Barcan Both Ways*

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August 3, 1997

1 Introduction

The Barcan formula made its first appearance in [1] as $\Diamond(\exists x)\Phi \mathrel{-\!\!\!3} (\exists x)\Diamond\Phi$, in the logic **S2**. This logic, **S2**, despite interesting properties, is not much seen these days, and the Barcan formula itself has been simplified to $\Diamond(\exists x)\Phi \supset (\exists x)\Diamond\Phi$. The paper [1] was one of the earliest serious investigations of first-order modal logic and, of necessity, it was axiomatic since a first-order modal semantics was still some years away. In the system as formulated, the converse of the Barcan formula was provable, but the Barcan formula itself was not. Consequently the Barcan formula was added as an axiom schema because it enabled an interesting development that otherwise seemed impossible.

The fact that the converse Barcan formula was provable axiomatically, and in Gentzen sequent calculi, led to a misjudgment about the relative importance of it and the Barcan formula. That this was a misjudgment did not become clear until Kripke, [8], when semantics clarified respective roles. Informally speaking, the Barcan formula says that nothing comes into existence on moving from a possible world to an alternative, while the converse Barcan formula says that nothing goes out of existence. To see this, a semantics making no assumptions about relationships between quantifier domains from world to world was needed. At this point it was realized that the axiomatic provability of the converse Barcan formula was a consequence of the particular choice of axiomatic treatment—in a sense the "standard" treatment had the converse Barcan formula built in.

Unfortunately, to this day effects of the early confusion are still with us. Many treatments of first-order modal logic, my own [4] included, use a semantics with monotonicity assumed: if world Δ is accessible from world Γ, everything in the domain of quantification at Γ is also present at Δ. This, in fact, is not a particularly natural assumption for most applications of modal logic, certainly not for most philosophical applications. The recent [7] is one of the few textbook treatments that takes a broader view.

*A few years ago George Gargov and his wife spent an afternoon at my house, and we talked of an article he, or possibly he and I, would write on the foundations and history of bilattices. Sadly George never wrote that article, and I cannot without him. He knew a vast amount about the prehistory of bilattices and its many anticipations, things I never knew, and without him, still don't. Bilattices would have been the ideal topic for a memorial paper, but I no longer work in the area. What follows reflects my current study and, I think, George would have found it of some interest.

The primary addition this paper makes to the literature is an alternative formulation of both the Barcan and the converse Barcan formulas, making use of equality. Ordinarily the Barcan formula and its converse are *schemes*—infinitely many formulas are involved. By reformulating them using equality, each becomes a single formula. While it is easy to verify this using semantical arguments, for novelty sake we give an axiomatic proof, based on systems from [7]. As a consequence, tableau systems for varying domain modal logics can be easily adapted to monotonic or anti-monotonic versions.

In addition we have some remarks to make about allowing domains in modal models to vary arbitrarily or, at the other extreme, insisting they all be the same. The point is made in [7] that these correspond to two well-known philosophical positions on quantification. In a certain sense, a choice between them makes no difference. Whichever we choose, the other version can be "discussed." In one direction, a simple embedding is available. In the other direction, taking the Barcan and converse Barcan formulas as premises allows discourse about constant domain logics using varying domain machinery. We feel this can not be said often enough, since there is much technical and philosophical confusion here.

2 Formulas and models

Atomic formulas are of the form $R(x_1, \ldots, x_n)$, where R is a relation symbol and the x_i are variables. We do not consider constant or function symbols here. We have an equality symbol as one of our relation symbols, and write it in infix position in the customary way. More complex formulas are built up as usual, using $\neg$, $\wedge$, $\Box$, and $\forall$. We take other propositional connectives, $\Diamond$ and $\exists$ as defined.

An *augmented frame* is a structure $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ where: $\mathcal{G}$ is a non-empty set of possible worlds; $\mathcal{R}$ is an accessibility relation on $\mathcal{G}$; $\mathcal{D}$ is a domain function, mapping possible worlds to non-empty sets. We generally drop the word "augmented." For $\Gamma \in \mathcal{G}$, we call $\mathcal{D}(\Gamma)$ the domain of the world Γ. By $\mathcal{D}(\mathcal{F})$ we mean $\cup\{\mathcal{D}(\Gamma) \mid \Gamma \in \mathcal{G}\}$, and we refer to this as the domain of the *frame*.

An *interpretation* in the frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is a mapping $\mathcal{I}$ that assigns, to each n-place relation symbol R and each possible world $\Gamma \in \mathcal{G}$, some n-place relation $\mathcal{I}(R, \Gamma)$ on the domain of the frame $\mathcal{F}$. A *model* is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ where $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is a frame and $\mathcal{I}$ is an interpretation in it; it is said to be *based on* the frame $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$. We assume that in all our models, the interpretation of the equality symbol is the equality relation on the domain of the underlying frame—that is, all our models are normal.

A *valuation* in a model $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ is a mapping v that assigns, to each variable x, some member $v(x)$ of the domain of the underlying frame. Note that, unlike interpretations, valuations are not world-dependent. We say a valuation w is an *x-variant of v at* Γ if v and w agree on all variables except possibly x and further, $w(x)$ is in the domain of Γ.

Finally, the central notion of truth is characterized as follows. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ be a model. We write $\mathcal{M}, \Gamma \Vdash_v X$ to symbolize that: formula X is true at world Γ of model $\mathcal{M}$ with respect to valuation v in that model. Its defining conditions are these:

1. For an atomic formula $R(x_1, \ldots, x_n)$, $\mathcal{M}, \Gamma \Vdash_v R(x_1, \ldots, x_n)$ if and only if $\langle v(x_1), \ldots, v(x_n) \rangle \in \mathcal{I}(R, \Gamma)$.
2. $\mathcal{M}, \Gamma \Vdash_v (X \wedge Y)$ if and only if $\mathcal{M}, \Gamma \Vdash_v X$ and $\mathcal{M}, \Gamma \Vdash_v Y$.
3. $\mathcal{M}, \Gamma \Vdash_v \neg X$ if and only if not-$\mathcal{M}, \Gamma \Vdash_v X$.
4. $\mathcal{M}, \Gamma \Vdash_v \Box X$ if and only if for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\mathcal{M}, \Delta \Vdash_v X$.

5. $\mathcal{M}, \Gamma \Vdash_v (\forall x)X$ if and only if $\mathcal{M}, \Gamma \Vdash_w X$ for every valuation w that is an x-variant of v at Γ.

We say a formula is *valid* with respect to a class of models if the formula is true at each world of each model in the class, with respect to each valuation. As usual, one can consider classes of models whose accessibility relation is transitive, or reflexive, or meets some other conditions. For simplicity here, we work with an unrestricted accessibility relation, which means the underlying propositional modal logic is **K**.

Above the propositional level, models divide themselves into categories depending on the behavior of the domain function. An augmented frame $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$, and a model $\langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ based on it, is:

1. *varying domain* if $\mathcal{D}$ meets no special conditions.
2. *monotonic* if $\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Delta)$.
3. *anti-monotonic* if $\Gamma \mathcal{R} \Delta$ implies $\mathcal{D}(\Delta) \subseteq \mathcal{D}(\Gamma)$.
4. *constant domain* if $\mathcal{D}(\Gamma) = \mathcal{D}(\Delta)$ for all $\Gamma, \Delta \in \mathcal{G}$.

3 Commentary

Monotonic models are common in the literature but are not natural philosophically. Anti-monotonic models are equally unnatural, which leaves constant and varying domain models. Why these? An important philosophical question is: in a modal context, what should we take quantifiers as quantifying over. The two common choices are: that which is, and that which could be. If we opt for that which is, varying domain semantics is the formal counterpart. Each possible world has its own collection of existent objects and, at each world quantifiers range over these existent objects. On the other hand, if quantifiers are taken to range over that which could be—over possibly existent objects—these are world independent, and we have constant domain models as the formal counterpart. Interestingly enough, either choice allows us to simulate the other in a precise sense, as we will see.

Note that in the interpretation of relation symbols, values assigned to variables are not restricted to those that exist at a world, that is, to those in the domain of that world. To make such a restriction would, in effect, rule out talk of non-existent objects, but this is talk we engage in every day. Of course for constant domain semantics, the effect of such a restriction is moot. And as it happens, imposing the restriction for monotonic semantics yields the same collection of valid closed formulas as monotonic semantics without the restriction. Nonetheless, for varying domain semantics it makes a difference, and we explicitly leave things unrestricted.

Kripke showed in [8] that the converse Barcan formula corresponded to monotonicity, and the Barcan formula to anti-monotonicity. Here is a precise statement of this—it's proof is omitted.

Proposition 3.1 *Let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a frame. $\mathcal{F}$ is monotonic if and only if every instance of the converse Barcan formula is valid in every model based on $\mathcal{F}$. $\mathcal{F}$ is anti-monotonic if and only if every instance of the Barcan formula is valid in every model based on $\mathcal{F}$.*

If we have both monotonicity and anti-monotonicity, this does not quite give us constant domains. It only says two possible worlds that are related by the accessibility relation have the same domains. Nonetheless, we have the following, whose proof is also omitted.

Proposition 3.2 *A formula X is valid in all models that are both monotonic and anti-monotonic if and only if X is valid in all constant domain models.*

4 An embedding

Think of the domain in a constant domain model as the set of possible existents. This suggests that relativizing quantifiers to "actual" existents should allow constant domain semantics to simulate varying domain semantics. The following, from [7], makes this precise.

Let $\mathcal{E}$ be a one-place relation symbol, which will not be used for purposes other than quantifier relativization. Think of $\mathcal{E}(x)$ as asserting that x denotes an object that "actually" exists. Formally, of course, it is just another atomic formula. Quantifier relativization is defined as usual.

1. If A is atomic, $A^{\mathcal{E}} = A$.
2. $(\neg X)^{\mathcal{E}} = \neg(X^{\mathcal{E}})$.
3. $(X \wedge Y)^{\mathcal{E}} = (X^{\mathcal{E}} \wedge Y^{\mathcal{E}})$.
4. $(\Box X)^{\mathcal{E}} = \Box X^{\mathcal{E}}$.
5. $((\forall x)\Phi)^{\mathcal{E}} = (\forall x)(\mathcal{E}(x) \supset \Phi^{\mathcal{E}})$.

Now, here is the main item, whose proof is once again omitted (it is straightforward).

Proposition 4.1 *Let Φ be a formula not containing the symbol $\mathcal{E}$. Then: Φ is valid in every varying domain model if and only if $\Phi^{\mathcal{E}}$ is valid in every constant domain model.*

This means that by using quantifier relativization, a modal logic based on constant domains can simulate one based on varying domains. Loosely, if I think quantifiers should range over possible existents, while you think quantifiers should range over actual existents, I can still manage to understand what you are saying by a suitable paraphrase.

Incidentally, the Proposition above remains correct if *monotonic model* replaces *constant domain model*. (The proof is similar.) This means that monotonic tableau systems as found in [4, 5] can be used, via quantifier relativization, to verify validity of formulas with respect to varying domain semantics.

5 The other way

If constant domains are taken as primary, a logic based on varying domains can still be treated, via the quantifier relativization embedding of the previous section. In the other direction, if varying domain semantics is basic, validity in constant domain models can be developed since, as is well-known (and is a consequence of Propositions 3.1 and 3.2), X is valid in all constant domain models if and only if it is valid in all varying domain models in which the Barcan and the converse Barcan formulas are valid.

The problem here is that both the Barcan and the converse Barcan formulas are *schemes* with infinitely many instances, and for a particular formula X whose validity is in question, which instances are needed may not be obvious. Fortunately the whole problem can be avoided, since we have equality available.

In constant domain semantics, the idea behind quantifier relativization was to use a predicate $\mathcal{E}$ that we intuitively thought of as an existence predicate. In varying domain semantics, an existence predicate can be *defined*, since quantifiers range over what actually exists rather than what might exist. The definition is standard.

Definition 5.1 The expression $\mathbf{E}(x)$ abbreviates $(\exists y)(y = x)$ (and should be thought of as an existence assertion).

Now, the following two propositions are not hard to verify semantically.

Proposition 5.2 *Let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a varying domain frame. The following are equivalent.*

1. *$\mathcal{F}$ is monotonic.*
2. *Every instance of the converse Barcan formula is valid in every model based on $\mathcal{F}$.*
3. *The open formula $\mathbf{E}(x) \supset \Box\mathbf{E}(x)$ is valid in every model based on $\mathcal{F}$.*
4. *The closed formula $(\forall x)\Box\mathbf{E}(x)$ is valid every model based on $\mathcal{F}$.*

Proposition 5.3 *Again let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be a varying domain frame. The following are equivalent.*

1. *$\mathcal{F}$ is anti-monotonic.*
2. *Every instance of the Barcan formula is valid in every model based on $\mathcal{F}$.*
3. *The open formula $\Diamond\mathbf{E}(x) \supset \mathbf{E}(x)$ is valid in every model based on $\mathcal{F}$.*

The equivalences in the Propositions above have a special significance that should be pointed out. First, the formulas displayed that involve $\mathbf{E}$ are *single* formulas, and are not schemes. And second, since the only predicate symbol in $\mathbf{E}(x)$ is $=$, and the interpretation of that is always the equality relation then, for instance, saying that $\mathbf{E}(x) \supset \Box\mathbf{E}(x)$ is valid in *every* model based on a frame is equivalent to saying that $\mathbf{E}(x) \supset \Box\mathbf{E}(x)$ is valid in *some* model based on that frame. That is, features of frames are determined by validity of formulas in single models based on them.

6 Axiomatics

As was remarked in the previous section, semantic proofs of the Propositions contained there are not hard to come by. But for tradition and exercise value, we actually present axiomatic arguments. We use the axiom system LPCK of Hughes and Cresswell, [7, Page 304], which is appropriate for a semantics that assumes neither monotonicity nor anti-monotonicity. We also need an equality relation, for which we assume standard axioms. Also, at one point we need an additional rule of inference. Here are the assumptions we make, along with the Hughes, Cresswell names for them. (Hughes and Cresswell omit *UG*, though it was probably intended to be included.)

$\mathbf{K}'$ Any substitution instance of a valid formula of the propositional modal logic $\mathbf{K}$.

$\forall^{\supset}$ All formulas of the form $(\forall x)(\Phi \supset \Psi) \supset ((\forall x)\Phi \supset (\forall x)\Psi)$.

VQ All formulas of the form $(\forall x)\Phi \equiv \Phi$, where x is not free in Φ.

N From Φ conclude $\Box\Phi$.

MP From Φ and $\Phi \supset \Psi$ conclude Ψ.

UG From Φ conclude $(\forall x)\Phi$.

$\forall$*1K* All formulas of the form $(\forall y)(\forall z)[(\forall x)\Phi(x) \supset \Phi(y)]$, where $\Phi(y)$ is like $\Phi(x)$ but with all free occurrences of x replaced with occurrences of y (with bound variable renaming, as necessary, to avoid unintended quantifier capturing).

Axiom scheme $\forall$*1K* is the significant one here. From it (and the rest) follow both $(\forall x)(\forall y)\Phi \supset (\forall y)(\forall x)\Phi$ and $(\forall x)[(\forall x)\Phi \supset \Phi]$, but *not* $(\forall x)\Phi \supset \Phi$. One would not expect this to be valid in varying domain models (which are rather like free logic models) because in the antecedent, $(\forall x)\Phi$, the quantifier ranges only over the domain of a particular world, while in the consequent, Φ, a free occurrence of x can be assigned a value in the domain of the frame that may not be in the domain of that world. It is the case, however, that every valid *closed* formula of classical logic is provable in this system—differences affect formulas with free variables.

In addition to the axioms and rules above, we need the following, about which Hughes and Cresswell say it is "not part of this basis, though it is in fact validity-preserving."

UGL$\forall$ From $\Phi_1 \supset \Box(\Phi_2 \supset \ldots \supset \Box(\Phi_n \supset \Box\Psi)\ldots)$ infer $\Phi_1 \supset \Box(\Phi_2 \supset \ldots \supset \Box(\Phi_n \supset \Box(\forall x)\Psi)\ldots)$, where x is not free in any Φ_i.

In fact we need only the following special case, which we also state in an equivalent version, which is what we actually use.

UGL$\forall$ **Special case** From $\Phi \supset \Box\Psi$ infer $\Phi \supset \Box(\forall x)\Psi$, where x is not free in Φ.

Equivalent version From $\Diamond\Psi \supset \Phi$ infer $\Diamond(\exists x)\Psi \supset \Phi$ where x is not free in Φ.

We also assume the following axioms for identity, [7, Page 313].

I1 $x = x$.

I2 $(x = y) \supset (\Phi(x) \supset \Phi(y))$, where $\Phi(y)$ has occurrences of y substituted for free occurrences of x in $\Phi(x)$ (with quantifier renaming as appropriate).

In the following, for convenience, we use *BF* to denote (any) Barcan formula, and *CBF* for the converse Barcan formula. We also use $\forall\Box\mathbf{E}$ to denote the closed formula $(\forall x)\Box\mathbf{E}(x)$, $\mathbf{E}\Box\mathbf{E}$ to denote $\mathbf{E}(x) \supset \Box\mathbf{E}(x)$, and $\Diamond\mathbf{EE}$ to denote $\Diamond\mathbf{E}(x) \supset \mathbf{E}(x)$. The axiomatic proofs presented are sketches; we assume the axiom system above can be used to prove standard formulas in standard ways, and make use of this as convenient.

6.1 Converse Barcan Formula

We show the equivalence of *CBF*, $\forall\Box\mathbf{E}$ and $\mathbf{E}\Box\mathbf{E}$.

Proposition 6.1 $\mathbf{E}\Box\mathbf{E}$ *implies* $\forall\Box\mathbf{E}$.

Proof

1.	$(\forall x)\,[(\forall y)\neg(y = x) \supset \neg(x = x)]$	$\forall 1K$ and VQ
2.	$(\forall x)\,[(x = x) \supset (\exists y)(y = x)]$	from 1
3.	$(\forall x)(x = x) \supset (\forall x)(\exists y)(y = x)$	from 2 by $\forall^{\supset}$
4.	$x = x$	axiom
5.	$(\forall x)(x = x)$	UG on 4
6.	$(\forall x)(\exists y)(y = x)$	from 3 and 5
7.	$\mathbf{E}(x) \supset \Box\mathbf{E}(x)$	$\mathbf{E}\Box\mathbf{E}$
8.	$(\forall x)\mathbf{E}(x) \supset (\forall x)\Box\mathbf{E}(x)$	from 7
9.	$(\forall x)(\exists y)(y = x) \supset (\forall x)\Box\mathbf{E}(x)$	from 8 and def. of $\mathbf{E}(x)$
10.	$(\forall x)\Box\mathbf{E}(x)$	MP on 6, 9

■

Now we start seriously abbreviating proofs.

Proposition 6.2 $\forall\Box\mathbf{E}$ *implies CBF.*

Proof

1.	$(\forall y)\,[(\forall x)A(x) \supset A(y)]$	$\forall 1K$ and VQ
2.	$(\forall y)\,[(y = z \wedge A(y)) \supset A(z)]$	equality theorem
3.	$(\forall y)\,[((\forall x)A(x) \wedge y = z) \supset A(z)]$	from 1, 2
4.	$(\exists y)\,[(\forall x)A(x) \wedge y = z] \supset A(z)$	from 3
5.	$[(\forall x)A(x) \wedge (\exists y)(y = z)] \supset A(z)$	follows from 4
6.	$\Box\,[(\forall x)A(x) \wedge (\exists y)(y = z)] \supset \Box A(z)$	5, regularity rule
7.	$[\Box(\forall x)A(x) \wedge \Box(\exists y)(y = z)] \supset \Box A(z)$	from 6
8.	$(\forall z)\,[\Box(\forall x)A(x) \wedge \Box(\exists y)(y = z)] \supset (\forall z)\Box A(z)$	from 7
9.	$[\Box(\forall x)A(x) \wedge (\forall z)\Box(\exists y)(y = z)] \supset (\forall z)\Box A(z)$	from 8
10.	$[\Box(\forall x)A(x) \wedge (\forall z)\Box\mathbf{E}(z)] \supset (\forall z)\Box A(z)$	9, reformulated
11.	$(\forall z)\Box\mathbf{E}(z)$	$\forall\Box\mathbf{E}$
12.	$\Box(\forall x)A(x) \supset (\forall x)\Box A(x)$	from 10, 11

■

Proposition 6.3 *CBF implies* $\forall\Box\mathbf{E}$.

Proof

1.	$(\forall x)(\exists y)(y = x)$	equality property
2.	$(\forall x)\mathbf{E}(x)$	1, reformulated
3.	$\Box(\forall x)\mathbf{E}(x)$	necessitation on 2
4.	$\Box(\forall x)\mathbf{E}(x) \supset (\forall x)\Box\mathbf{E}(x)$	*CBF*
5.	$(\forall x)\Box\mathbf{E}(x)$	3, 4, MP

■

Lemma 6.4 *CBF implies* $(\exists x)\Box A(x) \supset \Box(\exists x)A(x)$ *[That is, each instance of this formula is implied by some instance of CBF.] Hence* $\forall\Box\mathbf{E}$ *implies* $(\exists x)\Box A(x) \supset \Box(\exists x)A(x)$.

Proof

1.	$(\forall x)\,[A(x) \supset (\exists x)A(x)]$	$\forall 1K$
2.	$\Box(\forall x)\,[A(x) \supset (\exists x)A(x)]$	necessitation on 1
3.	$(\forall x)\Box\,[A(x) \supset (\exists x)A(x)]$	from 2 using *CBF*
4.	$(\forall x)\,[\Box A(x) \supset \Box(\exists x)A(x)]$	from 3
5.	$(\exists x)\Box A(x) \supset \Box(\exists x)A(x)$	from 4

■

Proposition 6.5 $\forall\Box\mathbf{E}$ *implies* $\mathbf{E}\Box\mathbf{E}$.

Proof

1.	$(y = x) \supset [\Box(y = y) \supset \Box(y = x)]$	*I2*
2.	$\Box(y = y) \supset [(y = x) \supset \Box(y = x)]$	from 1
3.	$\Box(y = y)$	*I1* and *N*
4.	$(y = x) \supset \Box(y = x)$	from 2, 3
5.	$(\exists y)(y = x) \supset (\exists y)\Box(y = x)$	from 4
6.	$(\exists y)(y = x) \supset \Box(\exists y)(y = x)$	Lemma 6.4
7.	$\mathbf{E}(x) \supset \Box\mathbf{E}(x)$	6 reformulated

■

6.2 Barcan Formula

This is where we need the special case of the rule of inference *UGL*$\forall$. We also find it convenient to use the Barcan formula in the equivalent version $\Diamond(\exists x)A(x) \supset (\exists x)\Diamond A(x)$.

Proposition 6.6 *BF implies* $\Diamond\mathbf{EE}$.

Proof

1.	$\Diamond(\exists x)(y = x) \supset (\exists x)\Diamond(y = x)$	*BF*
2.	$\Diamond(y = x) \supset (y = x)$	using equality axioms
3.	$\Diamond(\exists x)(y = x) \supset (\exists x)(y = x)$	1,2
4.	$\Diamond\mathbf{E}(y) \supset \mathbf{E}(y)$	3 rewritten

■

Lemma 6.7 *The following is provable:* $[\mathbf{E}(x) \wedge \varphi(x)] \supset (\exists x)\varphi(x)$.

Proof

1.	$[\mathbf{E}(x) \wedge \varphi(x)] \supset [(\exists y)(y = x) \wedge \varphi(x)]$	rewriting **E**
2.	$[\mathbf{E}(x) \wedge \varphi(x)] \supset (\exists y)\,[y = x \wedge \varphi(x)]$	from 1
3.	$[\mathbf{E}(x) \wedge \varphi(x)] \supset (\exists y)\varphi(y)$	using *I2* and 2
4.	$[\mathbf{E}(x) \wedge \varphi(x)] \supset (\exists x)\varphi(x)$	variable renaming, on 3

■

Lemma 6.8 *The following is provable:* $(\exists x)\varphi(x) \supset (\exists x)[\mathbf{E}(x) \wedge \varphi(x)]$.

Proof Exercise. ■

Proposition 6.9 $\Diamond\mathbf{EE}$ *implies BF.*

Proof

1.	$\Diamond[\mathbf{E}(x) \wedge A(x)] \supset [\Diamond\mathbf{E}(x) \wedge \Diamond A(x)]$	standard property of $\Diamond$
2.	$\Diamond[\mathbf{E}(x) \wedge A(x)] \supset [\mathbf{E}(x) \wedge \Diamond A(x)]$	using $\Diamond\mathbf{EE}$ on 1
3.	$\Diamond[\mathbf{E}(x) \wedge A(x)] \supset (\exists x)\Diamond A(x)$	Lemma 6.7 on 2
4.	$\Diamond(\exists x)[\mathbf{E}(x) \wedge A(x)] \supset (\exists x)\Diamond A(x)$	rule *UGL*$\forall$ on 3
5.	$\Diamond(\exists x)A(x) \supset (\exists x)\Diamond A(x)$	Lemma 6.8 on 4

■

7 Some observations

In a *symmetric* frame, the Barcan and the converse Barcan formulas should be equivalent. This is trivial to show, using the alternate formulations.

Proposition 7.1 *Assuming as an axiom* $X \supset \Box\Diamond X$*, or equivalently,* $\Diamond\Box X \supset X$ *(corresponding to frame symmetry, and generally called B), each of* $\mathbf{E}\Box\mathbf{E}$ *and* $\Diamond\mathbf{EE}$ *implies the other.*

Proof Assume $\mathbf{E}\Box\mathbf{E}$. Then

1.	$\mathbf{E}(x) \supset \Box\mathbf{E}(x)$	
2.	$\Diamond\mathbf{E}(x) \supset \Diamond\Box\mathbf{E}(x)$	
3.	$\Diamond\mathbf{E}(x) \supset \mathbf{E}(x)$	using axiom B

The other direction is similar. ■

Of course in the presence of axiom B, the Barcan and the converse Barcan formulas are equivalent also. More precisely, since they are schemes and not single formulas, each instance of the Barcan formula is a consequence of some instances of the converse Barcan formula, and the other way around as well. In one direction this is well-known, and is essentially due to Prior. Here is an axiomatic proof of an instance of the Barcan formula, in which converse Barcan is used for line 3.

1.	$(\forall x)[(\forall x)\Box\Phi(x) \supset \Box\Phi(x)]$	$\forall$*1K* and *VQ*
2.	$\Box(\forall x)[(\forall x)\Box\Phi(x) \supset \Box\Phi(x)]$	Nec. rule on 1
3.	$(\forall x)\Box[(\forall x)\Box\Phi(x) \supset \Box\Phi(x)]$	from 2 using Conv. Barcan
4.	$(\forall x)[\Diamond(\forall x)\Box\Phi(x) \supset \Diamond\Box\Phi(x)]$	from 3
5.	$(\forall x)[\Diamond(\forall x)\Box\Phi(x) \supset \Phi(x)]$	from 4 using B axiom
6.	$(\forall x)\Diamond(\forall x)\Box\Phi(x) \supset (\forall x)\Phi(x)$	from 5 using Univ. Dist.
7.	$\Diamond(\forall x)\Box\Phi(x) \supset (\forall x)\Phi(x)$	from 6 using *VQ*
8.	$\Box\Diamond(\forall x)\Box\Phi(x) \supset \Box(\forall x)\Phi(x)$	from 7 using Reg. Rule
9.	$(\forall x)\Box\Phi(x) \supset \Box(\forall x)\Phi(x)$	from 8 using axiom B

The other direction remains a minor mystery to me. A given instance of the converse Barcan formula should be provable assuming axiom B and certain instances of the Barcan formula. Which ones?

There is one more minor mystery, concerning the role of rule *UGL*$\forall$ in the proof of Proposition 6.9. As we noted earlier, it is *not* part of the Hughes and Cresswell axiomatic basis, though it is a sound rule. Can Proposition 6.9 be proved without it? If not, since it is not part of the Hughes, Cresswell system, and that system is without equality, is it necessary for completeness once a treatment of equality is added? Exactly what is its status?

8 Prefixed tableaus

In a broad sense there are two families of tableaus for modal logics: those that do not refer explicitly to possible worlds, and those that do. Earlier, in Section 4, tableaus were mentioned. That kind did not refer to possible worlds. Among those that do, *prefixed* tableau systems are fairly well-known. These were introduced in [3], further developed in [4], and given a more elegant formulation in [9, 10]—see also [6]. Here we confine discussion to the simplest case, first-order logics built on **K**.

A *prefix* is a finite sequence of positive integers. A *prefixed formula* is an expression of the form $\sigma\, X$, where σ is a prefix and X is a closed formula, or sentence. We will write prefixes using periods to separate integers, 1.2.3.2.1 for instance. Also, if σ is a prefix and n is a positive integer, $\sigma.n$ is σ followed by a period followed by n. Intuitively a prefix, σ, names a possible world in some model, and $\sigma\, X$ tells us that X is true at the world σ names. We intend that $\sigma.n$ names a world that is accessible from the one that σ names.

A tableau proof of Z is a tree. Construction of it begins by creating an initial tree with $1\,\neg Z$ at its root (and with no other nodes). Intuitively this says Z is false at a world named by 1. Next, branches are "grown" according to certain *Branch Extension Rules*, to be given shortly. This yields a succession of *tableaus for* $1\,\neg Z$. A tableau branch is *closed* if it contains both $\sigma\, X$ and $\sigma\,\neg X$. A branch that is not closed is *open*. A tableau is *closed* if every branch is closed. A closed tableau for $1\,\neg Z$ is a proof of Z.

Now the branch extension rules. Unlike with axiom systems, we give rules for all the usual connectives, modal operators, and quantifiers.

Definition 8.1 [Double Negation Rule] For any prefix σ,

$$\frac{\sigma\,\neg\neg X}{\sigma\, X}$$

Definition 8.2 [Conjunctive Rules] For any prefix σ,

$$\frac{\sigma\, X \wedge Y}{\begin{array}{c}\sigma\, X\\ \sigma\, Y\end{array}} \qquad \frac{\sigma\,\neg(X \vee Y)}{\begin{array}{c}\sigma\,\neg X\\ \sigma\,\neg Y\end{array}} \qquad \frac{\sigma\,\neg(X \supset Y)}{\begin{array}{c}\sigma\, X\\ \sigma\,\neg Y\end{array}} \qquad \frac{\sigma\, X \equiv Y}{\begin{array}{c}\sigma\, X \supset Y\\ \sigma\, Y \supset X\end{array}}$$

Definition 8.3 [Disjunctive Rules] For any prefix σ,

$$\frac{\sigma\, X \vee Y}{\sigma\, X \mid \sigma\, Y} \qquad \frac{\sigma\,\neg(X \wedge Y)}{\sigma\,\neg X \mid \sigma\,\neg Y}$$

$$\frac{\sigma\, X \supset Y}{\sigma\,\neg X \mid \sigma\, Y} \qquad \frac{\sigma\,\neg(X \equiv Y)}{\sigma\,\neg(X \supset Y) \mid \sigma\,\neg(Y \supset X)}$$

The modal operator rules obviously treat them as relatives of quantifiers.

Definition 8.4 [Possibility Rules] If the prefix $\sigma.n$ is new to the branch,

$$\frac{\sigma\,\Diamond X}{\sigma.n\, X} \qquad \frac{\sigma\,\neg\Box X}{\sigma.n\,\neg X}$$

Definition 8.5 [Necessity Rules] If the prefix $\sigma.n$ already occurs on the branch,

$$\frac{\sigma\,\Box X}{\sigma.n\,X} \qquad \frac{\sigma\,\neg\Diamond X}{\sigma.n\,\neg X}$$

Next we give the quantifier rules for the *varying domain* version. We need a family of *parameters*—these are additional free variables that are never quantified. They occur in proofs, but never in sentences being proved. More specifically, we assume that to each prefix σ there is associated an infinite list of parameters, in such a way that different prefixes never have the same parameter associated with them. We write p_σ to indicate that p is a parameter associated with the prefix σ. Now, here are the varying domain quantifier rules.

Definition 8.6 [Universal Rules—Varying Domain] In the following, p_σ is be any parameter that is associated with the prefix σ.

$$\frac{\sigma\,(\forall x)\Phi(x)}{\sigma\,\Phi(p_\sigma)} \qquad \frac{\sigma\,\neg(\exists x)\Phi(x)}{\sigma\,\neg\Phi(p_\sigma)}$$

Definition 8.7 [Existential Rules—Varying Domain] In the following, p_σ is a parameter associated with the prefix σ, subject to the condition that p_σ is new to the tableau branch.

$$\frac{\sigma\,(\exists x)\Phi(x)}{\sigma\,\Phi(p_\sigma)} \qquad \frac{\sigma\,\neg(\forall x)\Phi(x)}{\sigma\,\neg\Phi(p_\sigma)}$$

Finally, the rules for equality. And these are quite straightforward.

Definition 8.8 [Reflexivity Rule] If p_τ is a parameter and σ is a prefix, then $\sigma(p_\tau = p_\tau)$ can be added to the end of the branch. Briefly,

$$\frac{}{\sigma\,(p_\tau = p_\tau)}$$

Definition 8.9 [Substitutivity Rule] Let $\Phi(x)$ be a formula in which x occurs free, let $\Phi(p_{\tau_1})$ be the result of substituting occurrences of the parameter p_{τ_1} for all free occurrences of x in $\Phi(x)$, and similarly for $\Phi(q_{\tau_2})$. If $\sigma_1(p_{\tau_1} = q_{\tau_2})$ and $\sigma_2\Phi(p_{\tau_1})$ both occur on a tableau branch, $\sigma_2\Phi(q_{\tau_2})$ can be added to the end. Briefly,

$$\frac{\begin{array}{c}\sigma_1\,(p_{\tau_1} = q_{\tau_2})\\ \sigma_2\,\Phi(p_{\tau_1})\end{array}}{\sigma_2\,\Phi(q_{\tau_2})}$$

Here is an example of a tableau proof, of the sentence $(\forall x)\{\Diamond(\exists y)[x = y \wedge P(y)] \supset \Diamond P(x)\}$.

$$\begin{array}{ll}
1 & \neg(\forall x)\{\Diamond(\exists y)\,[x = y \wedge P(y)] \supset \Diamond P(x)\} \quad 1.\\
1 & \neg\{\Diamond(\exists y)\,[p_1 = y \wedge P(y)] \supset \Diamond P(p_1)\} \quad 2.\\
1 & \quad \Diamond(\exists y)\,[p_1 = y \wedge P(y)] \quad 3.\\
1 & \neg\Diamond P(p_1) \quad 4.\\
1.1 & (\exists y)\,[p_1 = y \wedge P(y)] \quad 5.\\
1.1 & p_1 = q_{1.1} \wedge P(q_{1.1}) \quad 6.\\
1.1 & p_1 = q_{1.1} \quad 7.\\
1.1 & P(q_{1.1}) \quad 8.\\
1 & \neg\Diamond P(q_{1.1}) \quad 9.\\
1.1 & \neg P(q_{1.1}) \quad 10.
\end{array}$$

In this, 2 is from 1 by an Existential Rule, 3 and 4 are from 2 by a Conjunctive Rule, 5 is from 3 by a Possibility Rule, 6 is from 5 by an Existential Rule, 7 and 8 are from 6 by a Conjunctive Rule, 9 is from 4 and 7 by Substitutivity, and 10 is from 9 by a Necessity Rule. Note that this is not the only proof available.

To incorporate monotonicity or anti-monotonicity, special conditions can be put on parameters and their subscripts—this is well-known. These conditions can be somewhat complicated for hand computation, though they can be implemented well for computer use. Perhaps a more elegant solution is to make use of the converse Barcan and Barcan formulas, in the versions introduced in Section 5. For the converse Barcan formula (monotonicity) this amounts to allowing $\sigma\ (\forall x)\Box\mathbf{E}(x)$ to be used as a tableau line, at any point, for any prefix σ that has already been introduced.

Here is an example: a tableau proof of $\Box(\exists x)\Diamond A(x) \supset \Box\Diamond(\exists x)A(x)$, *taking* $(\forall x)\Box\mathbf{E}(x)$ *as an assumption.* Thus in effect it is a monotonic proof of the formula.

$$
\begin{array}{lll}
1 & \neg\,[\Box(\exists x)\Diamond A(x) \supset \Box\Diamond(\exists x)A(x)] & 1.\\
1 & \quad\Box(\exists x)\Diamond A(x) & 2.\\
1 & \neg\Box\Diamond(\exists x)A(x) & 3.\\
1.1 & \neg\Diamond(\exists x)A(x) & 4.\\
1.1 & \quad(\exists x)\Diamond A(x) & 5.\\
1.1 & \quad\Diamond A(p_{1.1}) & 6.\\
1.1.1 & A(p_{1.1}) & 7.\\
1.1.1 & \neg(\exists x)A(x) & 8.\\
1.1 & \quad(\forall x)\Box\mathbf{E}(x) & 9.\\
1.1 & \quad\Box\mathbf{E}(p_{1.1}) & 10.\\
1.1.1 & \mathbf{E}(p_{1.1}) & 11.\\
1.1.1 & (\exists x)(x = p_{1.1}) & 12.\\
1.1.1 & q_{1.1.1} = p_{1.1} & 13.\\
1.1.1 & \neg A(q_{1.1.1}) & 14.\\
1.1.1 & \neg A(p_{1.1}) & 15.
\end{array}
$$

Items 2 and 3 are from 1 by a Conjunctive Rule; 4 is from 3 by a Possibility Rule; 5 is from 2 by a Necessity Rule; 6 is from 5 by an Existential Rule; 7 is from 6 by a Possibility Rule; 8 is from 4 by a Necessity Rule; 9 is our assumption; 10 is from 9 by a Universal Rule; 11 is from 10 by a Necessity Rule; 12 is 11 unabbreviated; 13 is from 12 by an Existential Rule; 14 is from 8 by a Universal Rule; and 15 is from 13 and 14 by Substitution.

To get anti-monotonicity, we make use of $\Diamond\mathbf{E}(x) \supset \mathbf{E}(x)$ as a tableau assumption. We must be careful here, though, since it is an open formula and these have no place in tableaus. It is used in the following way: we can introduce $\sigma\,\Diamond\mathbf{E}(p_\tau) \supset \mathbf{E}(p_\tau)$ at any point in a tableau provided: σ is a prefix that has already appeared, and p_τ is a parameter already introduced.

Here is a tableau proof of an instance of the Barcan formula, $(\forall x)\Box A(x) \supset \Box(\forall x)A(x)$, taking $\Diamond\mathbf{E}(x) \supset \mathbf{E}(x)$ as an assumed validity as just described.

$$
\begin{array}{lll}
1 & \neg\,[(\forall x)\Box A(x) \supset \Box(\forall x)A(x)] & 1.\\
1 & \quad(\forall x)\Box A(x) & 2.\\
1 & \neg\Box(\forall x)A(x) & 3.\\
1.1 & \neg(\forall x)A(x) & 4.\\
1.1 & \neg A(p_{1.1}) & 5.\\
1 & \quad\Diamond\mathbf{E}(p_{1.1}) \supset \mathbf{E}(p_{1.1}) & 6.
\end{array}
$$

Items 2 and 3 are from 1 by a Conjunctive Rule; 4 is from 3 by a Possibility Rule; 5 is from 4 by an Existential Rule; 6 is our assumed validity;

At this point the tableau branches, using item 6. We first give the left branch continuation.

$$
\begin{array}{ll}
1 & \neg\Diamond\mathbf{E}(p_{1.1}) \quad 10. \\
1.1 & \neg\mathbf{E}(p_{1.1}) \quad 11. \\
1.1 & \neg(\exists x)(x = p_{1.1}) \quad 12. \\
1.1 & \neg(p_{1.1} = p_{1.1}) \quad 13. \\
1.1 & p_{1.1} = p_{1.1} \quad 14.
\end{array}
$$

Item 10 is from 6 as the left formula in a Disjunctive Rule; 11 is from 10 by a Necessity Rule; 12 is 11 unabbreviated; 13 is from 12 by a Universal Rule; and 14 is by the Reflexivity Rule.

Now the right branch continuation.

$$
\begin{array}{ll}
1 & \mathbf{E}(p_{1.1}) \quad 20. \\
1 & (\exists x)(x = p_{1.1}) \quad 21. \\
1 & q_1 = p_{1.1} \quad 22. \\
1 & \Box A(q_1) \quad 23. \\
1 & \Box A(p_{1.1}) \quad 24. \\
1.1 & A(p_{1.1}) \quad 25.
\end{array}
$$

Item 20 is from 6 as the right formula in a Disjunctive Rule; 21 is 20 unabbreviated; 22 is from 21 by an Existential Rule; 23 is from 2 by a Universal Rule; 24 is from 22 and 23 by Substitutivity; and 25 is from 24 by a Universal Rule.

Then to get tableaus corresponding to constant domain semantics, we can just incorporate both of the tableau assumptions above. A much simpler tableau system is available, however: just drop subscripts from parameters, and use a single family for all worlds. This is a system that is fairly standard in the literature.

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