Here are some logic expressibility examples and problems

1. Suppose the intended domain is the rational numbers, the constant symbol **0** is intended to designate 0, the two place function symbol $\circ$ represents multiplication, and the two-place relation symbol L is intended to represent the less-than relation.

   (a) Write a formula saying that if x is negative (less than 0) and y is positive (greater than 0) then x is less than y. Solution:

   $$(\forall x)(\forall y)\{[L(x,\mathbf{0} \wedge L(\mathbf{0},y)] \supset L(x,y)\} \tag{1}$$

   (b) Write a formula saying the less than relation is transitive. Then use tableaus to show it implies (1). This is left to you.

   (c) Write a formula that says multiplying both sides of an inequality by a positive number preserves the inequality. Solution:

   $$(\forall x)(\forall y)(\forall z)\{[L(x,y) \wedge L(\mathbf{0},z)] \supset L(x \circ z, y \circ z)\}$$

   (d) Write a formula saying, in the intended model, between any two numbers there is another (the rationals are dense). Solution:

   $$(\forall x)(\forall y)\{L(x,y) \supset (\exists z)[L(x,z) \wedge L(z,y)]\} \tag{2}$$

   (e) Write a formula saying, in the intended model, between any two numbers there are two others. Solution:

   $$(\forall x)(\forall y)\{L(x,y) \supset (\exists z)(\exists w)[L(x,z) \wedge (L(z,w) \wedge L(w,y))]\} \tag{3}$$

   (f) Use tableaus to show formula (2) implies formula (3). This is also left to you.

Here are some tableau problems

1. Give, or attempt to give, tableau proofs of the following (they are not all theorems):

   (a) $(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)$

   (b) $(\forall x)(\exists y)R(x,y) \supset (\exists y)(\forall x)R(x,y)$

   (c) $(\exists x)[P(x) \supset (\forall x)P(x)]$

   (d) $(\exists x)[P(x) \vee Q(x)] \supset [(\exists x)P(x) \vee (\exists x)Q(x)]$

   (e) $(\exists x)[P(x) \wedge Q(x)] \supset [(\exists x)P(x) \wedge (\exists x)Q(x)]$

   (f) $[(\exists x)P(x) \wedge (\forall x)Q(x)] \supset (\exists x)[P(x) \wedge Q(x)]$

   (g) $(\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \vee \neg R(w,z)]$

   (h) $(\exists x)(\forall y)[P(y) \uparrow (P(x) \uparrow Q(x))] \subset (\forall x)Q(x)$

   (i) $(\forall x)[P(x) \supset Q] \supset [(\exists x)P(x) \supset Q]$ (where x does not occur free in Q)

   (j) $(\forall x)[P(x) \supset Q] \supset [(\forall x)P(x) \supset Q]$ (where x does not occur free in Q)

2. For convenience, we give the following names to sentences:

   $$\mathit{trans} = (\forall x)(\forall y)(\forall z)\{[R(x,y) \wedge R(y,z)] \supset R(x,z)\}$$

$sym = (\forall x)(\forall y)[R(x, y) \supset R(y, x)]$

$ref = (\forall x)R(x, x)$

$nontriv = (\forall x)(\exists y)R(x, y)$

(a) Show $(trans \wedge sym) \supset ref$ does not have a tableau proof, by producing a model in which *trans* and *sym* are true but *ref* is not.

(b) Give a tableau proof of $(trans \wedge sym \wedge nontriv) \supset ref$.

3. Prove the following using tableaus:

(a) $(\exists x)(\forall y)(\forall z)[(P(y) \supset Q(z)) \supset (P(x) \supset Q(x))]$

(b) $(\exists x)(\forall y)(\forall z)(\forall w)[(P(y) \vee Q(z) \vee R(w)) \supset (P(x) \vee Q(x) \vee R(x))]$