow[l, "h"'] \arrow[r, "g"] \arrow[d, "f_\lambda"] & A_{\lambda(p)} \arrow[d, "f_{\lambda(p)}"] \\
U_\infty \arrow[d, "\Phi_\infty"] & A_\lambda \arrow[l, "h"'] \arrow[r, "g"] \arrow[d, "\Phi_*"] & A_{\lambda(p)} \\
T_\infty^2 & T_\lambda^2 \arrow[l, "H"'] &
\end{tikzcd}$$

We may assume that H is quasiconformal with Beltrami differential $\nu_{T_\lambda^2}$. and use Φ_* to lift to a Beltrami differential ν on A_λ compatible with the dynamics. We set $\nu = 0$ on the complement of A_λ (the Julia set of f_λ), and note that because the map is hyperbolic, this set has measure zero. We now invoke the measurable Riemann mapping theorem, [AB], to obtain a quasiconformal homeomorphism $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0 and ∞, and so unique up to scale, such that $g \circ f_\lambda \circ g^{-1}$ is holomorphic. By Nevanlinna's theorem, theorem 2.1, we can assume g is normalized so that $g \circ f_\lambda \circ g^{-1}$ is of the form $f_{\lambda(p),\rho(p)}$ for some $\lambda(p)$ where $\lambda(p) = g(\lambda)$ and $\mu(p) = g(\mu)$ is on the boundary of $O_{\lambda(p)}$, the region of injectivity of the uniformizing map at the origin. Since g is compatible with the dynamics, and both tori T^2_λ and T^2_∞ have modulus ρ; it follows that $g'(0) = \rho$ also. Thus $\lambda(p) \in \mathcal{S}^0_\lambda$ and the map $e = g \circ h^{-1}$ is the required embedding.

To complete the proof we need to show that the correspondence $p \to \lambda(p)$ is an inverse of the map E.

$$\begin{tikzcd}
K_0 & & \\
U_0 \arrow[u, hook] \arrow[r, "i_\infty"] & U_\infty \arrow[r, "e"] & A_{\lambda(p)} \subset \mathbb{C} \arrow[ull, "\xi_{\lambda(p)}"']
\end{tikzcd}$$

By our construction, i_∞ is the direct limit of the maps i_n. It satisfies

$$e \circ i_\infty(p) = \lambda(p).$$

The second asymptotic value of $f_{\lambda(p)}$ is $\mu(p)$. By definition, $\xi_\lambda(\mu(\lambda(p))) = \lambda_0$ and $\xi_\lambda(\lambda(p)) = p \in U_0 \subset K_0$.

In the non-generic cases, the point p in $K_0 \setminus \Delta$ is either in the grand orbit of the fixed point q_0 or the other asymptotic value λ_0 and the quotient of U_∞ by the grand orbit relation is a once punctured torus. The construction of the inverse of E is analogous, but simpler in these cases and again yields a unique $f_\lambda \in \mathcal{S}^0_\lambda$.

If we choose p on $\partial\Delta$, the function $f_{\lambda(p)}$ will have both its asymptotic values on the boundary of $O_{\lambda(p)}$. Only one choice, however, preserves the marking.

By the Measurable Riemann Mapping Theorem, the quasiconformal map g depends holomorphically on the parameter p. Thus, as we vary p analytically along $\partial\Delta \setminus \{\lambda_0\}$, the image $e(p)$ defines an analytic curve $\mathcal{S}_*$ in $\mathcal{S}$. The construction fails if $p = \lambda_0$ because as p approaches λ_0, the limit point on the analytic curve in $\mathcal{S}$ is a parameter singularity; in the construction of f_λ from the model, as $p \to \lambda_0$, $\lambda \to 0$. Therefore we can extend E^{-1} by continuity so that $E(0) = \lambda_0$; therefore $\mathcal{S}_* \cup \{0\}$ is homeomorphic to a circle.

Since the model $K_0 \setminus \Delta$ is topologically an annulus $\mathbb{A}$, the above paragraph shows that E extends as a map from the boundary component $\mathcal{S}_*$ of $\mathcal{S}_\lambda$ to a boundary component of $\mathbb{A}$.

6.3. **Topology of the shift locus.** We are now ready to complete the proof of the Main Structure Theorem.

FIGURE 11. The λ plane with the regions $\mathcal{M}_\lambda$, $\mathcal{M}_\mu$ and the circle of inversion.

Theorem 6.7 (Topology of the shift Locus)**.** *$\mathcal{S}$ is homeomorphic to a punctured annulus; that is, there is a homeomorphism $\Phi : \mathcal{S} \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$.*

Proof. We begin by recalling the relation between $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ given by the inversion $I(\lambda) = -\mu$ defined in section 3.2 that shows

$$f_\lambda(z) = f_{-\mu}(-z).$$

It follows that if $\lambda \in \mathcal{M}_\lambda$ and $f_\lambda^m(\lambda)$ tends to a periodic orbit $\mathbf{z} = \{z_0, z_1, \ldots, z_n\}$ then $f_{-\mu}^m(-\mu)$ tends to the orbit $-\mathbf{z} = \{-z_0, -z_1, \ldots, -z_n\}$ and $-\mu \in \mathcal{M}_\mu$. This proves

Proposition 6.8. *The inversion $I : \mathcal{M}_\lambda \to \mathcal{M}_\mu$ defined by*

$$I(\lambda) = -\mu = \frac{\lambda}{2\lambda/\rho - 1}$$

maps shell components of period n in $\mathcal{M}_\lambda$ to shell components of period n in $\mathcal{M}_\mu$.

This is illustrated in figure 11. The large green region is the shift locus, $\mathcal{M}_\lambda$ is the complementary region on the right and $\mathcal{M}_\mu$ is the complementary region to the left, surrounded by the shift locus. The circle of inversion is drawn in figure 11 where $\rho = 2/3$. In this figure, since ρ is real, by proposition 6.3, it is $\mathcal{S}_*$. For arbitrary fixed ρ, $\mathcal{S}$ is the image of $\partial\Delta \setminus \{\lambda_0\}$ under E^{-1}; thus $\mathcal{S}_* \cup \{0\}$, which we still denote as $\mathcal{S}_*$, is a topological circle.

In theorem 6.4 we saw that $\mathcal{S}_\lambda$ is homeomorphic to an annulus. One of the complementary components is $\mathcal{M}_\lambda$. The other complementary component is bounded by the curve $\mathcal{S}_*$. By proposition 6.8, I maps $\mathcal{S}_\lambda \cup \mathcal{M}_\lambda$ to $\mathcal{S}_\mu \cup \mathcal{M}_\mu$; since $I(\mathcal{M}_\lambda) = \mathcal{M}_\mu$, $I(\mathcal{S}_\lambda) = \mathcal{S}_\mu$ so that $\mathcal{S}_\mu$ is also an annulus. Because I maps $\mathcal{S}_*$ to itself, these annuli share a common boundary component.

Note that although both the circle of inversion and $\mathcal{S}_*$ are invariant under inversion, unless ρ is real, they are not necessarily the same.

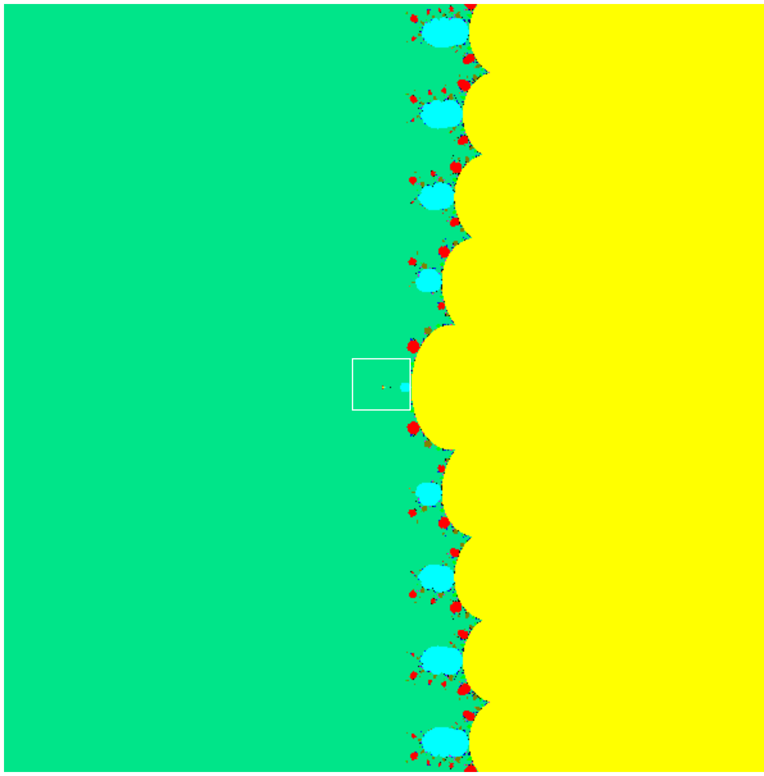

FIGURE 12. The λ plane when $\rho = -2/3$. Note the position of the period 2 components.

Therefore $\mathcal{S} \cup \{0\} = \mathcal{S}_\lambda \cup \mathcal{S}_\mu \cup \mathcal{S}_* \cup \{0\}$ is topologically an annulus. Removing the parameter singularity $\lambda = 0$ completes the proof. □

Immediate corollaries of this theorem are:

Corollary 6.9. *The sets $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$ are connected.*

Corollary 6.10. *The full shift locus in $\mathcal{F}_2$ has the product structure $\mathbb{D}^* \times \mathbb{C} \setminus \{0, 1\}$.*

Figure(12) shows the λ plane when $\rho = -2/3$. This is another slice in the fibration and shows how the fibers change as the argument of ρ changes. The picture is similar to figure (2) except that we see that the $\mathcal{M}_\lambda$ is translated vertically and there is a period 2 component budding off Ω_1 on the real axis instead of a cusp.

6.4. **Single valued inverse branches.** We now prove the lemma we assumed for the proof of the Combinatorial Structure Theorem in section 4.2

Lemma 6.11. *There is a simply connected domain $\Sigma \in \mathbb{C} \setminus \{0, \rho/2\}$ in which, after a choice of basepoint and branch of the logarithm, the pole functions $p_k(\lambda)$ and the inverse branches $g_{\lambda,k}$ can be defined as single valued functions of λ.*

Proof. In the proof of theorem 6.7 we showed that $\mathcal{S}_\mu$ and $\mathcal{S}_\lambda$ are homeomorphic to annuli with a common boundary component that contains the singularity at the origin. It follows that in a neighborhood of the origin both asymptotic values are attracted to zero.

Now consider the period one shell component Ω_1' of $\mathcal{M}_\mu$. It has a virtual center at $\lambda = \rho/2$. Since it is a virtual center, it is on the boundary of both $\mathcal{M}_\mu$ and $\mathcal{S}$ and so a neighborhood V of $\rho/2$ contains points in $\mathcal{S}_\mu$.

Applying the inversion, $I(V)$ is a neighborhood of infinity intersecting the period one component $\Omega_1 \in \mathcal{S}_\lambda$ and an open set in $\mathcal{S}_\lambda$. Thus infinity is on the boundary $\mathcal{S}_\lambda$. Since a neighborhood of any point in $\mathcal{S}_*$ only contains points in $\mathcal{S}_\lambda$ and $\mathcal{S}_\mu$, infinity and 0 are on different boundary components. Hence because $\mathcal{S}_\lambda$ is an annulus, we can find a curve $\gamma \subset \mathcal{S}_\lambda$ joining 0 and infinity. Let W be the component of $\mathbb{C} \setminus \mathcal{S}_*$ containing $\mathcal{M}_\mu$ and set $\Sigma = \mathbb{C} \setminus (W \cup \gamma)$. This is a simply connected domain. It contains all the virtual cycle parameters belonging to $\mathcal{M}_\lambda$ and none of the virtual cycle parameters belonging to $\mathcal{M}_\mu$. Therefore, choosing a basepoint $\lambda_0 \in \Sigma$, and a branch for Log, we can define $p_k(\lambda_0)$ as in equation (5) by

$$p_k(\lambda_0) = \frac{1}{2}\mathrm{Log}(\frac{\rho - 2\lambda_0}{\rho}) + ik\pi,$$

and extend analytically to all of Σ as single valued functions of λ. Then, as we did in section 4.2, we can define the inverse branches of f_λ as single valued functions of λ. □

7. Concluding Remarks

There are many more questions one can address about the space of functions we have been studying. Below we list some of them and leave an investigation of them to future work.

- An important tool in studying the Mandelbrot set is the use of the level curves where the escape rate of the critical value is constant and their gradient "external rays". Can we define the analogue for the set $\mathcal{S}_\lambda$ and $\mathcal{S}_\mu$ using the level curves of ϕ_0 defined on K_0? There will be infinitely many curves for each level so the structure will be much more complicated. This would lead to more questions such as
  - i- In [CJK2], we used the level curves and their gradients to prove that the virtual centers are accessible points from inside both

the shell components and the shift locus. Can we also use it to characterize other types of boundary points of $\mathcal{S}$ such as cusps, root points for bud components or Misiurewicz points where an asymptotic value lands on a repelling cycle.

ii- Can we describe primitive and satellite components in terms of rays in a manner analogous to the discussion for rational maps.

iii- In [CJK] we showed there is a renormalization operator defined for the family $it \tan z$ where t is real. Are there renormalization operators that can be defined in $\mathcal{F}_2$?

- We know that at the virtual centers and Misiurewicz points the only Fatou component is the attracting basin of the origin. Is the Julia set a Cantor bouquet in the sense of Devaney? Does it have positive measure? area?
- In [GK] the mapping class group of the Teichmüller space Rat_2 is analyzed. The analogous space here is $\mathcal{F}_2$ from which points with orbit relations have been removed. Describe the mapping class group of this space.
- How do the results here extend to parameter spaces of families of meromorphic functions with more than two asymptotic values, or those with both critical values and asymptotic values.

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