Subformula Results in Some Propositional Modal Logics

1. Introduction. Say the modal logic \( K \) has been formulated axiomatically, with a rule of necessitation, but without a rule of substitution. To turn \( K \) into \( T \) one may add to the axioms of \( K \) all formulas of the form \( \Box A \Rightarrow A \). But, in demonstrating that some particular formula \( X \) is a theorem of \( T \), not all of these new axioms will be used. We show one needs only those in which \( \Box A \) is a subformula of \( X \), the formula being demonstrated in \( T \). We also establish a similar relationship between \( T \) and \( S4 \), and between \( S4 \) and \( S5 \).

Our proof methods make use of Kripke's model theory [2]. Unfortunately, our methods are not general. Each result seems to require an argument with its own peculiarities, and so the techniques apparently do not even cover the relationship between \( B \) and \( S5 \). It would be interesting to know if our theorems could be established by a more uniform approach.

2. Preliminaries. Formulas are built up as usual, with \( \land, \sim \) and \( \Box \) as primitive, and \( \Rightarrow \) defined.

Let \( L \) be a set of formulas. By a derivation from \( L \) we mean a sequence \( X_1, X_2, \ldots, X_n \), of formulas, such that, for each \( X_i \), one of:

1) \( X_i \) is a classical tautology
2) \( X_i \) is a member of \( L \)
3) For some \( j, k < i \), \( X_j = (X_k \Rightarrow X_i) \)
4) For some \( j < i \), \( X_i = \Box X_j \).

We say \( X \) is derivable from \( L \) if \( X \) is the last term of a derivation from \( L \). (Note that there is no rule of substitution; the members of \( L \) themselves are the axioms. This differs from, say [3].)

We write \( \vdash_L X \) to mean \( X \) is derivable from \( L \). Let \( S \) also be a set of formulas. We write \( S \vdash_L X \) to mean \( X \) is derivable from \( S \cup L \). Thus \( S \vdash_L X \) and \( \vdash_{S \cup L} X \) mean the same, but the different notation is useful for emphasis.

We adopt the usual abbreviations, and write

\[ X \vdash_L Y \quad \text{for} \quad \{X\} \vdash_L Y \]
\[ S, X_1, \ldots, X_n \vdash_L Y \quad \text{for} \quad S \cup \{X_1, \ldots, X_n\} \vdash_L Y \]

Let \( \rightarrow \) be a new symbol. Its use is given by the following.

\( S \vdash_L \emptyset \rightarrow X \) means \( S \vdash_L X \)
\( S \vdash_L \{A_1, \ldots, A_n\} \rightarrow X \) means \( S \vdash_L (A_1 \land \ldots \land A_n) \Rightarrow X \)
And, if $\Gamma$ is an infinite set of formulas, $S \vdash_L \Gamma \rightarrow X$ means, for some finite subset $A$ of $\Gamma$, $S \vdash_L A \rightarrow X$. (Note that $\vdash_L \Gamma \rightarrow X$ in our notation would be written $\Gamma \vdash L X$ in [3], except for the omission of the substitution rule.)

Let $f$ be some fixed false statement, say $(A \land \neg A)$. Call a set $\Gamma$ of formulas $S$-inconsistent in $L$, if $S \vdash_L \Gamma \rightarrow f$. If $\Gamma$ is not $S$-inconsistent in $L$, call $\Gamma S$-consistent in $L$. The following results hold for this notion.

1) If $A$ is $S$-consistent in $L$ it can be extended to a set $\Gamma$ $S$-consistent in $L$-having no proper $S$-consistent extension, that is, to a maximal $S$-consistent set.

2) If $\Gamma \cup \{X\}$ and $\Gamma \cup \{\neg X\}$ are both $S$-inconsistent in $L$, so is $\Gamma$.

3) If $\Gamma$ is maximal $S$-consistent in $L$, $(X \land Y) \in \Gamma$ iff $X \in \Gamma$ and $Y \in \Gamma$.

4) If $\Gamma$ is maximal $S$-consistent in $L$, $S \subseteq \Gamma$. More generally, $S \vdash_L X$ implies $X \in \Gamma$.

5) Suppose all formulas of the form $\Box(\Box A \supset \Box B)$ or $(\Box A \supset \Box B)$ are in $L$. Then if $\{\Box X_1, \ldots, \Box X_n, \neg \Box Y\}$ are $S$-consistent in $L$, so is $\{X_1, \ldots, X_n, \neg Y\}$.

We will be using Kripke models. We assume the basic results about them are known [1], [2]. We use the following notation. A frame is a pair $\langle G, R \rangle$ where $G$ is a non-empty set and $R$ is a binary relation on $G$. A model is a triple $\langle G, R, \vdash \rangle$ where $\langle G, R \rangle$ is a frame and $\vdash$ is a relation between members of $G$ and formulas such that, for $\Gamma \in G$,

$$
\Gamma \vdash (X \land Y) \text{ iff } \Gamma \vdash X \text{ and } \Gamma \vdash Y
$$

$$
\Gamma \vdash \neg X \text{ iff } \not \Gamma \vdash X
$$

$$
\Gamma \vdash \Box X \text{ iff } A \vdash X \text{ for all } A \in G \text{ for which } \Gamma \vdash A.
$$

A formula $X$ is valid in a model $\langle G, R, \vdash \rangle$, if, for each $\Gamma \in G$, $\Gamma \vdash X$.

3. Results. Let $K$ be the set of all formulas of the form $\Box(\Box A \supset \Box B) \supset (\Box A \supset \Box B)$. Let $T$ be the set consisting of the members of $K$ together with all formulas of the form $\Box A \supset A$.

THEOREM 1. For each formula $X$, let $T(X)$ be the set of formulas of the form $\Box A \supset A$ where $\Box A$ is a subformula of $X$. Then $\vdash_T X$ iff $T(X) \vdash_K X$.

PROOF. Let $X$ be a formula, fixed for the proof. Trivially, if $T(X) \vdash_K X$ then $\vdash_T X$. Now suppose not-$T(X) \vdash_K X$. Let $G$ consist of all maximal $T(X)$-consistent sets in $K$. If $\Gamma$, $A \in G$, let $\Gamma \vdash A$ mean, for each subformula $\Box A$ of $X$, if $\Box A \in \Gamma$ then $A \in \Gamma$. Then $\langle G, R, \vdash \rangle$ is a frame.

Suppose $\Box A$ is a subformula of $X$, and $\Box A \in \Gamma \in G$. By item 4 of 2, $(\Box A \supset A) \in \Gamma$ and it follows that $A \in \Gamma$. Hence $\Gamma \vdash A$, so $R$ is reflexive.

If $A$ is any atomic formula, set $\Gamma \vdash A$ if $A \in \Gamma$. Then $\vdash$ extends uniquely to all formulas to make $\langle G, R, \vdash \rangle$ a model. Suppose this done. Let $\Box A$ be any subformula of $X$, and $\Gamma \in G$. We claim $\Gamma \vdash A$ if $A \in \Gamma$. This is shown by induction on the degree of $A$. If $A$ is atomic, the result
is true by definition. If $A$ is $(B \land C)$ or $\neg B$ the result is true by item 3) in 2. If $A$ is $\Box B$ and $A$ is a subformula of $X$, so is $B$. Suppose the result known for $B$; we show it holds for $\Box B$, that is, for $A$.

Suppose $\Box B \in \Gamma$. Let $\Gamma A$. By definition of $R, B \in \Delta$ so $\Delta \vdash B$. It follows that $\Gamma \vdash \Box B$.

Suppose $\Box B \not\in \Gamma$. Then $\neg \Box B \in \Gamma$. Let $\Gamma^\omega = \{Z | \Box Z \in \Gamma\}$. By item 5 in 2, $\Gamma^\omega \cup \{\neg B\}$ is $T(X)$-consistent in $K$. Extend it to a maximal $T(X)$-consistent set in $K$, call it $\Delta$. Clearly $\Gamma A$; $\neg B \in \Delta$, so $B \not\in \Delta$. Then not-$\Delta \vdash B$ by induction hypothesis, so not-$\Gamma \vdash \Box B$.

Thus for subformulas $A$ of $X$, $\Gamma \vdash A$ iff $A \in \Gamma$. Now, we are supposing not-$T(X) \vdash_K X$. Then $\{\sim X\}$ is $T(X)$-consistent in $K$. Extend it to a maximal $T(X)$-consistent set $\Delta$. Then $\Delta \not\in \Delta$, and $\Box \not\in \Delta$, so not-$\Delta \vdash X$. Thus is not valid in the model $\langle G, R, \vdash \rangle$. But $R$ is reflexive, so by now-standard results [1], [2] not-$\Gamma \vdash X$. This completes the proof.

Let $S4$ be the set of formulas consisting of the members of $T$ together with all formulas of the form $\Box A \Rightarrow \Box \Box A$.

**Theorem 2.** For each formula $X$, let $S4(X)$ be the set of formulas of the form $\Box A \Rightarrow \Box \Box A$ where $\Box A$ is a subformula of $X$. Then $\Gamma \vdash X$ iff $S4(X) \vdash_T X$.

**Proof.** Let $X$ be some fixed formula. Suppose not-$S4(X) \vdash T X$. Let $G$ consist of all maximal $S4(X)$-consistent sets in $T$. If $\Gamma, A \in G$, let $\Gamma A$ mean, for each subformula $\Box A$ of $X$, if $\Box^n A \in \Gamma$ then $\Box^{n+1} A \in \Delta$. We claim $R$ is transitive (as well as reflexive).

Suppose $\Gamma A$ and $\Delta B \Omega$. Let $\Box A$ be a subformula of $X$, and suppose $\Box^n A \in \Gamma$. We show $\Box^{n+1} A \in \Omega$. Well, $(\Box A \Rightarrow \Box \Box A) \in S4(X)$, so $S4(X) \vdash_T (\Box A \Rightarrow \Box \Box A)$. Using the necessitation rule, and $\Box (Z \Rightarrow W), S4(X) \vdash_T (\Box^n A \Rightarrow \Box^{n+1} A)$ and so, by item 4 of 2, $(\Box^n A \Rightarrow \Box^{n+1} A) \in \Gamma$. Since $\Box^n A \in \Gamma$ then $\Box^{n+1} A \in \Gamma$. Since $\Gamma A$, $\Box^n A \in \Delta$, and since $\Delta B \Omega$, $\Box^{n-1} A \in \Omega$. Thus $\Gamma A$, and $R$ is transitive.

The rest of the proof is a simple modification of that of Theorem 1, which we omit.

Let $S5$ be the set of formulas consisting of the members of $S4$ together with all formulas of the form $\neg \Box A \Rightarrow \Box \neg \Box A$.

**Lemma 3.** For each formula $X$, let $S5^*(X)$ be the set of all formulas of the form $\neg \Box M A \Rightarrow \Box \neg \Box M A$ where $M$ is any string (possibly empty) of $\Box$ and $\neg$ symbols, and $\Box A$ is a subformula of $X$. Then $\Gamma \vdash X$ iff $S5^*(X) \vdash S4 X$.

**Proof.** Let $X$ be some fixed formula. Suppose not-$S5^*(X) \vdash S4 X$. Let $G$ consist of all maximal $S5^*(X)$-consistent sets in $S4$. If $\Gamma, A \in G$, let $\Gamma A$ mean, for each subformula $\Box A$ of $X$, if $\Box M A \in \Gamma$ then $MA \in \Delta$, where $M$ is any string of $\Box$ and $\neg$ symbols. We claim $R$ is symmetric (as well as reflexive and transitive).
Suppose \( T \models A \). Let \( \Box A \) be a subformula of \( X \), and suppose \( \Box MA \in \Delta \). We claim \( MA \in \Gamma \). For if not, \( \neg MA \in \Gamma \). Then \( \neg \Box MA \in \Gamma \), so \( \Box \neg \Box MA \in \Gamma \). Then \( \Box \neg MA \in \Delta \) since \( T \models A \), and this is a contradiction. Thus \( MA \in \Gamma \); \( \Delta \models \Gamma \), so \( R \) is symmetric.

The rest of the proof is similar to that of Theorem 1, so we omit it.

**Lemma 4.** Let \( A^* \) be the formula \((\neg \Box A \supset \Box \neg \Box A) \land (\neg \Box \neg A \supset \Box \neg \Box \neg A)\). Then each of the following is derivable from \( S4 \):

\[
A^* \supset (\neg \Box MA \supset \Box \neg \Box MA) \tag{1}
\]

\[
A^* \supset (\neg \Box \neg MA \supset \Box \neg \Box \neg MA) \tag{2}
\]

where \( M \) is any string of \( \Box \) and \( \neg \) symbols.

**Proof.** By induction on the length of \( M \). If \( M \) is of length 0, the result is immediate. Now suppose the result is known for \( M \) of length \( m \), and suppose \( M' \) is of length \( m + 1 \). Then either \( M' = \neg M \) or \( M' = \Box M \).

- case a) \( M' = \neg M \). Then (1) for \( M' \) is the same as (2) for \( M \). And (2) for \( M' \) follows from (1) for \( M \) on insertion of two double-negations.

- case b) \( M' = \Box M \). Then (1) for \( M' \) follows from (1) for \( M \) on replacing two \( \Box \) symbols by \( \Box \Box \). And (2) for \( M' \) follows from (1) for \( M \) using the following \( S4 \) theorem:

\[
(\neg \Box Z \supset \Box \neg \Box Z) \supset (\neg \Box \neg \Box Z \supset \Box \neg \Box \neg \Box Z).
\]

**Theorem 5.** For each formula \( X \), let \( S5(X) \) consist of all formulas of the forms \( \neg \Box A \supset \Box \neg \Box A \) and \( \neg \Box \neg A \supset \Box \neg \Box \neg A \), where \( \Box A \) is a subformula of \( X \). Then \( \models S5(X) \iff S5(X) \models S4 \).

**Proof.** Immediate from the above two lemmas.

Unfortunately, the above techniques do not seem to extend very far. We give the following as an example of the difficulties.

Let \( B \) be the set of formulas consisting of the members of \( T \) together with all formulas of the form \( \neg A \supset \Box \neg \Box A \).

**Theorem 6.** For each formula \( X \), let \( B^*(X) \) consist of all formulas of the form \( \neg MA \supset \Box \neg \Box MA \) where \( M \) is any string of \( \Box \) and \( \neg \) symbols, and \( \Box A \) is a subformula of \( X \). Then \( \models B^*(X) \iff B^*(X) \models T \).

**Proof.** As usual. This time let \( T \models A \) mean, whenever \( \Box MA \in \Gamma \) then \( MA \in \Delta \) where \( M \) is any string of \( \Box \) and \( \neg \) symbols, and \( \Box A \) is a subformula of \( X \). We claim \( R \) is symmetric. For, let \( T \models A \), and suppose \( \Box MA \in \Delta \). If \( MA \notin \Gamma \), then \( \neg MA \in \Gamma \), so \( \Box \neg \Box MA \in \Gamma \), so \( \Box \neg MA \in \Delta \), a contradiction. Now finish as in earlier proofs.

This is the analog of Lemma 3. The trouble is, there seems to be no analog to Lemma 4. The function of Lemma 4 was to replace the generally infinite set \( S5^*(X) \) by the finite set \( S5(X) \). Some comparable way of
replacing $B^*(X)$ by a finite set would be nice. Even better would be a uniform approach to the above. Each result was obtained by a suitable complication of a standard completeness proof, but each result used a different complication. A uniform approach, whatever it may be like, should allow the extension of the above to other logics as well.

References


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