

Resolution for Intuitionistic Logic¹

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A resolution style theorem prover is presented for first-order Intuitionistic logic. Soundness and completeness proofs are sketched. Relationships with Classical resolution are considered.

§1 Introduction.

Most automated theorem provers have been built around some version of resolution [4]. But resolution is an inherently Classical logic technique. Attempts to extend the method to other logics have tended to obscure its simplicity. In this paper we present a resolution style theorem prover for Intuitionistic logic that, we believe, retains many of the attractive features of Classical resolution. It is, of course, more complicated, but the complications can be given intuitive motivation. We note that a small change in the system as presented here causes it to collapse back to a Classical resolution system.

We present the system in some detail for the propositional case, including soundness and completeness proofs. For the first order version we are sketchier.

§2 Background.

In Classical logic, X and $\neg X$ are duals. To deny X is to assert $\neg X$, and conversely. Intuitionistic logic does not behave this way though. Loosely speaking, when an Intuitionist says “ X is true,” what is meant is “I have a proof of X .” To deny this is not to assert $\neg X$, that is “I have a *refutation* of X ,” but only “I don’t have a proof of X .” This is the source of the failure of the law of excluded middle in Intuitionistic logic. We need formal notation that can capture this proof/no proof distinction. Following [1, 2, 5] we use signed formulas.

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Definition. Let T and F be two new symbols. By a signed formula we mean TX or FX , where X is a formula.

Think of TX as asserting that X is Intuitionistically true (X has been proved), and FX as asserting the opposite (X has not been proved).

In Classical resolution one generally begins by putting formulas into clause form. Some variations of Classical resolution allow resolutions at non-clausal stages, and it is possible to construct Classical systems in which the reduction to clause form is interwoven with resolution applications. This is the approach we adopt here, but with a twist. In Intuitionistic logic, not every formula is equivalent to one in clause form (indeed, all of \wedge , \vee , \neg and \supset are independent). The effect is that the analog of reduction to clause form used here proceeds by a series of implications, not equivalences. And there may be incompatible reductions possible; consequently backtracking may be necessary. Once the rules have been presented it will be clear that the changes that turn the system into a Classical one also remove the need for this backtracking.

We continue to use the terminology of clause and clause set, but because of the problems mentioned in the previous paragraph we broaden their usual meaning. In particular we do not confine ourselves to the literal level. (*Literal* now means a *signed atomic* formula.) Basically, we retain only the disjunctive interpretation for clauses, and the conjunctive one for clause sets. In the following definition we assume the notion of Kripke Intuitionistic model is known [3].

Definitions. A clause, denoted $[Z_1, \dots, Z_n]$, is a finite set of signed formulas. A clause set is a finite set of clauses, denoted $[C_1, \dots, C_n]$. Let Γ be a possible world of a Kripke model. The signed formula TX is true at Γ if X is forced at Γ ; the signed formula FX is true at Γ if X is not forced at Γ . A clause is true at Γ if one of its members is. A clause set is true at Γ if each of its members is. Finally, a clause set is satisfiable if it is true at some possible world of some Kripke model.

Clauses and clause sets are officially sets, but in practice they will probably be represented by lists. In stating our rules the following notation will be handy. $[X|Y]$ denotes the clause, or clause set, containing X , with Y as the set (list) of remaining members. We assume X does not occur in Y . Likewise, if Y and Z are sets (lists) we use $Y * Z$ to denote the result of appending Y and Z (and removing repetitions).

§3 The propositional system.

The resolution rule itself retains its usual form, after adjusting in the obvious way for the use of signed formulas. The major change is that the formula being resolved on need not be a literal but can be more complex.

Resolution Rule. A clause set containing the clauses $[TX|A]$ and $[FX|B]$ may be extended by adding the clause $A * B$.

This rule may be schematically stated as follows.

$$\frac{[TX|A] \quad [FX|B]}{A * B}$$

The rest of the system consists of rules that are the counterparts of Classical steps reducing to conventional clause form. These rules fall into two categories, which we call *regular* and *special*.

Regular Reduction Rules. A clause set that contains the clause shown above the line may have the clause(s) below the line added.

$$\begin{array}{cc} \frac{[TX \wedge Y|A]}{[TX|A], [TY|A]} & \frac{[FX \wedge Y|A]}{[FX, FY|A]} \\ \frac{[TX \vee Y|A]}{[TX, TY|A]} & \frac{[FX \vee Y|A]}{[FX|A], [FY|A]} \\ \frac{[T\neg X|A]}{[FX|A]} & \frac{[TX \supset Y|A]}{[FX, TY|A]} \end{array}$$

Some of these rules are, in effect, equivalences. For example, $FX \wedge Y$ is true at a possible world Γ iff $X \wedge Y$ is not forced at Γ iff X is not forced at Γ or Y is not forced at Γ iff FX or FY is true at Γ . Thus it is easy to see $[FX \wedge Y|A]$ and $[FX, FY|A]$ are true at the same possible worlds. The $T\neg$ and $T \supset$ rules are implicational in nature however, not equivalences.

Special Reduction Rules. A clause set S that contains the clause shown above the line may have the clause(s) below the line added, but first all clauses in S that contain any F -signed formulas must be removed. (This includes the clauses displayed above the lines as well.)

$$\frac{[F\neg X|A]}{[TX|A]} \quad \frac{[FX \supset Y|A]}{[TX|A], [FY|A]}$$

Both of these rules represent implications, not equivalences. The intuitive motivation is not difficult, though. Consider the rule for $F \supset$; the other rule is similar.

Informally, for me to assert $FX \supset Y$ is to say that I do not have a proof of $X \supset Y$. From an Intuitionistic point of view this means that, as far as I know, someday someone might present me with a proof of X which I am unable to convert into a proof of Y . Well, imagine this has happened: I am at some hypothetical future time when X has been proved but Y has not been. Thus I have both TX and FY . In passing from today to that future time I am certainly entitled to retain positive knowledge: what has been proved remains proved. But negative knowledge is more problematical. Even if I don't know Z today, I might discover a proof of it tomorrow. Thus in moving to a hypothetical

future time I can not retain clauses that contain F -signed formulas: F 's may not remain F 's.

The motivation above is loose. Nonetheless, a formal version of it is embodied in the soundness argument in the next section.

Definition. A derivation from a clause set S is a sequence of clause sets, beginning with S , each of which comes from the preceding using one of the rules above (or a rule to be stated below). A refutation of a clause set S is a derivation from S of a clause set containing the empty clause. A proof of the formula X is a refutation of the clause set $\llbracket FX \rrbracket$.

Example. The following is a proof of $(X \supset Y) \supset (\neg Y \supset \neg X)$. We omit the reasons for the steps.

$\llbracket F((X \supset Y) \supset (\neg Y \supset \neg X)) \rrbracket$
 $\llbracket T(X \supset Y), [F(\neg Y \supset \neg X)] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [F\neg X] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [TX] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [TX], [FX, TY] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY], [TY] \rrbracket$
 $\llbracket T(X \supset Y), [T\neg Y], [TX], [FX, TY], [FY], [TY], [] \rrbracket$

Unfortunately the rules above, though sound, are not complete. With them alone we are unable to prove the Intuitionistically valid formula $(P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)$ for instance. The following rule completes the system.

Special Case Rule. Let $\llbracket [A|R]|S \rrbracket$ be a clause set. (Here $[A|R]$ is a clause containing the signed formula A , with R as the list of other signed formulas making up the clause. S is the list consisting of the remaining clauses of the clause set.) If there is a refutation of the clause set $\llbracket [A]|S \rrbracket$ then the clause set $\llbracket R|S \rrbracket$ follows from $\llbracket [A|R]|S \rrbracket$. Schematically,

$$\frac{\llbracket [A|R]|S \rrbracket}{\llbracket R|S \rrbracket}$$

provided $\llbracket [A]|S \rrbracket$ has a refutation.

The following is a proof of $(P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)$ using this rule.

$\llbracket F((P \vee Q) \supset (\neg\neg P \vee \neg\neg Q)) \rrbracket$
 $\llbracket T(P \vee Q), [F(\neg\neg P \vee \neg\neg Q)] \rrbracket$
 $\llbracket [TP, TQ], [F(\neg\neg P \vee \neg\neg Q)] \rrbracket$
 $\llbracket [TP, TQ], [F\neg\neg P], [F\neg\neg Q] \rrbracket \quad (*)$

Now we consider the case

$\llbracket [TP], [F\neg\neg P], [F\neg\neg Q] \rrbracket$
 $\llbracket [TP], [T\neg P] \rrbracket$
 $\llbracket [TP], [T\neg P], [FP] \rrbracket$
 $\llbracket [TP], [T\neg P], [FP], [] \rrbracket$

Having derived the empty clause in this case, by the Special Case Rule, the following clause set follows from (*):

[[TQ], [$F\neg\neg P$], [$F\neg\neg Q$]]
 [[TQ], [$T\neg Q$]]
 [[TQ], [$T\neg Q$], [FQ]]
 [[TQ], [$T\neg Q$], [FQ], []]

Remarks. The proof procedure presented above is inherently more complex than the Classical one because there is hidden branching. Suppose we have a situation in which Special Reduction Rules can be applied to more than one clause of a clause set. Then applying a Special Rule to one clause will cause the deletion of the other clause. In a complete implementation, if a proof is not found after making such a choice of Special Reduction Rule application, backtracking to the choice point must occur, and the other choice must be tried. Of course, such choices are independent, and the derivations could be constructed in parallel.

If we modify the Special Reduction Rules into Regular ones by removing the clause deletion requirements, a sound and complete proof procedure for Classical logic results. Indeed, the Special Case Rule now becomes a derivable rule and need not be taken as primitive. Notice too that now rule applications are never destructive, so there is no need for a backtracking mechanism.

All the Regular Reduction Rules are addition rules: clauses get added, not replaced. In fact the system remains complete if the rules for $T\wedge$, $F\wedge$, $T\vee$ and $F\vee$ are modified to read: the clause above the line may be removed and those below the line added. (This is not the case with the $T\neg$ and $T\supset$ rules however.) Demonstrating completeness is more work with this change, and we do not do so.

Finally, in Classical resolution one reduces to the literal level (the usual clause form) before beginning to use the Resolution Rule. This is still possible here in the sense that, if X has a proof, it has one in which all resolutions are on literals. This is not hard to show, but again we do not prove it here.

§4 Soundness and Completeness.

To show soundness it is enough to establish that refutable clause sets are not satisfiable. Then, if X is provable the clause set $[[FX]]$ will be refutable, hence not satisfiable, and so X must be forced at every world of every Kripke model; i.e. X must be Intuitionistically valid.

Proposition. *The Resolution Rule and each of the Regular and Special Reduction Rules turns a satisfiable clause set into another satisfiable clause set.*

Proof. This is straightforward for the Resolution Rule and for the Regular Reduction Rules. We consider one of the Special Reduction Rules in more detail.

Suppose S is a satisfiable clause set containing the clause $[FX \supset Y|A]$, and the corresponding Special Reduction Rule is applied, producing the new clause set S^* , which contains those clauses from S having no F -signed formulas, and also contains $[TX|A]$ and $[FY|A]$. We show S^* is satisfiable.

Say the clauses in S are all true at the world Γ of some Kripke model. In particular, $[FX \supset Y|A]$ is true at Γ . Suppose first that some member of A is true at Γ . That member is still in both $[TX|A]$ and $[FY|A]$, so both clauses are true at Γ . And the remaining members of S^* all were in S , hence were all true at Γ . Thus all members of S^* are true at Γ ; S^* is satisfiable in this case.

Next suppose no member of A is true at Γ . Since $[FX \supset Y|A]$ is true at Γ , $FX \supset Y$ must be true at Γ , that is, $X \supset Y$ is not forced at Γ . Then by the definition of Kripke model there must be a possible world Δ , accessible from Γ , at which X is forced but Y is not. That is, both TX and FY are true at Δ , hence both $[TX|A]$ and $[FY|A]$ are true at Δ . Finally, if a formula is forced at a possible world of a Kripke model, it is forced at any world accessible from it. Then any clause set with no F -signed formulas that is true at Γ is also true at Δ . It follows that S^* is satisfiable, but this time at the world Δ .

Proposition. *A clause set that has a refutation is not satisfiable.*

Proof. This is by an induction on the number of applications of the Special Case Rule in the refutation. The initial case of zero applications is taken care of by the preceding proposition together with the fact that the empty clause is not satisfiable. Details of the induction step are straightforward and are omitted.

The simplest way to establish completeness is to use an Intuitionistic version of the Model Existence Theorem. We do not give details, but refer to [1, 2] for them. In brief, a completeness proof goes as follows. Call a finite set $\{Z_1, \dots, Z_n\}$ of signed formulas *consistent* if there is no refutation of the clause set $[[Z_1], \dots, [Z_n]]$. This meets the requirements of being a (propositional) Intuitionistic consistency property. Then by the Model Existence Theorem, every set that is consistent in this sense is satisfiable.

Now, if X is Intuitionistically valid, X must be provable. For if it were not, there would be no refutation of $[[FX]], \{FX\}$ would be consistent, hence satisfiable, and X would not have been valid.

§5 A First Order System.

The lack of an adequate Intuitionistic version of Herbrand's theorem is no real bar to extending the system just presented to a first-order setting. We briefly sketch such an extension.

We use a first-order language with quantifiers \forall and \exists , function symbols, but no $=$ relation. Function symbols are interpreted rigidly in the sense that, whatever a closed term is taken to designate in a possible world, it must designate the same thing in all accessible worlds.

We add reduction rules for the four quantifier cases. And once again they fall into Regular and Special groupings.

Regular Reduction Rules. A clause set that contains the clause shown above the line may have the clause below the line added.

$$\frac{[T(\forall x)\phi(x)|A]}{[T\phi(y)|A]} \quad \frac{[F(\exists x)\phi(x)|A]}{[F\phi(y)|A]}$$

where y is a previously unused free variable,

$$\frac{[T(\exists x)\phi(x)|A]}{[T\phi(f(y_1, \dots, y_n))|A]}$$

where f is a previously unused function symbol and y_1, \dots, y_n are all the free variables introduced thus far in the proof.

Special Reduction Rule. A clause set S that contains the clause shown above the line may have the clause below the line added but first all clauses in S that contain any F -signed formulas must be removed.

$$\frac{[F(\forall x)\phi(x)|A]}{[F\phi(f(y_1, \dots, y_n))|A]}$$

where f is a previously unused function symbol and y_1, \dots, y_n are all the free variables introduced thus far in the proof.

Finally, the Resolution Rule becomes the following.

Resolution Rule. Suppose S is a clause set containing the clauses $[TX|A]$ and $[FY|B]$. Let σ be a most general unifier for X and Y . Then from S we may obtain the clause set $[A*B|S]\sigma$, that is, the result of adding $A*B$ to S and applying σ to every formula. (This tacitly assumes bound variable renaming is carried out when necessary to prevent unintended free variable capture.)

Soundness may be shown as before, by establishing that all rules preserve satisfiability in a suitable sense. Completeness for a ground level analog of the system above may be shown using the Model Existence Theorem, and completeness of the system with free variables can be derived from that by using a suitable version of the Lifting Lemma.

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