FOIL Axiomatized

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Abstract

In an earlier paper, [5], I gave semantics and tableau rules for a simple first-order intensional logic called FOIL, in which both objects and intensions are explicitly present and can be quantified over. Intensions, being non-rigid, are represented in FOIL as (partial) functions from states to objects. Scoping machinery, *predicate abstraction*, is present to disambiguate sentences like that asserting the necessary identity of the morning and the evening star, which is true in one sense and not true in another.

In this paper I address the problem of axiomatizing FOIL. I begin with an interesting sublogic with predicate abstraction and equality but no quantifiers. In [2] this sublogic was shown to be undecidable if the underlying modal logic was at least K4, though it is decidable in other cases. The axiomatization given is shown to be complete for standard logics without a symmetry condition. The general situation is not known. After this an axiomatization for the full FOIL is given, which is straightforward after one makes a change in the point of view.

1 Introduction

There are many varieties of first-order modal logic. One way in which they differ is in what they quantify over. Should it be objects, in which case $(x = y) \supset \Box(x = y)$ is a plausible validity, or should it be intensions, in which case it may not be. It seems unreasonable to insist on the primacy of either of these, since both have a natural role. In [5] I gave semantics and tableau rules for a simple first-order intensional logic called FOIL, in which both are allowed. FOIL is a two-sorted modal logic, with quantifiers over extensions (objects) and quantifiers over intensions, where intensions are modeled by partial functions from worlds to extensions. It is a rich formalism, with many natural sublogics of independent interest. The logics examined in [6] fall into this category, for instance. A key piece of machinery present in FOIL is *predicate abstraction*, which originated in [15, 16], and which I have discussed at length elsewhere, including [6, 1, 2, 4], and of course [5]. I will say something about this notion here too, to make the present paper a bit more self-contained.

There were some items left incomplete in [5]. Most notably, tableau rules were given, but an axiomatization was not. Here I will begin by giving an axiomatization for a sublogic of FOIL that has predicate abstraction and can model intensions, but does not have quantification. Then I will provide an axiomatization for the entire of FOIL, but along quite different lines. In this alternative approach predicate abstraction is taken to be a defined rather than a primitive notion. Since this involves quantifiers essentially, it cannot be applied to the weak sublogic of FOIL just mentioned. However, a secondary outcome of this alternative axiomatization is that it also provides us with a different tableau system for FOIL besides that of [5].

I should say something about the nuances of FOIL semantics, before we get properly started. Kripke style models will be used. Domains will be constant, with members thought of as objects. That is, we have a possibilist semantics. There will be a special relation symbol, E, to play the role of an existence predicate, thus allowing an indirect actualist approach, by using quantifiers relativized to this predicate. One could, of course, allow domains to vary and have a direct actualist semantics, though the present approach seems the simpler one. Intensions will be modeled by partial functions from possible worlds to the object domain, in the Carnap style—any such partial function picks out, at each state, the object the intension designates at that state, if any.

There are distinctions we make all the time in natural language that require proper care when a formalization is attempted. For instance when intensions are involved, sometimes we speak of the intensions as such, sometimes of the objects the intensions pick out. "The monarch of England," is a natural candidate for an intension, picking out different people at different times. When speaking of it in its intensional role, we might use an alternative phrase such as "the throne of England," or "the English monarchy," but these differences are not significant for what I am saying here. Similar things apply to "the monarch of Denmark," of course. Now, if I say "the English monarchy is more significant to Americans than the Danish monarchy," I probably am speaking intensionally. For Americans, the former concept is more significant than the later, no matter who is on the throne of either country. If I say "the monarch of England is older than the monarch of Denmark," I am clearly speaking of individuals and not of concepts. In FOIL there is machinery to formalize both ways of speaking.

Given the expressiveness of FOIL, we should expect that problems might arise when designation by intensions is intermixed with modality. If I say "someday the monarch of England may be younger," I probably mean (1) that at some point in the future, whoever is ruler then will be younger than the person who is ruler now—there is a future in which the ruler has the *younger than 79* property. But I also might be a science fiction author suggesting a plot in which (2) the present English monarch undergoes a 'youngification' process, and thus the present monarch has the *younger than 79* property in the future. Suppose $Y(x)$ is intended to be the *younger than 79* predicate, and modal operators are read temporally, so that $\Diamond$ corresponds to *at some time in the future*. Then, we need to distinguish between $\Diamond$(having the $Y(x)$ property), and having the $\Diamond Y(x)$ property. It is for this purpose that predicate abstraction was introduced. Think of $\langle\lambda x.\varphi(x)\rangle(f)$ as expressing that the object designated by the intension f has the property specified by $\varphi(x)$. Thus we distinguish between a formula $\varphi(x)$ and the property it determines, represented by $\langle\lambda x.\varphi(x)\rangle$. Then if m is the intension *English monarch*, alternative (1) can be formalized as $\Diamond\langle\lambda x.Y(x)\rangle(m)$, and (2) by $\langle\lambda x.\Diamond Y(x)\rangle(m)$. See [5] or [6] for more discussion of this issue, which was originally introduced in [15, 16].

2 Syntax

FOIL has two sorts of variables, *object variables*, x, y, $\ldots$, and *intension variables*, f, g, $\ldots$. There are no constant or function symbols. These could be added if desired, but at this point they would distract from the essential issues. As usual there is a family of relation symbols, but rather than an arity, I'll assume each relation symbol has a *type* associated with it, where a type is an n-tuple whose entries are in $\{O, I\}$. An *atomic formula* is an expression of the form $P(\alpha_1, \ldots, \alpha_n)$ where P is a relation symbol whose type is $\langle t_1, \ldots, t_n\rangle$ and, for each i, if $t_i = O$ then α_i is an object variable, and if $t_i = I$ then α_i is an intension variable. I will use lower case Greek letters, α, β, $\ldots$, to stand for variables when the exact type is not significant.

Among the relation symbols there is one, E, whose type is $\langle O\rangle$, intended to represent *actual* existence of objects—this is to take care of actualist quantification, since our basic formulation is possibilist. There is also a two place relation symbol $=$ of type $\langle O, O\rangle$, intended to represent equality of objects. I will write it in its customary infix position. One might also introduce an equality relation for intensions, though this is a bit more problematic.

FOIL formulas are built up from atomic formulas using propositional connectives, including the 0-place *false* and *true*, the modal operator $\Box$, with $\Diamond$ defined in the usual way, and two sorted quantification, with $\forall$ primitive and $\exists$ defined. In addition predicate abstraction is allowed, as noted above. If Φ is a formula, x is an object variable, and f is an intension variable, then $\langle\lambda x.\Phi\rangle(f)$ is a formula, in which the free variable occurrences are those of Φ except for x, together with the displayed occurrence of f. I may abbreviate $\langle\lambda x.\langle\lambda y.\Phi\rangle(g)\rangle(f)$ by $\langle\lambda x, y.\Phi\rangle(f, g)$, and so on. The idea is $\langle\lambda x.\Phi\rangle(f)$ says that, at a particular world, the object designated by f at that world has the Φ property.

An occurrence of intension variable f is free in A if it is not within the scope of a quantifier $(\forall f)$. And g is free for f in A if no free occurrences of f in A are within the scope of $(\forall g)$. An occurrence of object variable x is free in A if it is not within the scope of a quantifier $(\forall x)$ or an abstract λx. Variable y is free for variable x in A if no free occurrences of x in A are within the scope of a quantifier $(\forall y)$ or an abstract λy.

3 Semantics

FOIL is, properly speaking, a family, depending on a choice of the underlying propositional modal logic. When necessary to be more specific, I will write FOIL-S4, FOIL-S5, and the like. When I simply write FOIL, I mean any one of the general family of logics. FOIL is an outgrowth of the system developed in [6], and can be seen as the first-order part of the system of [3], which itself is a continuation of the intensional logic of Montague and Gallin, [12, 13, 14, 7] with influences from [15, 16].

A FOIL model is a structure $\mathcal{M} = \langle\mathcal{G}, \mathcal{R}, \mathcal{D}_O, \mathcal{D}_I, \mathcal{I}\rangle$ meeting the following conditions. $\langle\mathcal{G}, \mathcal{R}\rangle$ is a frame, as usual. A model will be referred to as FOIL-K4 if the frame is transitive, and so on in the obvious way. $\mathcal{D}_O$ is a non-empty set called the *object domain*. $\mathcal{D}_I$ is a non-empty set of partial functions from $\mathcal{G}$ to $\mathcal{D}_O$, called the *intension domain*. Finally, $\mathcal{I}$ is an *interpretation* function. If P is a relation symbol of type $\langle t_1, \ldots, t_n\rangle$ then $\mathcal{I}(P)$ is a mapping from $\mathcal{G}$ to subsets of $\mathcal{D}_{t_1} \times \cdots \times \mathcal{D}_{t_n}$. $\mathcal{I}(\mathsf{E})$ is required to map members of $\mathcal{G}$ to *non-empty* subsets of $\mathcal{D}_O$. It is also required that $\mathcal{I}(=)$ is the constant function mapping each world to the identity relation on $\mathcal{D}_O$.

A *valuation* in FOIL model $\mathcal{M}$ is a mapping that assigns to each object variable a member of $\mathcal{D}_O$ and to each intension variable a member of $\mathcal{D}_I$.

Given a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}_O, \mathcal{D}_I, \mathcal{I} \rangle$ and a valuation v, I will say the intension variable f *designates at* $\Gamma \in \mathcal{G}$ *with respect to* v provided Γ is in the domain of the partial function $v(f)$.

The truth definition is mostly as usual. I write $\mathcal{M}, \Gamma \Vdash_v \Phi$ to symbolize that formula Φ is true at possible world Γ of model $\mathcal{M}$ with respect to valuation v. The conditions for this are as follows. In the quantifier conditions, α is either an extensional or an intensional variable, thus covering two cases together.

Atomic $\mathcal{M}, \Gamma \Vdash_v P(\alpha_1, \ldots, \alpha_n) \Leftrightarrow \langle v(\alpha_1), \ldots, v(\alpha_n) \rangle \in \mathcal{I}(R)(\Gamma)$.

Propositional

$\mathcal{M}, \Gamma \nVdash_v \mathit{false} \quad \mathcal{M}, \Gamma \Vdash_v \mathit{true}$

$\mathcal{M}, \Gamma \Vdash_v X \supset Y \Leftrightarrow \mathcal{M}, \Gamma \nVdash_v X$ or $\mathcal{M}, \Gamma \Vdash_v Y$

And similarly for other propositional connectives.

Necessity $\mathcal{M}, \Gamma \Vdash_v \Box X \Leftrightarrow \mathcal{M}, \Delta \Vdash_v X$ for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$.

Universal Quantifier $\mathcal{M}, \Gamma \Vdash_v (\forall \alpha)\Phi \Leftrightarrow \mathcal{M}, \Gamma \Vdash_w \Phi$ for every valuation w that is like v except possibly on α.

Predicate Abstraction

If f designates at Γ with respect to v, $\mathcal{M}, \Gamma \Vdash_v \langle \lambda x.\Phi \rangle(f)$ if $\mathcal{M}, \Gamma \Vdash_w \Phi$ where w is like v except that $w(x) = v(f)(\Gamma)$.

If f does not designate at Γ with respect to v, $\mathcal{M}, \Gamma \nVdash_v \langle \lambda x.\Phi \rangle(f)$.

A formula X is valid in a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}_O, \mathcal{D}_I, \mathcal{I} \rangle$ provided $\mathcal{M}, \Gamma \Vdash_v X$ for every $\Gamma \in \mathcal{G}$ and every valuation v. A formula is valid if it is valid in every FOIL model. Likewise a formula is K4 valid if it is valid in every transitive FOIL model, and so on.

As noted earlier, the idea behind the predicate abstraction condition is that $\mathcal{M}, \Gamma \Vdash_v \langle \lambda x.\Phi \rangle(f)$ says the object designated by f at Γ has the property specified by Φ at Γ, provided there is such an object, that is, provided f designates. The second part of the condition above is simply to the effect that no property can correctly be ascribed to the object designated by f if f does not designate. Note that, if f does not designate at Γ with respect to v, we must have $\mathcal{M}, \Gamma \nVdash_v \langle \lambda x.\Phi \rangle(f)$, and hence $\mathcal{M}, \Gamma \Vdash_v \neg \langle \lambda x.\Phi \rangle(f)$. On the other hand, we will also have $\mathcal{M}, \Gamma \nVdash_v \langle \lambda x.\neg\Phi \rangle(f)$, by the second part of the Predicate Abstraction condition again. Thus $\neg\langle\lambda x.\Phi\rangle(f)$ and $\langle\lambda x.\neg\Phi\rangle(f)$ need not be equivalent.

Since the equality relation symbol must be interpreted by equality of objects, we have the validity of $(x = y) \supset \Box(x = y)$ and, $\neg(x = y) \supset \Box\neg(x = y)$. But also, since $\langle\lambda x, y.(x = y)\rangle(f, g)$ asserts intension variables f and g denote the same object, at a world, we do *not* have validity of $\langle\lambda x, y.(x = y)\rangle(f, g) \supset \Box\langle\lambda x, y.(x = y)\rangle(f, g)$. This says that if f and g designate the same object in the present world, they will designate the same object in all accessible worlds (they will be synonymous), and this need not be the case. On the other hand, we do have validity of $\langle\lambda x, y.(x = y)\rangle(f, g) \supset \langle\lambda x, y.\Box(x = y)\rangle(f, g)$. This says that if the objects denoted by f and g are identical, these objects are necessarily identical. This is so because, as just noted, identity between objects is necessary identity. Similar remarks apply to $\langle\lambda x, y.\neg(x = y)\rangle(f, g) \supset \Box\langle\lambda x, y.\neg(x = y)\rangle(f, g)$, which is valid.

In a model a member of $\mathcal{D}_I$, an intension, is a partial function from worlds to objects, which are members of $\mathcal{D}_O$. Thus an intension designates objects (or nothing) at worlds. This can be captured within the language itself. (In the following it is assumed that y is a variable distinct from x. I won't mention this kind of condition in the future.)

$$\mathsf{D}(f, x) \text{ abbreviates the formula } \langle\lambda y.y = x\rangle(f) \tag{1}$$

If $\mathsf{D}(f, x)$ were atomic, rather than being an abbreviation, it would be of type $\langle I, O\rangle$. Working through the semantics, $\mathcal{M}, \Gamma \Vdash_v \mathsf{D}(f, x)$ is true just in case $v(f)(\Gamma) = v(x)$. In words, it says that at Γ the intension f designates, and further that it designates the object $v(x)$. Then we can also say that intension f designates, at a world, using the formula $(\exists x)\mathsf{D}(f, x)$. I do not want to make this the 'official' formalization however, because it involves quantification, and there is an alternative formula that does the same thing without using quantifiers.

$$\mathsf{D}(f) \text{ abbreviates the formula } \langle\lambda y.\mathit{true}\rangle(f) \tag{2}$$

(I am overloading the symbol D; one can tell from context whether it is being used as a one-place or a two-place relation symbol.) Working through the semantics, $\mathcal{M}, \Gamma \Vdash_v \mathsf{D}(f)$ just in case f designates at Γ with respect to v.

The domain $\mathcal{D}_I$ of intensions in a model is not required to be the set of *all* functions from $\mathcal{G}$ to $\mathcal{D}_O$. Not everything mathematically possible should always be considered to be an intension. Also, if we take the entire set of functions from $\mathcal{G}$ to $\mathcal{D}_O$ as $\mathcal{D}_I$, a complete proof procedure is almost certainly beyond reach, just as an axiomatization of true second order logic is.

4 Without Quantifiers

FOIL has a great deal of machinery. So before considering it in its full complexity, let us look at a simpler sublogic, in which we concentrate on predicate abstraction. By the $\{\lambda, =\}$ *part of* FOIL I mean the sublogic whose language does not contain quantifiers or E. For a semantics we use that of FOIL, except that the truth conditions concerning quantification and E are no longer relevant. In the remainder of this section I present an axiomatization for the $\{\lambda, =\}$ part of FOIL, and show soundness and completeness. The completeness proof assumes an underlying propositional modal logic of K, and extends to some other cases, though how far is not clear. Dropping quantifiers still leaves us with predicate abstraction and object equality, and this is a powerful and expressive system in its own right. Indeed, if the underlying modal logic is between K4 and S5, the $\{\lambda, =\}$ part of FOIL is an undecidable system, though for K, T, and D it is decidable, [2].

4.1 An Axiomatization

I assume the underlying modal logic is K. I'll say something about other systems in Section 4.3. To begin, here are the rules of inference; two are standard, one is new.

Modus Ponens

$$\frac{X \qquad X \supset Y}{Y}$$

Necessitation

$$\frac{X}{\Box X}$$

λ Regularity

$$\frac{A \supset B}{\langle \lambda x.A \rangle(f) \supset \langle \lambda x.B \rangle(f)}$$

Next, here are the axioms, or rather, the axiom schemes. Some of them make use of D, whose definition is in (2).

1. All classical tautologies
2. $\Box(A \supset B) \supset (\Box A \supset \Box B)$
3. $\langle \lambda x.A \supset B \rangle(f) \supset [\langle \lambda x.A \rangle(f) \supset \langle \lambda x.B \rangle(f)]$
4. if x is not free in A then $\langle \lambda x.A \rangle(f) \supset A$

5. $\langle\lambda x.\varphi(x)\rangle(f) \supset \langle\lambda y.\varphi(y)\rangle(f)$ provided y is free for x in $\varphi(x)$
6. $\mathsf{D}(f) \supset [\langle\lambda x.A\rangle(f) \vee \langle\lambda x.\neg A\rangle(f)]$
7. $x = x$
8. $x = y \supset [P(\ldots, x, \ldots) \equiv P(\ldots, y, \ldots)]$ where P is a relation symbol of type $\langle\ldots, O, \ldots\rangle$
9. $(x = y) \supset \Box(x = y)$
10. $\neg(x = y) \supset \Box\neg(x = y)$
11. $\mathsf{D}(f) \supset \langle\lambda x, y.x = y\rangle(f, f)$

It is interesting to note that predicate abstraction itself behaves like a modality. Axiom 3 is an analog of the usual K axiom, while the λ Regularity Rule is an analog of the common modal rule of regularity. Strengthening it to an analog of the Necessitation Rule amounts to adding an assumption that intensions always designate. Axiom 11 embodies the idea that intensions are partial functions.

As usual, formulas asserting symmetry and transitivity of equality follow using axioms 7 and 8. Here are some more interesting consequences of these axioms.

Proposition 4.1 *The following are theorems in the $\{\lambda, =\}$ part of FOIL.*

1. $[\mathsf{D}(f) \wedge A] \supset \langle\lambda x.A\rangle(f)$ *provided x is not free in A*
2. $\langle\lambda x.A\rangle(f) \supset \mathsf{D}(f)$
3. $\langle\lambda x.A\rangle(f) \equiv [\mathsf{D}(f) \wedge A]$ *provided x is not free in A*
4. $\langle\lambda x.A \wedge B\rangle(f) \equiv [\langle\lambda x.A\rangle(f) \wedge \langle\lambda x.B\rangle(f)]$
5. $\langle\lambda x.\neg A\rangle(f) \supset \neg\langle\lambda x.A\rangle(f)$.
6. $\mathsf{D}(f) \supset [\langle\lambda x.\neg A\rangle(f) \equiv \neg\langle\lambda x.A\rangle(f)]$
7. $\mathsf{D}(f) \supset [\langle\lambda x.A \supset B\rangle(f) \equiv [\langle\lambda x.A\rangle(f) \supset \langle\lambda x.B\rangle(f)]$
8. $\langle\lambda x.A \vee B\rangle(f) \equiv [\langle\lambda x.A\rangle(f) \vee \langle\lambda x.B\rangle(f)]$
9. $[\langle\lambda y.x = y\rangle(f) \wedge \langle\lambda y.z = y\rangle(f)] \supset (x = z)$

Proof

1. By axiom 6 $\mathsf{D}(f) \supset [\langle \lambda x.A \rangle(f) \vee \langle \lambda x.\neg A \rangle(f)]$, so if x is not free in A, by axiom 4, $\mathsf{D}(f) \supset [\langle \lambda x.A \rangle(f) \vee \neg A]$. Then, of course, $[\mathsf{D}(f) \wedge A] \supset \langle \lambda x.A \rangle(f)$.

2. This follows immediately from $A \supset \mathit{true}$, the λ Regularity Rule, and the definition of D.

3. By parts 1, 2, and axiom 4.

4. Since predicate abstraction behaves like a modal operator, we should expect this theorem, and with the usual proof. First, $(A \wedge B) \supset A$ so by the λ Regularity Rule, $\langle \lambda x.A \wedge B \rangle(f) \supset \langle \lambda x.A \rangle(f)$. In a similar way $\langle \lambda x.A \wedge B \rangle(f) \supset \langle \lambda x.B \rangle(f)$, and so $\langle \lambda x.A \wedge B \rangle(f) \supset [\langle \lambda x.A \rangle(f) \wedge \langle \lambda x.B \rangle(f)]$.

   Second, $A \supset [B \supset (A \wedge B)]$ so by λ Regularity, $\langle \lambda x.A \rangle(f) \supset \langle \lambda x.B \supset (A \wedge B) \rangle(f)$. Then, using axiom 3, $\langle \lambda x.A \rangle(f) \supset [\langle \lambda x.B \rangle(f) \supset \langle \lambda x.A \wedge B \rangle(f)]$, and hence $[\langle \lambda x.A \rangle(f) \wedge \langle \lambda x.B \rangle(f)] \supset \langle \lambda x.A \wedge B \rangle(f)$.

5. Since $\neg A \supset (A \supset \mathit{false})$, using λ Regularity we have $\langle \lambda x.\neg A \rangle(f) \supset \langle \lambda x.A \supset \mathit{false} \rangle(f)$, and then by axiom 3, $\langle \lambda x.\neg A \rangle(f) \supset [\langle \lambda x.A \rangle(f) \supset \langle \lambda x.\mathit{false} \rangle(f)]$. Since x is not free in *false*, by axiom 4, $\langle \lambda x.\mathit{false} \rangle(f) \supset \mathit{false}$, hence we have $\langle \lambda x.\neg A \rangle(f) \supset [\langle \lambda x.A \rangle(f) \supset \mathit{false}]$ or equivalently $\langle \lambda x.\neg A \rangle(f) \supset \neg \langle \lambda x.A \rangle(f)$.

6. From part 5, $\mathsf{D}(f) \supset [\langle \lambda x.\neg A \rangle(f) \supset \neg \langle \lambda x.A \rangle(f)]$, and from axiom 6, $\mathsf{D}(f) \supset [\neg \langle \lambda x.A \rangle(f) \supset \langle \lambda x.\neg A \rangle(f)]$.

7. Since $P \supset Q$ is equivalent to $\neg(P \wedge \neg Q)$, this can be derived using parts 4 and 6.

8. $P \vee Q$ is equivalent to $\neg(\neg P \wedge \neg Q)$ so, using parts 4 and 6 one easily gets $\mathsf{D}(f) \supset \{\langle \lambda x.A \vee B \rangle(f) \equiv [\langle \lambda x.A \rangle(f) \vee \langle \lambda x.B \rangle(f)]\}$. Eliminating $\mathsf{D}(f)$ can be done using part 2.

9. $[\langle \lambda y.x = y \rangle(f) \wedge \langle \lambda y.z = y \rangle(f)] \supset \langle \lambda y.x = y \wedge z = y \rangle(f)$ by part 4. Also $(x = y \wedge z = y) \supset x = z$, so $\langle \lambda y.x = y \wedge z = y \rangle(f) \supset \langle \lambda y.x = z \rangle(f)$ using λ Regularity. Finally, $\langle \lambda y.x = z \rangle(f) \supset (x = z)$ using axiom 4.

■

Proposition 4.2 *Suppose $\varphi(z)$ is a formula such that z is free for both x and y. Then the following is provable in the $\{\lambda, =\}$ part of FOIL.*

$$(x = y) \supset [\varphi(x) \equiv \varphi(y)]$$

Proof This is by induction on the complexity of φ. Axiom 8 takes care of the atomic case, propositional connective cases are handled in the usual way, and the modal case is by a straightforward argument using axiom 9. This leaves one case remaining: $\varphi(z)$ is $\langle\lambda w.\psi(z)\rangle(f)$, where the result is known for ψ. Since z is required to be free for both x and y in $\varphi(z)$, w cannot be either x or y. Then we can proceed as follows.

1. $(x = y) \supset [\psi(x) \equiv \psi(y)]$
   induction hypothesis
2. $\langle\lambda z.x = y\rangle(f) \supset \langle\lambda z.\psi(x) \equiv \psi(y)\rangle(f)$
   from 1 by λ Regularity
3. $[\mathsf{D}(f) \wedge x = y] \supset \langle\lambda z.\psi(x) \equiv \psi(y)\rangle(f)$
   from 2 by Proposition 4.1 part 1
4. $[\mathsf{D}(f) \wedge x = y] \supset [\langle\lambda z.\psi(x)\rangle(f) \equiv \langle\lambda z.\psi(y)\rangle(f)]$
   from 3 using axiom 3 and Proposition 4.1 part 4
5. $\neg\mathsf{D}(f) \supset \neg\langle\lambda z.\psi(x)\rangle(f)$
   from Proposition 4.1 part 2
6. $\neg\mathsf{D}(f) \supset \neg\langle\lambda z.\psi(y)\rangle(f)$
   also from Proposition 4.1 part 2
7. $\neg\mathsf{D}(f) \supset [\langle\lambda z.\psi(x)\rangle(f) \equiv \langle\lambda z.\psi(y)\rangle(f)]$
   from 5 and 6
8. $(x = y) \supset [\langle\lambda z.\psi(x)\rangle(f) \equiv \langle\lambda z.\psi(y)\rangle(f)]$
   from 4 and 7

■

4.2 Soundness and Completeness

Soundness is quite straightforward. I'll leave it to you to check that the axioms are valid and the rules preserve validity. The rest of this section is devoted to a proof of completeness. I remind you again that the underlying propositional modal logic is K.

Given a set S of formulas, I'll write $S \vdash X$ if there is a finite subset $\{X_1, \ldots, X_n\} \subseteq S$ such that $(X_1 \wedge \ldots \wedge X_n) \supset X$ is a theorem of the $\{\lambda, =\}$ part of FOIL-K. As usual, if this is not the case it is symbolized by $S \nvdash X$. By definition, S is *consistent* if $S \nvdash$ *false*. Here is an important preliminary item.

Lemma 4.3 *Suppose S is a set of formulas, and the object variable x does not occur in S. If $S \cup \{\langle\lambda y.x = y\rangle(f)\}$ is inconsistent, then $S \vdash \neg\mathsf{D}(f)$.*

Proof Suppose $S \cup \{\langle \lambda y.x = y\rangle(f)\}$ is inconsistent. Then there is a conjunction of members of S, call it C, such that $(C \wedge \langle \lambda y.x = y\rangle(f)) \supset \mathit{false}$ is a theorem. Then, proceed as follows (recall, x does not occur in C).

1. $(C \wedge \langle \lambda y.x = y\rangle(f)) \supset \mathit{false}$
2. $\langle \lambda x.(C \wedge \langle \lambda y.x = y\rangle(f))\rangle(f) \supset \langle \lambda x.\mathit{false}\rangle(f)$
   from 1 by λ Regularity
3. $(\langle \lambda x.C\rangle(f) \wedge \langle \lambda x, y.x = y\rangle(f, f)) \supset \langle \lambda x.\mathit{false}\rangle(f)$
   from 2 by Proposition 4.1 part 4
4. $(\mathsf{D}(f) \wedge C \wedge \langle \lambda x, y.x = y\rangle(f, f)) \supset \langle \lambda x.\mathit{false}\rangle(f)$
   from 3 by Proposition 4.1 part 1
5. $(\mathsf{D}(f) \wedge C) \supset \langle \lambda x.\mathit{false}\rangle(f)$
   from 4 by axiom 11

6 $(\mathsf{D}(f) \wedge C) \supset \mathit{false}$
   from 5 by axiom 4

7. $C \supset \neg\mathsf{D}(f)$

■

With Henkin-style completeness arguments, the language must be enlarged to provide “witnesses” for existential statements. In our case we must provide designations for intension variables, which is a similar problem. Extra object variables will be added that are not part of the original language. I will call these added variables *parameters*. By convention, they will not be subject to λ-binding.

Let S be a consistent set of formulas (possibly including parameters). And let P be a countable set of new parameters—new means they do not occur in formulas of S. I describe the notion of *completing* S with respect to P.

Let us say f_1, f_2, ... is an enumeration of all intension variables. Enlarge S by adding one of $\mathsf{D}(f_1)$ or $\neg\mathsf{D}(f_1)$ so that the resulting set is consistent. Then enlarge the resulting set with $\mathsf{D}(f_2)$ or $\neg\mathsf{D}(f_2)$, again preserving consistency. And so on. Let S^* be the limit set—it is consistent and contains exactly one of $\mathsf{D}(f)$ or $\neg\mathsf{D}(f)$ for every intension variable f.

Let p_1, p_2, ... be an enumeration of P. Let S^{**} be the result of enlarging S^* as follows. For each intension variable f_n, if $\mathsf{D}(f_n) \in S^*$, put $\langle \lambda y.p_n = y\rangle(f_n)$ into S^{**}, and otherwise don't. Using Lemma 4.3, S^{**} is consistent.

Finally, let S^{***} be an extension of S^{**} that is a maximally consistent subset of the set of all $\{\lambda, =\}$ formulas with object variables from P, as well as the usual object and intension variables. (Recall that parameters are not

λ-bound.) S^{***} will be called *a P-completion* of S, and will be said to be *P-complete*.

Again, let S be a consistent set of formulas, with parameters allowed. I describe the construction of what I will call a successor set for S. Let $S^\sharp = \{X \mid \Box X \in S\}$. If $\neg\Box Y \in S$, by a standard modal logic argument $S^\sharp \cup \{\neg Y\}$ is consistent. Now, enumerate all formulas in S that are negated necessitations: $\neg\Box Y_1$, $\neg\Box Y_2$, ... And let P_1, P_2, ..., be a sequence of sets of parameters such that: each P_i is infinite; if $i \neq j$ then $P_i \cap P_j = \emptyset$; and no member of any P_i occurs in S. For each $i = 1, 2, \ldots$, let S_i be a P_i completion of $S^\sharp \cap \{\neg Y_i\}$ (in particular, then, S_i is consistent). The set $\{S_1, S_2, \ldots\}$ is *a successor set* for S, and for each n, the set P_n is the *new parameter set* of S_n.

Finally, an extension of the notion above. Let $\{S^1, S^2, \ldots\}$ be a collection of consistent sets of formulas. A *successor set* for this is a collection $\{S^1_1, S^1_2, \ldots, S^2_1, S^2_2, \ldots\}$ where for each i, $\{S^i_1, S^i_2, \ldots\}$ is a successor set for S^i. In addition to the successor set conditions above, we also impose the following additional requirements. Let P^i_n be the new parameter set of S^i_n. Then no member of P^i_n occurs in any S_m, and $P^i_n \cap P^j_m = \emptyset$.

Let X be a parameter-free non-theorem of the $\{\lambda, =\}$ part of FOIL-K. X is fixed for what follows. I will produce a counter-model for X, thus establishing completeness.

As usual, $\{\neg X\}$ is consistent. Let C^0 be a P completion of $\{\neg X\}$, for some set P of new parameters. I'll call $\{C^0\}$ *stage 0*. Then, having defined stage n, $\{C^n_1, C^n_2, \ldots\}$, let *stage* $n+1$ be some successor set to stage n, $\{C^{n+1}_1, C^{n+1}_2, \ldots\}$. In this way, stage n is defined for all n. Let $\mathcal{G}$ be the union of all the stages: $\{C^0, C^1_1, C^1_2, \ldots, C^2_1, C^2_2, \ldots\}$. For $\Gamma, \Delta \in \mathcal{G}$, set $\Gamma\mathcal{R}\Delta$ provided Δ is a member of the successor set for Γ, in the stage construction just described. Thus we have a frame $\langle\mathcal{G}, \mathcal{R}\rangle$. Note that if $\Gamma\mathcal{R}\Delta$, for $\Gamma, \Delta \in \mathcal{G}$, then $\Gamma^\sharp \subseteq \Delta$.

Next we must define the object domain of our model, and this takes a bit of work. We want to factor the set of object variables into equivalence classes, where the equivalence relation is obtained using the $=$ symbol and its behavior in members of $\mathcal{G}$. But producing the equivalence relation requires some effort.

Object variables are either from the original language, x, y, ..., or they are parameters, p_1, p_2, ... For the rest of this discussion I'll use v, w, v_1, v_2, ... for object variables of either kind. For each $\Gamma \in \mathcal{G}$, let $d(\Gamma)$ be the set of object variables (parameters included) that occur in formulas in Γ, and let $\mathcal{V} = \cup_{\Gamma \in \mathcal{G}} d(\Gamma)$. Thus $\mathcal{V}$ is the set of all object variables from the original language together with those parameters introduced in the construction of $\mathcal{G}$.

Let $\leq_{\mathcal{R}}$ be the transitive, reflexive closure of $\mathcal{R}$. Note that by our construction, if $\Gamma \leq_{\mathcal{R}} \Delta$, then $d(\Gamma) \subseteq d(\Delta)$. By the newness conditions imposed on parameters, each parameter in $\mathcal{V}$ was introduced as a new parameter in the stages of the construction of $\mathcal{G}$ exactly once. Object variables from the original language appear in C_0, of course. So for each $v \in \mathcal{V}$ there is a unique member of $\mathcal{G}$, call it $[v]$, that saw v introduced and, if Δ is any member of $\mathcal{G}$ with $v \in d(\Delta)$, we must have $[v] \leq_{\mathcal{R}} \Delta$. The following important fact will be used several times below: If $v, w \in d(\Delta)$ for some $\Delta \in \mathcal{G}$, then either $[v] \leq_{\mathcal{R}} [w]$ or $[w] \leq_{\mathcal{R}} [v]$ (because the 'ancestry' of Δ is linearly ordered.)

Lemma 4.4 *If $(v = w)$ is in some member of $\mathcal{G}$ then $(v = w) \in \Gamma$ for every $\Gamma \in \mathcal{G}$ such that $v, w \in d(\Gamma)$.*

Proof Let Δ be a fixed member of $\mathcal{G}$ and suppose $(v = w) \in \Delta$.

If $\Delta \mathcal{R} \Omega$ then $(v = w) \in \Omega$ as well; let us call this the preservation of equality under successor. Here is the simple argument for this. Since $(v = w) \in \Delta$, then also $\Box(v = w) \in \Delta$ by axiom 9 and the maximal consistency of Δ. Then $(v = w) \in \Delta^\sharp \subseteq \Omega$. By a similar argument, if $\Omega \mathcal{R} \Delta$, and $v, w \in d(\Omega)$, then $(v = w) \in \Omega$; this time we use axiom 10. Let us call this the preservation of equality under predecessor.

Since $v, w \in d(\Delta)$, we have that $[v]$ and $[w]$ are comparable under $\leq_{\mathcal{R}}$; say $[v] \leq_{\mathcal{R}} [w]$. Then if Γ is any member of $\mathcal{G}$ with $v, w \in d(\Gamma)$, $[w] \leq_{\mathcal{R}} \Gamma$. (In particular, $[w] \leq_{\mathcal{R}} \Delta$.) Then by repeated application of the preservation of equality under predecessor from the previous paragraph, we must have $(v = w) \in [w]$, and by the preservation of equality under successor, we must also have $(v = w) \in \Gamma$. ■

Define a binary relation on $\mathcal{V}$ by setting $v \sim w$ if the formula $(v = w)$ occurs in some member of $\mathcal{G}$. The Lemma above says this does not depend on particular choices of members of $\mathcal{G}$. The relation is reflexive and symmetric, but it need not be transitive since $(v_1 = v_2)$ and $(v_2 = v_3)$ might occur in different members of $\mathcal{G}$ while v_1 and v_3 do not appear together in any member, and so of course $(v_1 = v_3)$ does not occur. We do, however, have the following weak version of transitivity.

Lemma 4.5 *Let v_1, v_2, v_3, v_4 be members of $\mathcal{V}$ and suppose that $v_1 \sim v_2$, $v_2 \sim v_3$, and $v_3 \sim v_4$. Then either $v_1 \sim v_3$ or $v_2 \sim v_4$.*

Proof Suppose we can show that v_1, v_2, and v_3 are all in $d(\Gamma)$ for some $\Gamma \in \mathcal{G}$. Using Lemma 4.4, that member must contain $(v_1 = v_2)$ and $(v_2 = v_3)$. Then using the axiomatically provable transitivity of $=$, we can conclude that $(v_1 = v_3) \in \Gamma$, and hence $v_1 \sim v_3$. Similarly if v_2, v_3, and v_4 are all in $d(\Gamma)$ for some $\Gamma \in \mathcal{G}$ we can conclude that $v_2 \sim v_4$. So it is enough to show that either v_1, v_2, v_3 or else v_2, v_3, v_4 are all in $d(\Gamma)$ for some $\Gamma \in \mathcal{G}$. And doing this involves a case analysis.

Since $v_1 \sim v_2$ we must have $v_1, v_2 \in d(\Gamma)$ for some $\Gamma \in \mathcal{G}$, and so $[v_1]$ and $[v_2]$ are comparable. Similarly $[v_2]$ and $[v_3]$ are comparable, and $[v_3]$ and $[v_4]$ are comparable. The overall proof divides into two main cases, depending on which way $[v_1]$ and $[v_2]$ are ordered.

Suppose first that $[v_1] \leq_{\mathcal{R}} [v_2]$. We have that $[v_2]$ and $[v_3]$ are comparable. If $[v_2] \leq_{\mathcal{R}} [v_3]$, then all of v_1, v_2, and v_3 are in $d([v_3])$. If $[v_3] \leq_{\mathcal{R}} [v_2]$, then all of v_1, v_2, and v_3 are in $d([v_2])$.

Next suppose $[v_2] \leq_{\mathcal{R}} [v_1]$. Again, $[v_2]$ and $[v_3]$ are comparable. If $[v_3] \leq_{\mathcal{R}} [v_2]$, then all of v_1, v_2, and v_3 are in $d([v_1])$. Now suppose $[v_2] \leq_{\mathcal{R}} [v_3]$. We also have that $[v_3]$ and $[v_4]$ are comparable. If $[v_3] \leq_{\mathcal{R}} [v_4]$ then all of v_2, v_3 and v_4 are in $d([v_4])$. If $[v_4] \leq_{\mathcal{R}} [v_3]$, then all of v_2, v_3 and v_4 are in $d([v_3])$. ■

Let $\sim^*$ be the transitive closure of $\sim$; it is also reflexive and symmetric since this is the case with $\sim$. Of course if $v \sim w$, then $v \sim^* w$. There is also a kind of converse. It is what the Lemmas above have been leading up to.

Proposition 4.6 *If $v, w \in d(\Gamma)$ for some $\Gamma \in \mathcal{G}$, and $v \sim^* w$, then $v \sim w$.*

Proof Assume $v \sim^* w$, and $v, w \in d(\Gamma)$ for some $\Gamma \in \mathcal{G}$. Since $\sim^*$ is the transitive closure of $\sim$, there must be $r_1, \ldots, r_k$ such that $v \sim r_1$, $r_1 \sim r_2$, $\ldots$, $r_k \sim w$. The proof is by induction on k.

The ground case, $k = 1$, is direct. Since $v \sim r_1$, then $v, r_1 \in d(\Delta)$ for some $\Delta \in \mathcal{G}$, and so $[v]$ and $[r_1]$ are comparable under $\leq_{\mathcal{R}}$. Similarly $[r_1]$ and $[w]$ are comparable, as are $[v]$ and $[w]$ since $v, w \in d(\Gamma)$. Then the set $S = \{[v], [r_1], [w]\}$ is linearly ordered. It follows that all three of v, r_1, w are in a single member of $\mathcal{G}$, namely whichever of S is biggest under the $\leq_{\mathcal{R}}$ ordering. That member must contain $(v = r_1)$ and $(r_1 = w)$ by Lemma 4.4. Then, using the provable transitivity of $=$, and the maximality of members of $\mathcal{G}$, that member must also contain $(v = w)$, and hence we have $v \sim w$.

Next, suppose the result is known for $k = n$, and now we have $k = n+1$. In this case, $k \geq 2$; for notational convenience, if $k = 2$, we'll identify r_3 with w. Now the situation is, we have a chain of relations, $v \sim r_1$, $r_1 \sim r_2$,

$r_2 \sim r_3, \ldots, r_k \sim w$. Using Lemma 4.5, either $v \sim r_2$ or else $r_1 \sim r_3$. Thus we also have one of the following two chains of relations: $v \sim r_2$, $r_2 \sim r_3$, $\ldots$, $r_k \sim w$ or $v \sim r_1$, $r_1 \sim r_3$, $\ldots$, $r_k \sim w$. In either case we have a shorter chain, so the induction hypothesis applies, and we conclude $v \sim w$. ∎

Corollary 4.7 *If $v, w \in d(\Gamma)$, and $v \sim^* w$, then $(v = w) \in \Gamma$.*

Proof By Proposition 4.6 and Lemma 4.4. ∎

Moving ahead with the definition of our model, we have $\mathcal{G}$ and $\mathcal{R}$. Let the object domain, $\mathcal{D}_O$, be the set of equivalence classes of $\mathcal{V}$ using the equivalence relation $\sim^*$. The equivalence class containing v will be denoted $\overline{v}$.

Next we define the intension domain, $\mathcal{D}_I$. For each intension variable f define a partial function, denoted $\overline{f}$, on $\mathcal{D}_O$ as follows. The domain of $\overline{f}$ is the set of $\Gamma \in \mathcal{G}$ containing $\mathsf{D}(f)$. If Γ is in the domain of $\overline{f}$, by the way members of $\mathcal{G}$ were constructed, Γ contains $\langle \lambda y.p = y\rangle(f)$ for some parameter p. By Proposition 4.1 part 9, if $\langle \lambda y.v = y\rangle(f)$ is also in Γ, then $(p = v)$ is in Γ, and hence $p \sim^* v$. Now set $\overline{f}(\Gamma) = \overline{v}$, where $\langle \lambda y.v = y\rangle(f) \in \Gamma$. And let $\mathcal{D}_I$ be the set of $\overline{f}$ for all intension variables f.

Finally we define an interpretation $\mathcal{I}$, a definition that splits into two cases. First, for the equality symbol, set $\mathcal{I}(=)$ to be the constant function assigning to each member of $\mathcal{G}$ the equality relation on $\mathcal{D}_O$. Then $\langle \overline{v}, \overline{w}\rangle \in \mathcal{I}(=)(\Gamma)$ if and only if $v \sim^* w$. Next, let P be a relation symbol other than $=$; for notational convenience, assume that P is of type $\langle O, O, \ldots, I, I, \ldots\rangle$. Let $\Gamma \in \mathcal{G}$; we set $\langle \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots\rangle \in \mathcal{I}(P)(\Gamma)$ just in case there are $w_1, w_2, \ldots$ in $d(\Gamma)$ with $w_i \in \overline{v_i}$, such that $P(w_1, w_2, \ldots, f_1, f_2, \ldots) \in \Gamma$.

We have now completed defining our model, $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}_O, \mathcal{D}_I, \mathcal{I}\rangle$. What we need next is an appropriate version of the usual 'truth lemma.' Notation can get somewhat cluttered; to keep things relatively simple, I will write $\varphi(v_1, \ldots, v_n, f_1, \ldots, f_k)$ to indicate a formula with the displayed variables free, and I will list the object variables before the intension variables. More importantly, I will write $\mathcal{M}, \Gamma \Vdash \varphi(\overline{v_1}, \overline{v_2}, \ldots, \overline{g_1}, \overline{g_2}, \ldots)$ instead of writing $\mathcal{M}, \Gamma \Vdash_v \varphi(x_1, x_2, \ldots, f_1, f_2, \ldots)$ where v is the valuation such that $v(x_i) = \overline{v_i}$ and $v(f_i) = \overline{g_i}$.

Proposition 4.8 *Let $\Gamma \in \mathcal{G}$, and let $\varphi(v_1, v_2, \ldots, f_1, f_2, \ldots)$ be a formula. If $v_1, v_2, \ldots \in d(\Gamma)$ then $\mathcal{M}, \Gamma \Vdash \varphi(\overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)$ if and only if $\varphi(v_1, v_2, \ldots, f_1, f_2, \ldots) \in \Gamma$.*

Proof The proof is by induction on the complexity of φ. The ground, atomic, case is by definition, for relation symbols other than $=$, and by Corollary 4.7 for $=$.

For the induction step, the propositional connective cases are simple and are omitted. For the case where φ is of the form $\Box\psi$ and the result is in known for ψ, we rely on the fact that, by construction, if $\Box\psi$ is in Γ then ψ is in every member of the successor set of Γ, and if $\neg\Box\psi$ is in Γ then $\neg\psi$ is in some member of the successor set. (Recall that $\Gamma\mathcal{R}\Delta$ was defined to mean that Δ is in the successor set of Γ.)

Finally assume $\varphi(v_1, v_2, \ldots, f_1, \ldots)$ is $\langle\lambda x.\psi(x, v_1, v_2, \ldots, f_1, f_2, \ldots)\rangle(f)$ and the result is known for ψ. Assume $v_1, v_2, \ldots \in d(\Gamma)$. Of course x is not one of v_1, v_2, $\ldots$, which are assumed to be free in φ.

Suppose first that $\mathcal{M}, \Gamma \Vdash \langle\lambda x.\psi(x, \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)\rangle(\overline{f})$. Then f must designate at Γ, that is, Γ is in the domain of $\overline{f}$, and $\mathcal{M}, \Gamma \Vdash \psi(\overline{w}, \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)$ where $\overline{f}(\Gamma) = \overline{w}$. Since $\overline{f}(\Gamma) = \overline{w}$, we must have that $\langle\lambda y.p = y\rangle(f) \in \Gamma$ for some parameter p such that $p \in \overline{w}$. Since $\overline{p} = \overline{w}$, we have $\mathcal{M}, \Gamma \Vdash \psi(\overline{p}, \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)$. Obviously $p \in d(\Gamma)$, so by the induction hypothesis, $\psi(p, v_1, v_2, \ldots, f_1, f_2, \ldots) \in \Gamma$. Since $\langle\lambda y.p = y\rangle(f) \in \Gamma$, we also have $\langle\lambda x.p = x\rangle(f) \in \Gamma$, using axiom 5. Now the following is axiomatically provable (I'll leave it as an exercise):

$$[\langle\lambda x.p = x\rangle(f) \wedge \psi(p, v_1, v_2, \ldots, f_1, f_2, \ldots)] \supset \\ \langle\lambda x.\psi(x, v_1, v_2, \ldots, f_1, f_2, \ldots)\rangle(f)$$

It follows that $\langle\lambda x.\psi(x, v_1, v_2, \ldots, f_1, f_2, \ldots)\rangle(f) \in \Gamma$, that is, $\varphi(v_1, v_2, \ldots, f_1, f_2, \ldots) \in \Gamma$.

Next, suppose $\varphi(v_1, v_2, \ldots, f_1, f_2, \ldots) \in \Gamma$, that is, $\langle\lambda x.\psi(x, v_1, v_2, \ldots, f_1, f_2, \ldots)\rangle(f) \in \Gamma$. By Proposition 4.1 part 2, $\mathsf{D}(f) \in \Gamma$. Since Γ is a P-complete set, $\langle\lambda y.p = y\rangle(f) \in \Gamma$ for some parameter p, and so of course we also have $\langle\lambda x.p = x\rangle(f) \in \Gamma$. We also have the following axiomatically provable formula (another exercise):

$$[\langle\lambda x.\psi(x, v_1, v_2, \ldots, f_1, f_2, \ldots)\rangle(f) \wedge \langle\lambda x.p = x\rangle(f)] \supset \\ \psi(p, v_1, v_2, \ldots, f_1, f_2, \ldots)$$

It follows that $\psi(p, v_1, v_2, \ldots, f_1, f_2, \ldots) \in \Gamma$, so by the induction hypothesis, $\mathcal{M}, \Gamma \Vdash \psi(\overline{p}, \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)$. Also since $\mathsf{D}(f) \in \Gamma$, then Γ is in the domain of $\overline{f}$, and since $\langle\lambda x.p = x\rangle(f) \in \Gamma$, then $\overline{f}(\Gamma) = \overline{p}$. It follows that $\mathcal{M}, \Gamma \Vdash \langle\lambda x.\psi(x, \overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)\rangle(\overline{f})$, that is, $\mathcal{M}, \Gamma \Vdash \varphi(\overline{v_1}, \overline{v_2}, \ldots, \overline{f_1}, \overline{f_2}, \ldots)$. ■

With Proposition 4.8 established, completeness is immediate. Recall that the construction of the model $\mathcal{M}$ began with a non-theorem, X. One of the members of $\mathcal{G}$, it was denoted C^0, contains $\neg X$, and hence by Proposition 4.8, there is a state in $\mathcal{G}$ at which X fails. More precisely, we have $\mathcal{M}, C^0 \not\Vdash_v X$ where v is the valuation mapping each object variable x to $\overline{x} \in \mathcal{D}_O$ and each intension variable f to $\overline{f} \in \mathcal{D}_I$.

4.3 Remarks

There are several comments that should be made concerning the completeness proof above. First, the model constructed in the course of the proof is, at the end, treated as a constant domain model—$\mathcal{D}_O$ is the same for each world. In fact the construction is a kind of monotonic varying domain construction, with each world Γ having its own domain, denoted $d(\Gamma)$. The reason we can get away with this discrepancy is that no quantifiers are present, so there is no way we can 'see' the difference between constant and varying domains. This suggests that the addition of quantifiers will require a completeness proof of a different sort.

The completeness proof just given was only for an underlying logic K, as was stated at the beginning of Section 4.1. It should be clear that a similar argument will work for T, D, K4, and S4. It will *not* work once symmetry comes into the picture—in particular, it will not work for S5. This is related to the point mentioned above—the structure of the model constructed during the completeness argument is essentially varying domain, while symmetry forces constant domains on us. Without quantifiers available, I don't know how to state something like the Barcan formula, and so I don't know how to reconcile symmetry requirements with the model construction given. Completeness of the $\{\lambda, =\}$ part of FOIL, with an underlying logic of, say, S5 is an interesting question.

Finally, as noted, we do have completeness of the $\{\lambda, =\}$ part of FOIL with an underlying logic of K, T, D, K4, and S4. What the model construction does not bring out is that even here there are significant differences. $\{\lambda, =\}$ logics based on K, T, and D are decidable, while those based on K4 and S4 are not; this was proved in [2]. The decidability part makes use of tableaus, while the undecidability part involves a direct argument about the semantics. Thus both transitivity and symmetry have strong, though different, effects on the $\{\lambda, =\}$ part of FOIL.

5 With Quantifiers

Now that the $\{\lambda, =\}$ part of FOIL has been axiomatized, at least for some choices of underlying modal logics, it is time to turn our attention to the entire of FOIL. But as was noted in Section 4.3, the completeness argument given when quantifiers are not present is not likely to extend to encompass them. So, the axiomatization presented now is along quite different lines. What is done is to cheat a bit. In (1), from Section 3, we saw that a designation predicate can be defined using predicate abstraction and equality. One can go in the other direction as well. Suppose we had a *primitive* designation predicate of type $\langle I, O\rangle$. We could then take the following as a definition.

$$\langle \lambda x.\varphi(x)\rangle(f) \text{ abbreviates the formula } (\exists x)[\mathsf{D}(f,x) \wedge \varphi(x)] \tag{3}$$

This is the route followed here. Rather than taking predicate abstraction as primitive, I will assume there is a special relation symbol D, satisfying an axiom that just says no more than one object can be designated by an intension. Predicate abstraction will be defined from it, using (3).

Let's begin with the routine part of the axiomatization, and get it out of the way. Some of the axioms also appeared earlier, when the $\{\lambda, =\}$ part of FOIL was being considered, but I'll repeat them to keep the section self-contained—I have not tried to make the numbering of axioms in the two sections match up. Recall, there are two types of variables, but since many of the intensional and extensional axioms are similar, I'll use α and β as metavariables, of either type. Now, here are the 'classical' axioms.

1. All tautologies
2. $(\forall\alpha)\varphi(\alpha) \supset \varphi(\beta)$ where β is free for α in φ (and α and β are the same type variables)
3. $(\forall\alpha)[\psi \supset \varphi(\alpha)] \supset [\psi \supset (\forall\alpha)\varphi(\alpha)]$ where α is not free in ψ

And of course, we take the following as rules.

Modus Ponens

$$\frac{X \quad X \supset Y}{Y}$$

Universal Generalization Rules

$$\frac{\varphi(\alpha)}{(\forall\alpha)\varphi(\alpha)}$$

Next we need axioms for equality, as before.

4. $x = x$

5. $x = y \supset [P(\ldots, x, \ldots) \equiv P(\ldots, y, \ldots)]$ where P is a relation symbol of type $\langle \ldots, O, \ldots \rangle$

6. $(x = y) \supset \Box(x = y)$

7. $\neg(x = y) \supset \Box\neg(x = y)$

Of course we need the usual modal axioms and rules.

8. $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$

9. Axioms for T, $\mathsf{K4}$, $\mathsf{S4}$, as desired

Necessitation Rule

$$\frac{X}{\Box X}$$

We have decided to use possibilist, constant domain, quantification, so Barcan formulas are needed for both types. We have also decided to have an existence predicate, and we (probably) want to assume some objects actually exist.

10. $(\forall \alpha)\Box\varphi(\alpha) \supset \Box(\forall \alpha)\varphi(\alpha)$

11. $(\exists x)\mathsf{E}(x)$

And finally, there is an axiom covering the behavior of a primitive designation relation, D, asserting its partial functionality.

12. $[\mathsf{D}(f, x) \wedge \mathsf{D}(f, z)] \supset x = z$

If we wanted to restrict things to models in which intensions are total functions, as was the case for part of [5], we could add the following as an axiom: $(\forall f)(\exists x)\mathsf{D}(f, x)$. This will not be assumed here.

We have now completed giving the axiomatization. I note in passing that the idea of treating predicate abstraction as defined instead of primitive can be adapted to tableau systems too—this leads to a simpler system than that presented in [5]. Since this is rather straightforward, I skip any further discussion.

6 Soundness and Completeness

As usual, soundness can be given short shrift. The axioms are valid and the rules preserve validity. The only minor complication comes with axiom 12, because D is primitive in the axiomatization, but defined in the semantics. Even so, using (1), validity of axiom 12 is straightforward. What must be shown valid is $[\langle \lambda y.y = x\rangle(f) \wedge \langle \lambda y.y = z\rangle(f)] \supset (x = z)$, which is simple. I note that this is essentially the formula that was proved in the axiom system of Section 4.1, in Proposition 4.1 part 9.

Completeness, of course, is more work. However, modal completeness proofs using the Barcan formula, and producing constant domain models, are rather standard in the literature, see [9, 10, 8], so I will just sketch an argument rather than giving it in detail.

As axiomatized in Section 5, we have a two-sorted modal system, with axiom 12 being the only postulated connection between the sorts. Each sort obeys the usual axioms and rules of first order modal logic, along with the Barcan formula. As noted, the usual completeness arguments apply in the two-sorted setting just as well as in the one-sorted case. Thus, if a formula X is not provable in the axiom system of Section 5, a two-sorted modal counter-model for X exists that is constant domain in both sorts.

Next, objects obey axioms for equality, and so we can factor the object domain of the model into equivalence classes to produce a new model in which the interpretation of the symbol $=$ is the equality relation. This requires a little care, but is relatively straightforward. If we restrict our original model to those states accessible from a single state, Γ, then using axiom 7 one shows that if any state 'thinks' x and y are equal, Γ will 'think' that, and using axiom 6, if Γ 'thinks' $x = y$ then every state will also 'think' that. It follows that the behavior of $=$ is state independent, and so can be used to define an equivalence relation globally.

What was described above are standard constructions, though they are applied in a two-sorted setting. Thus I have omitted specifics, and start the detailed work from here. We begin from the following position. If X is not a theorem of the axiom system of Section 5, it is falsified at some world of a two-sorted model of the following kind. The model is $\widehat{\mathcal{M}} = \langle \widehat{\mathcal{G}}, \widehat{\mathcal{R}}, \widehat{\mathcal{D}_O}, \widehat{\mathcal{D}_I}, \widehat{\mathcal{I}}\rangle$. In it, $\langle \widehat{\mathcal{G}}, \widehat{\mathcal{R}}\rangle$ is a frame, as usual. $\widehat{\mathcal{D}_O}$ and $\widehat{\mathcal{D}_I}$ are (constant) domains of quantification, for type O and type I respectively. $\widehat{\mathcal{I}}$ is an interpretation that, in particular, assigns to the symbol $=$ of type $\langle O, O\rangle$ the equality relation on $\widehat{\mathcal{D}_O}$. $\widehat{\mathcal{M}}$ is *not* a FOIL model, since there is no requirement that $\widehat{\mathcal{D}_I}$ consist of partial functions (the usual completeness proofs will result in

$\widehat{\mathcal{D}_I}$ being a set of variables). This needs a small fix. I will use the notation $\widehat{\mathcal{M}}, \Gamma \Vdash_v \varphi$ in the obvious way, even though $\widehat{\mathcal{M}}$ is not quite a FOIL model. By construction, every axiom is validated in this model.

Now the final steps, to convert $\widehat{\mathcal{M}}$ into a proper FOIL model. Construct a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}_O, \mathcal{D}_I, \mathcal{I} \rangle$ as follows. $\mathcal{G} = \widehat{\mathcal{G}}$, $\mathcal{R} = \widehat{\mathcal{R}}$, $\mathcal{D}_O = \widehat{\mathcal{D}_O}$. For each $d \in \widehat{\mathcal{D}_I}$, let f_d be the function whose domain is

$$\{\Gamma \in \mathcal{G} \mid \widehat{\mathcal{M}}, \Gamma \Vdash_v (\exists x)\mathsf{D}(f, x) \text{ where } v(f) = d\}$$

and for each Γ in the domain of f_d, let

$$\begin{aligned} f_d(\Gamma) = {} & \text{the } o \in \mathcal{D}_O \\ & \text{such that } \widehat{\mathcal{M}}, \Gamma \Vdash_v \mathsf{D}(f, x) \\ & \text{where } v(f) = d \text{ and } v(x) = o \end{aligned}$$

This does define a partial function, using the validity in $\widehat{\mathcal{M}}$ of axiom 12. Let $\mathcal{D}_I$ be the set of all f_d for $d \in \widehat{\mathcal{D}_I}$. Finally we must define $\mathcal{I}$ and for this it is convenient to introduce a mapping, t, as follows. The domain of t is $\widehat{\mathcal{D}_O} \cup \widehat{\mathcal{D}_I}$. For $o \in \widehat{\mathcal{D}_O}$, set $t(o) = o$. And for $d \in \widehat{\mathcal{D}_I}$, set $t(d) = f_d$, where this is defined as above. Now, for a relation symbol P, set $\mathcal{I}(P)(\Gamma) = \{\langle t(s_1), \ldots, t(s_n)\rangle \mid \langle s_1, \ldots, s_n\rangle \in \widehat{\mathcal{I}}(P)(\Gamma)\}$. This ends the description of $\mathcal{M}$. It is a proper FOIL model, and invalidates the same formulas that $\widehat{\mathcal{M}}$ invalidates. And this completes the brief sketch of completeness.

7 Conclusion

I conclude by setting out various topics that are, I think, worth exploration, which have been suggested by the investigations here. I invite others to work on these.

Two quite different styles of axiomatization have been given, one for the $\{\lambda, =\}$ part of FOIL, and one for the entire of it. Obviously the version for FOIL itself, from Section 5, does not scale down to the $\{\lambda, =\}$ part, since the definition of predicate abstraction in terms of the designation relation makes essential use of quantification, which is not available in the $\{\lambda, =\}$ part. On the other hand, the axiomatization of the sublogic, from Section 4.1, could scale up to the full FOIL. What does not scale up is the particular completeness proof given here. It would be interesting to know if a complete axiomatization of FOIL can be given with predicate abstraction primitive, using various choices of underlying propositional modal logics, in particular, logics involving a symmetric accessibility relation.

The relationship between predicate abstraction and quantification is real, but not well understood. In [2] some undecidability results were established by using predicate abstraction, semantically, to simulate classical quantification. A deeper investigation of the quantifier/abstraction connection is, I think, warrented.

In [5], an embedding from counterpart semantics into FOIL was sketched. This was worked out in some detail in [11]. In [5], the particular version of counterpart semantics presented was somewhat restricted. And the embedding was hard to establish, largely because *partial* intensions were not brought in. I believe that with them a broader range of counterpart semantics can be embedded into FOIL, with an easier proof. But this too remains for the future.

Finally, what is the significance of FOIL for investigations in intensionality? It is, in a way, a rather bland system, containing everything indiscriminately, objects, intensions, equality, designation, and so on. Everybody should find something to dislike about it. On the other hand, everybody should find some portion of it to their particular liking. If one takes the intersection of various approaches to intensionality one is left, I believe, with the empty set. In a sense, FOIL represents the union. That is its primary virtue.

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