Corrections to

Set Theory and the Continuum Problem (revised edition)

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These are corrections to the edition published by Dover in 2010.

Page 23 (Error found by David Feuer) Exercise 5.5(d) in Chapter 2 asks the reader to show
\[ B - (A - B) = \emptyset. \] It should be to show \[ B - (A - B) = B. \]

Page 23 (Error found by David Feuer) Text in §6 reads \[ A_6 \] [Power set axiom]. It should read \[ A_6 \] [Power set axiom].

Page 51 (Error found by Grigory Olkhovikov) Exercise 1.2 is incorrect as asked. Take \( A \) to be the
set of negative integers under the natural linear order \( \leq \). Every proper lower section of \( A \) has
a strict upper bound, but its ordering is not a well ordering. The exercise can be corrected
by including the condition that \( A \) has a least element.

Page 60 (Error found by Allen David Boozer) Exercise 4.3 should be deleted. There is a reference
to this Exercise in the Remark at the top of page 63, and this reference should also be deleted.

Page 66 (Error found by Stuart Newberger) Definition 7.1 is incorrect as stated. It should read
as follows.
For any sets \( y \) and \( x \), we will say that \( y \) is closed (under \( g \)) relative to \( x \) provided, for any
\( z \in y \), if \( g(z) \in \mathcal{P}(x) \) then \( g(z) \in y \). (Thus \( (z \in y \land g(z) \subseteq x) \supset g(z) \in y. \))

Page 225-226 (Error found by Chang Soon Choi.) In the Remarks, it is not the case that
\( (f \approx_{\lambda} g) \supset [f \approx_{\lambda} g] \) is S4 valid generally, but it is valid in the particular S4 models
being constructed. Here is the argument. Suppose as an induction hypothesis that it is
known for ordinals less than \( \lambda \). It follows from the definition of \( \approx_{\lambda} \) that if \( \alpha < \lambda \) then
\( (f \approx_{\alpha} g) \supset (f \approx_{\lambda} g) \) is valid in these models. It follows from this, using general S4 reasoning,
that \( \Box \Diamond (f \approx_{\alpha} g) \supset \Box \Diamond (f \approx_{\lambda} g) \) is also valid in these models, that is, \([f \approx_{\alpha} g] \supset [f \approx_{\lambda} g]\). Now if \( p \models (f \approx_{\lambda} g) \), then \( p \models (f \approx_{\alpha} g) \) for some \( \alpha < \lambda \) (by definition). This implies
\( p \models [f \approx_{\alpha} g] \), and hence \( p \models [f \approx_{\lambda} g] \).

Page 227 (Error found by Chang Soon Choi.) In the proof of Proposition 1.5, the limit ordinal
case is incorrect. It uses an inference from \( p \models [f \approx_{\lambda} g] \), where \( \lambda \) is a limit ordinal, to
\( p \models [f \approx_{\alpha} g] \), for some \( \alpha < \lambda \), and this is not justified. Replace the limit ordinal case by the
following.
Assume \( \lambda \) is a limit ordinal and every \( \alpha < \lambda \) is good. Now suppose \( p \models [f \approx_{\lambda} g] \) and \( \lambda < \beta \);
we must show \( p \models [f \approx_{\beta} g] \).
We first show \((f \approx_{\lambda} g) \supset \[[f \approx_{\beta} g]]\) is valid in the model that has been constructed (note the absence of double square brackets in the antecedent). Well, suppose \(q \models (f \approx_{\lambda} g)\). Then \(q \models (f \approx_{\alpha} g)\) for some \(\alpha < \lambda\), by definition of \(\approx_{\lambda}\). It follows by the Remarks at the bottom of page 225 and the top of 226 that \(q \models [f \approx_{\alpha} g]\). Since \(\alpha\) is good, \(q \models [f \approx_{\beta} g]\). Since \(q\) was arbitrary, we have shown the validity of \((f \approx_{\lambda} g) \supset [f \approx_{\beta} g]\) in the model.

It now follows, by standard modal manipulations, that \(\Box \Diamond (f \approx_{\lambda} g) \supset \Box \Diamond [f \approx_{\beta} g]\) is also valid in the model, and hence we have the validity of \([f \approx_{\lambda} g] \supset [f \approx_{\beta} g]\), making use of Proposition 4.3, part 2. Since \(p \models [f \approx_{\lambda} g]\), then \(p \models [f \approx_{\beta} g]\).

Page 228 (Found by Grigori Mints) In the Remark just before Definition 1.8 it is asserted that \((f \in g) \equiv [f \in g]\). The equivalence is not correct, but \((f \in g) \supset [f \in g]\) is.

Page 229 (Problem found by Chang Soon Choi.) Lemma 2.1 says that if \(p \models [f \approx_{\alpha} g]\) and \(p \models [g \approx_{\lambda} h]\) then \(p \models [f \approx_{\alpha} h]\). The proof is by induction on \(\alpha\). It begins by saying the cases where \(\alpha\) is 0 or a limit ordinal are simple. In fact 0 is simple, but the limit ordinal case is not. Here is a proof for the limit ordinal case.

Let \(\lambda\) be a limit ordinal and assume the result holds for smaller ordinals. We begin by showing that if \(p \models (f \approx_{\lambda} g)\) and \(p \models [g \approx_{\lambda} h]\) then \(p \models [f \approx_{\lambda} h]\) (note the difference in the first item). So, suppose \(p \models (f \approx_{\lambda} g)\) and \(p \models [g \approx_{\lambda} h]\). To show \(p \models [f \approx_{\lambda} h]\) we show \(p \models \Box \Diamond (f \approx_{\lambda} h)\). Let \(q\) be any member of \(\mathcal{G}\) such that \(p R q\); we must show there is some \(r\) with \(q R r\) so that \(r \models (f \approx_{\lambda} h)\).

Since \(p \models \Box \Diamond (g \approx_{\lambda} h)\) then \(q \models \Diamond (g \approx_{\lambda} h)\) and hence there is some \(r\) with \(q R r\) so that \(r \models (g \approx_{\lambda} h)\). By definition, \(r \models (g \approx_{\alpha} h)\) for some \(\alpha < \lambda\). Without loss of generality we can assume \(\alpha\) is a successor ordinal. Then \(r \models [g \approx_{\alpha} h]\) by the remarks on pages 225-226.

Since \(p \models (f \approx_{\lambda} g)\) then \(p \models (f \approx_{\beta} g)\) for some \(\beta < \lambda\) and again without loss of generality we can assume \(\beta\) is a successor ordinal. Then \(p \models [f \approx_{\beta} g]\) by the remarks on pages 225-226 again. Since this formula begins with \(\Box\), \(r \models [f \approx_{\beta} g]\). Let \(\gamma\) be the larger of \(\alpha\) and \(\beta\). By Proposition 1.5, \(r \models [f \approx_{\gamma} g]\) and \(r \models [g \approx_{\gamma} h]\). Since \(\gamma < \lambda\), by the induction hypothesis for the overall Lemma, \(r \models [f \approx_{\gamma} h]\). Since \(\gamma\) is a successor ordinal, by the remarks on pages 225-226 again, \(r \models (f \approx_{\gamma} h)\), which is what we wanted.

Since \(p\) was arbitrary, we have shown the validity in our model of

\((f \approx_{\lambda} g) \supset ([g \approx_{\lambda} h] \supset [f \approx_{\lambda} h]).\)

Then by standard S4 manipulations, this gives us the validity in our model of

\(\Box \Diamond (f \approx_{\lambda} g) \supset \Box \Diamond ([g \approx_{\lambda} h] \supset [f \approx_{\lambda} h]).\)

By Proposition 4.4 of Chapter 16 we then have

\(\Box \Diamond (f \approx_{\lambda} g) \supset [g \approx_{\lambda} h \supset f \approx_{\lambda} h]\)

and hence

\([f \approx_{\lambda} g] \supset ([g \approx_{\lambda} h] \supset [f \approx_{\lambda} h])\)

by using Proposition 4.5 of Chapter 16.

Page 234 (Problem found by Chang Soon Choi.) In line 8 of the proof of Proposition 3.3, “equivalently, \([s \in t]\)” should be changed to “and so \([s \in t]\)”. Also in line 11 of the same proof, “But then \(p \models (a \in t)\), so \(a\) is \(\hat{x} \ldots\)” should be changed to “So \(a\) is \(\hat{x} \ldots\)”. 

Pages 239–240 (Problem found by Chang Soon Choi.) In the proof of Lemma 5.2 it is said that “(Recall that $x \approx_\alpha y$ and $[x \approx_\alpha y]$ are equivalent.)” This is not the case. One should modify the condition that we need to express by a first-order formula so that the last part reads $\square \Diamond (x \approx_\alpha y)$. Then, in the formula following “Now let $F(\mathcal{A}, p, f, g)$ be the formula:” the final clause should be changed from “$(s', x, y) \in \mathcal{A}$” to “$(r', x, y) \in \mathcal{A}$”.

Page 241 (Problem found by Chang Soon Choi.) In the proof of Theorem 5.4, the atomic case should be modified. We know that $p \models \big[ f \in g \big]$ iff $p \models \big[ (\exists w)(w \approx f \land w \in g) \big]$ iff $p \models \square \Diamond (\big[ w \approx f \big] \land \big[ w \in g \big])$, so in the atomic case, $F_\varphi(z, x, y)$ should be

$$(\forall z'\vec{R}z)(\exists z''\vec{R}z')(\exists w \in D)(\mathsf{Equals}(z'', w, x) \land (\forall z'''\vec{R}z''')(\exists z''''\vec{R}z''') (z'''', w) \in y)$$

Also in the final part of the proof take $F_\varphi(z, x_1, \ldots, x_n)$ to be the following:

$$(\forall z'\vec{R}z)(\exists z''\vec{R}z') \neg F_\varphi(z'', x_1, \ldots, x_n).$$

Page 264 (Problem found by Jason Parker) The remarks at the end of the first paragraph are incorrect. First, a few useful observations: using Definition 3.1 on page 233, one has the following.

$$\hat{0} = 0$$
$$\hat{1} = \mathcal{G} \times \{\hat{0}\}$$
$$\hat{2} = \mathcal{G} \times \{\hat{0}, \hat{1}\}$$
$$\hat{3} = \mathcal{G} \times \{\hat{0}, \hat{1}, \hat{2}\}$$
$$\vdots$$
$$\hat{\omega} = \mathcal{G} \times \{\hat{0}, \hat{1}, \hat{2}, \ldots\}$$

Now, here is Parker’s argument.

“It is claimed that we can show that if $p \models [a \subseteq \hat{\omega}]$, then for some $b \subseteq \mathcal{G} \times \hat{\omega}$, $p \models [a \approx b]$. But this would entail that $b \in D^\varphi$, which does not seem possible. For suppose $b \subseteq \mathcal{G} \times \hat{\omega}$. If $b$ were a member of $D^\varphi$, then $b \in R^\varphi_{\alpha+1}$ for some least ordinal $\alpha$. Then $b \subseteq \mathcal{G} \times R^\varphi_\alpha$. Now since $b \subseteq \mathcal{G} \times \hat{\omega}$, it follows that any $x \in b$ is of the form $\langle p, \langle q, \hat{n} \rangle \rangle$ for some $p, q \in \mathcal{G}$ and $n \in \omega$. So $\langle q, \hat{n} \rangle \in R^\varphi_\alpha$. So there is some least ordinal $\beta$ such that $\langle q, \hat{n} \rangle \in R^\varphi_\beta$, whereby $\langle q, \hat{n} \rangle \subseteq \mathcal{G} \times R^\varphi_\beta$, which is clearly false. So it seems that it cannot be that $b \in D^\varphi$ if $b \subseteq \mathcal{G} \times \hat{\omega}$.”

The problem sentences at the end of paragraph 1, page 264, should be replaced with the following. “Now, this result can be improved, to establish that if $[a \subseteq \hat{\omega}]$ is true at $p$ then $[a \approx b]$ is true at $p$ for some $b \subseteq \mathcal{G} \times \{\hat{n} | n \in \omega\}$ (equivalently, for some $b \subseteq \hat{\omega}$). Consequently, to investigate the size of the power set of $\hat{\omega}$ in the modal model, we begin by investigating the actual power set of $\hat{\omega}$.”

Here is the argument for the revised assertion above. Throughout, assume that $p \models [a \subseteq \hat{\omega}]$, meaning $p \models [((\forall x)(x \in a \supset x \in \hat{\omega})]$

1. If $pR^p'$ and $p' \models [x \in a]$, then for some $p''$ with $p'R^p''$, $p'' \models [x \approx \hat{n} \land \hat{n} \in a]$ for some $n \in \omega$. 
Proof: Suppose \( pRp' \), and \( p' \models [x \in a] \). Then \( p' \models [x \in \hat{\omega}] \), and so for some \( p'' \) with \( p''Rp'' \), \( p'' \models x \in \hat{\omega} \), and hence for some \( h \), \( p'' \models [x \approx h] \) and \( p'' \models [h \in \hat{\omega}] \) (Definition 1.6 Chapter 17). Then for some \( p''' \) with \( p'''Rp''' \), \( p''' \models h \in \hat{\omega} \), and so \( \langle p'''', h \rangle \in \hat{\omega} = \mathcal{G} \times \{0,1,\ldots\} \). Then \( h = \hat{n} \) for some \( n \in \omega \). It follows that \( p''' \models [x \approx \hat{n}] \) and \( p''' \models [\hat{n} \in a] \).

2. Now let \( b = \{ \langle q, \hat{n} \rangle \mid n \in \omega, q \models [\hat{n} \in a] \} \). Trivially \( b \subseteq \mathcal{G} \times \{0,1,\ldots\} = \hat{\omega} \).

3. \( p \models [b \subseteq a] \). The proof is by contradiction. Suppose not; then for some \( h \) and for some \( p' \) with \( p'Rp' \), \( p' \models [h \in a] \) and \( p' \models [\neg(h \in b)] \) (\( P_9 \), Page 226). By item 1, for some \( p'' \) with \( p''Rp'' \), \( p'' \models [h \approx \hat{n}] \) and \( p'' \models [\hat{n} \in a] \) for some \( n \in \omega \). Let \( q \) be an arbitrary member of \( \mathcal{G} \) with \( p''Rq \). Then \( q \models [\hat{n} \in a] \), hence \( \langle q, \hat{n} \rangle \in b \), and so \( q \models [\hat{n} \in b] \). Since \( q \) was arbitrary, \( p'' \models \Box(\hat{n} \in b) \), and so \( p'' \models \Box(\hat{n} \in b) \), or \( p'' \models [\hat{n} \in b] \). Then \( p'' \models [\hat{n} \in b] \) (Corollary 2.4, Chapter 17). But we also have \( p'' \models [\neg(\hat{n} \in b)] \), and this is our contradiction.

4. \( p \models [b \subseteq a] \). Again the proof is by contradiction. If not, then for some \( h \) and for some \( p' \) with \( p'Rp' \), \( p' \models [h \in b] \) and \( p' \models [\neg(h \in a)] \). Then for some \( p'' \) with \( p''Rp'' \), \( p'' \models h \in b \) and hence for some \( k \), \( p'' \models [h \approx k] \) and \( p'' \models [k \in b] \). Then for some \( p''' \) with \( p'''Rp''' \), \( p''' \models k \in b \). But then \( (p''', k) \in b \), and so \( k = \hat{n} \) for some \( n \in \omega \), and \( p''' \models [\hat{n} \in a] \) (definition of \( b \)). We also have \( p''' \models [h \approx \hat{n}] \), and it follows that \( p''' \models [\neg(\hat{n} \in a)] \), a contradiction.

Additional changes resulting from the correction described above.

Page 264, second paragraph should begin: “Let \( C = \{ a \mid a \in \mathcal{G} \times \{0,1,\ldots\} \} \)”.

Page 264, last paragraph of the Proof of Lemma 5.2, second sentence. This should begin: “Since \( a \subseteq \mathcal{G} \times \{0,1,\ldots\} \)”.

Page 265, paragraph following Proposition 5.5. This should read: “We are finished investigating \( \mathcal{P}(\mathcal{G} \times \{0,1,\ldots\}) \) and its subset \( C_0 \).

Page 272 (Problem found by Grigori Mints). The proof of Proposition 20.4.1 begins by noting that \( f \in g \) and \( [f \in g] \) are equivalent. This is not the case, see correction to Page 228. However, the atomic case is still straightforward. For \( f, g \in D_G^P \), \( p \models f \in g \) if and only if \( p \models [f \in g] \) if and only if \( p \models [\neg(\hat{n} \in a)] \). It follows that \( p \models \Box(\hat{n} \in b) \) if and only if \( p \models \Box(\hat{n} \in b) \) for every \( p \).

The following errors and typos were reported by Jonathan Farley.

Page 23, line 17 “aet” should be “set”

Page 25, line 14 “A2” should be “A5”

Page 30, line 19 “qualify” should be “qualify”

Page 38, line 2 It is true as written, but “\( y \subseteq x \)” should be “\( y \subseteq x \)”

Page 40, line 11 “bounded” should be “non-empty bounded”

Page 44, line 15 Do we know \( c \) is a set? Response: standard mathematical practice treats this as a set, but technically it is not justified until the Axiom of Substitution is introduced, Ax 8 on page 170.

Page 49, line 14 “\( x' Rb' \)” should be “\( y' Rf' \)”

Page 49, line 15 “\( x \leq b \)” should be “\( b \leq x \)”
Page 58, line -2 "M" should be "N"
Page 59, line -5 "M" should be "S"
Page 60, line -8 "N should be “∪N”
Page 60, line -5 Both occurrences of “N” be “∪N”

Page 60, Exercise 4.3 Delete this exercise. Here is a counter-example. Take any set x, and consider S = \{x\}. The axiom of choice is not needed to say S has a choice function, but \(\cup S = x\), and this need not have a choice function

Page 62, line 9 The semicolon should be a comma

Page 62, Lemma 5.4 Add the assumption that \(A \neq \emptyset\)

Page 62, line -2 “5.3” should be “5.2”

Page 63, line 5 Exercise 4.3 has been deleted

Page 63, line 7 After “denumerable” add “or finite”

Page 64, line -9 Before “set of finite character” insert “non-empty”

Page 66 It would have helped to point out at the beginning of §7 that g is defined on all sets

Page 73, line 3 “10.2” should be “10.3”

Page 79, following Definition 1.1 Should begin, “In general, \(F(x) \neq F''(x)\)”

Page 79, Second paragraph following Definition 1.1 Should contain “whereas \(F''(x)\) is”

Page 80, line 2 Add that \(\varphi_2\) is 1 – 1

Page 80, proof of Proposition 1.3, second line “L” should be “\(L_\prec\)”

Page 80, line -2 “onto” should be “into”

Page 88, line -7 “isomorphich” should be “isomorphic”

Page 89, Theorem 6.1 This should begin “For any functions \(f(x), g(x)\) on ordinals, and any function \(h(x, y, z)\), where y and z are ordinals, …

Page 93, O6 “Since x” should be “Since S”

Page 97, last line of Proof of Q4 “has rank < \(\alpha\)” should be “has rank \(\leq \alpha\)”

Page 97, line -8 “\(\mathbb{F}\)” should be “\(F\)”

Page 99, next to last line of Example “every subclass” should be “every non-empty subclass”

Page 99, line -14 “Zermelo Fraenkel” needs a hyphen

Page 101, line -16 “x of A” should be “x of \(A - B\)”

Page 102, line -6 “set” should be “class”

Page 306, Tarski, A. (1955) “lattice-theoretical theorem” should be “lattice-theoretical fixpoint theorem”