CSc72010

Homework Answers (Thursday March 10)

1. Prove the following invariant for OptFloodMax:
For any round r and any i,j where j is an out-nbr of i, then if $max\text{-}uid_j < max\text{-}uid_i$ after r rounds, then new_i = true.

Proof by induction:
See the book, page 55.

2. Prove the following invariant for SynchBFS:
After d rounds, for $d \geq 1$, every process within d of the start process has a parent process.

To prove this, we prove a somewhat stronger invariant:
(*) After d rounds, for $d \geq 1$, a process is marked if and only if it is distance d or less from the start process.

Preliminary note: The definition of distance(i,j) for nodes i and j of a graph is the length of the shortest directed path from i to j.
If distance(i,j) is n, then j has an in-neighbor k such that distance(i,k) is n-1 and distance(k,j) is 1, i.e., k is an in-neighbor of j. This follows immediately from the definition by noticing that if a shortest path from i to j follows edges (i, i_1), (i_1, i_2), ..., (i_{n-1}, j) then i_{n-1} is an in-neighbor of j and the distance of i_{n-1} from i is n-1. (Clearly, the distance is at least n-1. If the distance were less than n-1, then there would be a shorter path to j.)

Proof of (*) by induction:
Induction base (d=1):
At the beginning of the first round, all processes are unmarked. Also, the start process sends out a search message to all of its out-neighbors.
Any process at distance 1 from the start process is an out-neighbor of the start process, so all processs at distance 1 receive a search message. Any unmarked process that receives a search message marks itself and chooses the start node as its parent.
If the distance of a process from the start node is greater than 1, then it does not receive a message during round 1 and thus does not become marked.
Therefore the statement holds for d=1.

Induction hypothesis: The invariant holds for $d < d_0$.

Induction step ($d = d_0$):
By the definition of distance in a directed graph, each process p at distance d_0 from the start process has at least one in-neighbor n at distance d_0-1 from the start process.
By the induction hypothesis, process n was first marked in round d_0-1 (in round d_0-2, it was not marked); it has a parent process; and at the end of round d_0-1 no process p at distance d_0 or greater has been marked.

Because process n was newly marked in round d_0-1, it sends a search message to all its out-neighbors at the beginning of round d_0. Since p is an out-neighbor of n, it receives the search message in round d_0, marks itself, and chooses a parent from all the in-neighbors sending it a search message.
Therefore, any process p at distance d_0 from the search node was marked during round d_0 and has a parent process.

By the induction hypothesis, no process p at distance $> d_0$ was marked prior to round d_0. Also, no process p at distance $> d_0$ receives a search message, because the only search messages are sent by processes at distance d_0-1, and these can only be received by processes at distance d_0 from the start node.