Triangulation complexity of fibred 3-manifolds

Jessica Purcell, joint with M. Lackenby

CUNY 2020

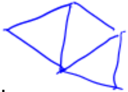
Notes:

1. Thanks
2. Caveate -
3. Pitch - students

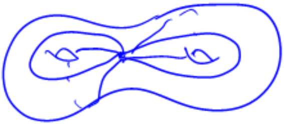
Part I: Triangulations

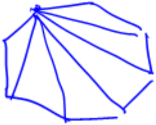
Definition. A *triangulation* of a surface is a gluing of triangles such that:

- edges glue to edges,
- vertices to vertices,
- interiors of triangles are disjoint.



> ### Theorem
> *Every surface can be triangulated.*





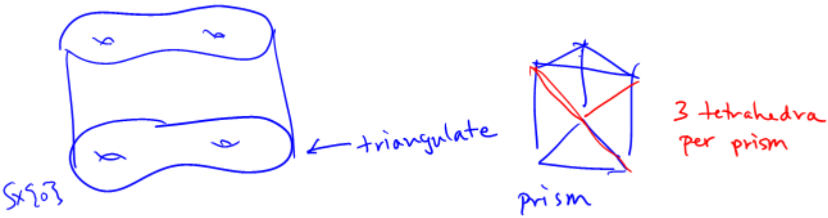
$4g$-gon

$4g-2$ triangles

3-manifold triangulations

Theorem (Moise 1952)
Every 3-manifold can be triangulated.

(Example: $S \times I$)

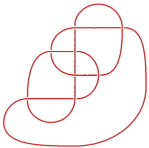


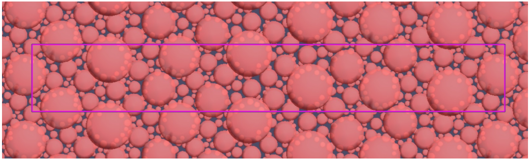
How is it used?

Computer:
3-manifold software:

- Regina (Burton, Budney, Petersson)
- SnapPy (Culler, Dunfield, Goerner, Weeks)

  Manifolds represented by triangulations.
  More "complicated" triangulations lead to slow algorithms, long processing time.





Measuring “complexity”

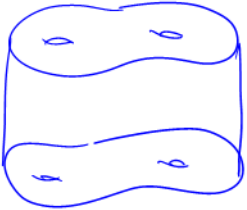
Simplest way: How many tetrahedra?

Measuring "complexity"

Simplest way: How many tetrahedra?

Definition. $\Delta(M) = \min$ number of tetrahedra in a triangulation of M.

(Example: $S \times I$)



← min triangulation

4g-2 triangles

$\longrightarrow$

prisms

subdivide

3(4g-2) tetrahedra

$$\Delta(S \times I) \leq 3(4g-2).$$

$\boxed{\text{Q: Can we do better?}}$

Problem:

Given M, find $\Delta(M)$.

Known results:

Problem:

Given M, find $\Delta(M)$.

Known results:

- Enumerations of manifolds built with up to k tetrahedra:
  - Matveev–Savvateev 1974: up to $k = 5$
  - Martelli–Petronio 2001: up to 9
  - Matveev–Tarkaev 2005: up to 11.
  - Regina: Includes all 3-manifolds up to 13 tetrahedra.

Problem:

Given M, find $\Delta(M)$.

Known results:

- Enumerations of manifolds built with up to k tetrahedra:
  - Matveev–Savvateev 1974: up to $k = 5$
  - Martelli–Petronio 2001: up to 9
  - Matveev–Tarkaev 2005: up to 11.
  - Regina: Includes all 3-manifolds up to 13 tetrahedra.
- Infinite families:
  - Anisov 2005: some punctured torus bundles
  - Jaco–Rubinstein–Tillmann 2009, 2011: infinite families of lens spaces
  - Jaco–Rubinstein–Spreer–Tillmann 2017, 2018: some covers, all punctured torus bundles, ...

Finding $\Delta(M)$

Finding exact value of $\Delta(M)$: HARD

Finding bounds: Maybe not as hard?

Upper bound: Typically easy.

Lower bound: Hard

Previous 2–sided bounds for families: Matveev–Petronio–Vesnin...

Löbell manifolds
Fibonacci mflds

Today: 2–sided bounds for fibred 3-manifolds.

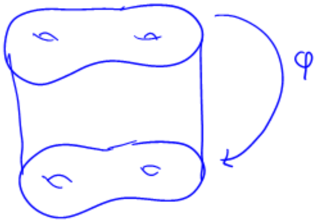
Fibred 3-manifold

Definition. Let S be a closed surface, $\phi : S \to S$ orientation preserving homeomorphism.

$$M_\phi = (S \times I)\,/\,(x,0) \sim (\phi(x),1)$$

Say M_ϕ *fibres over the circle* S^1 with fibre S.

ϕ is the *monodromy*.



Main theorem

Theorem (Lackenby – P)

Let M_ϕ be a closed 3-manifold that fibres over the circle with pseudo-Anosov monodromy ϕ. Then the following are within bounded ratios of each other, where the bound depends only on the genus of the fibre:

- $\Delta(M)$
- *Translation length of ϕ in the mapping class group.*
- *(Additional)* $Tr(S)$, $Sp(S)$

  lengths in

To do:

- Define terms
- Explain why this is the "right" theorem — comparisons with geometry
- Ideas of proof

Part II: Surfaces and their homeomorphisms

Definition. $\mathrm{MCG}\,(S)$ *Mapping class group* of S

Orientation preserving homeomorphisms of S up to isotopy.

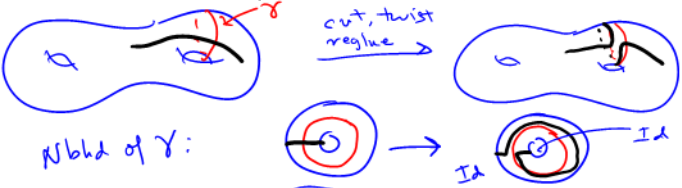
(Example: hyperelliptic involution)

Generators of MCG

Theorem (Dehn 1910-ish, Lickorish 1963)

$MCG(S)$ is finitely generated, generated by Dehn twists about a finite number of curves.

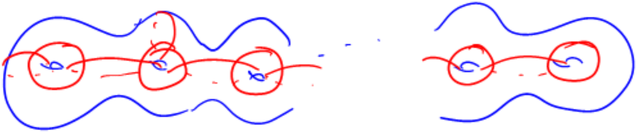
<ins>Dehn twist</ins> about simple closed curve γ:



Nbhd of γ:

<ins>Humphries generators 1977:</ins>

min collection of curves s.t. Dehn twists generate MCG.



Types of elements of MCG

1. *Periodic* $\varphi^n = \mathrm{Id}$ for some n
   E.g. hyperelliptic involution.

2. *Reducible*: Fixes a curve γ.
   E.g. power of a single Dehn twist. γ fixed

3. *Pseudo-Anosov*: Everything else.
   our theorem

Theorem (Thurston)
M_ϕ admits a complete hyperbolic metric if and only if ϕ is pseudo-Anosov.

Part III: Complexes and translation lengths

Definition. Let (X, d) be a metric space, ϕ an isometry. The *translation length* $\ell_X(\phi)$ is

$$\ell_X(\phi) = \inf\{d(\phi(x), x) : x \in X\}$$

(Example: MCG)

$X = MCG(S)$ d = word length in a fixed set of generators.
e.g. Humphries generators.

$\varphi \in MCG(S)$ written $\varphi : \tau_1 \tau_2 \cdots \tau_n$,
$d(\varphi, \psi)$ = word length $\varphi \psi^{-1}$

$\varphi \in MCG$ acts on $MCG(S)$ by isometry:
$d(x,y) = d(\varphi x, \varphi y)$

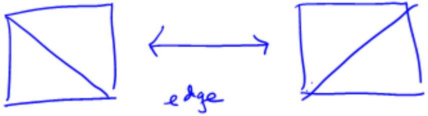
$\ell_{MCG}(\varphi) = \inf\{d(\varphi(x), x) : x \in MCG\}$
$\leq n$

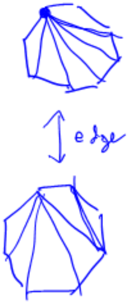
If $x = Id$
then $d(\varphi, Id) = n$

Example 2: Triangulation complex

$X = \mathrm{Tr}(S)$ complex of *1-vertex triangulations* of S.

- Vertices in $\mathrm{Tr}(S)$ = 1-vertex triangulations of S
- Edges: $\exists$ edge between two triangulations
  $\Leftrightarrow$ $\exists$ 2-2 Pachner move = diagonal exchange





Metric: Set each edge in $\mathrm{Tr}(S)$ to have length 1.
d is distance under path metric.
(Connected geodesic metric space)

$\phi \in \mathrm{MCG}(S)$ acts by isometry. ϕ(triang) = new triang.

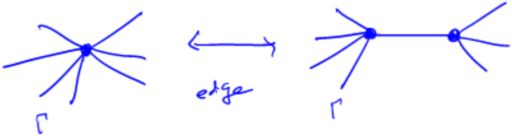
Therefore $\ell_{\mathrm{Tr}(S)}(\phi)$ defined.

Example 3: Spine complex

$X = \mathrm{Sp}\,(S)$ complex of *spines* on S.

Spine: Embedded graph $\Gamma \subset S$, with $S - \Gamma$ a disc, and no vertices of valence 0, 1, 2.

- Vertices in $\mathrm{Sp}\,(S)$ = spines of S
- Edges: $\exists$ edge between two spines $\Leftrightarrow \exists$ edge contraction/expansion



Metric: Each edge has length 1, d is path metric.
(Connected geodesic metric space)

$\phi \in \mathrm{MCG}\,(S)$ acts by isometry. $\varphi(\text{spine}) = \text{new spine}$

Therefore $\ell_{\mathrm{Sp}\,(S)}(\phi)$ defined.

Quasi-isometries

Lemma

$Tr(S)$, $Sp(S)$, $MCG(S)$ are all quasi-isometric.

(Proof, for experts: Svarc–Milnor lemma)

Quasi-isometric: $\exists f : (X, d_X) \to (Y, d_Y)$ and constants $A \geq 1$, $B \geq 0$, $C \geq 0$ such that:

i.e. within bounded ratios of one another.

1. $\forall x, y \in X$,

$$\frac{1}{A} \cdot d_X(x, y) - B \leq d_Y(f(x), f(y)) \leq A \cdot d_X(x, y) + B$$

2. $\forall y \in Y, \exists x \in X$ such that $d_Y(y, f(x)) \leq C$.

Example 4: Pants complex

$X = P(S)$ complex of *pants decompositions* of S.
Pants decomposition: Collection of $3g - 3$ disjoint simple closed curves on S.

- Vertices in $P(S)$ = pants decompositions of S
- Edges: $\exists$ edge between two pants
  $\Leftrightarrow \exists$ pants differ by one curve

edge.

Metric: Each edge has length 1, d is path metric.
(Connected geodesic metric space)

$\phi \in \mathrm{MCG}(S)$ acts by isometry.

$\varphi(\text{pants}) = $ new pants decomp.

$\ell_{P(S)}(\varphi)$

MCG is NOT quasi-isometric to P(S)

Proof.

Let $x, y \in P(S)$. Let ϕ Dehn twist about curve in x.

φ Dehn twist about γ. $\Rightarrow$ fix $x \in P(S)$

$\underbrace{d_{P(S)}(x, \phi^n(y))} = \underbrace{d_{P(S)}(\phi^n(x), \phi^n(y))} = \underbrace{d_{P(S)}(x, y)}$: Independent of n.

$\underbrace{d_{\text{MCG}}(x, \phi^n(y))}_{\text{word length}}$ growing with n.

Main theorem revisited

Theorem (Lackenby–P)

For ϕ pseudo-Anosov, and $\underline{M_\phi = (S \times I)/\phi}$, the following are within bounded ratios:

- $\Delta(M_\phi)$ ⟵ New & hard.
- $\ell_{MCG}(\phi)$
- $\ell_{Tr}(\phi)$
- $\ell_{Sp}(\phi)$

$\left.\begin{array}{l}\ell_{MCG}(\phi)\\ \ell_{Tr}(\phi)\\ \ell_{Sp}(\phi)\end{array}\right\}$ By Svarc-Milnor

Const: depend only on genus of S.

Compare to older theorem

Theorem (Brock 2003)

For ϕ pseudo-Anosov, $M_\phi = (S \times I)/\phi$, the following are within bounded ratios of each other:

- *$Vol(M_\phi)$ hyperbolic volume*
- *$\ell_P(\phi)$ translation length in pants complex*

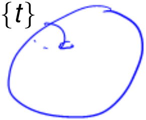
Consts depend on sfce.

Why ours is the "right" theorem (for experts)

Suppose ϕ is a word in a very high power of a Dehn twist about some curve γ:

$$\phi = \tau_1 \tau_2 \ldots \tau_k^N \ldots \tau_\ell$$

Geometrically, M_ϕ contains a deep tube about $\gamma \times \{t\}$



Deep tubes and volume:

$$\mathrm{vol}(\text{tube}) \leq \underbrace{\mathrm{vol}(\text{cusp})}_{\text{finite bounded.}}$$

Deep tubes and triangulations: Layered solid tori (Jaco–Rubinstein)

LST : # tetrahedra grows as tube gets "deeper"

Unbounded.

Why ours is not yet the "most right" theorem

- Pseudo-Anosov shouldn't be required.
- Closed manifolds shouldn't be required.
- Brock extended volumes to Heegaard splittings. We should too.

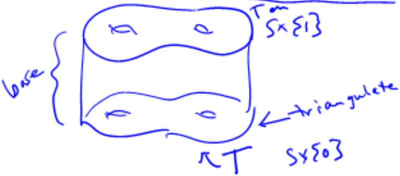
Part IV: Proof of upper bound

Theorem (Upper bound)

There exist constants C, D, depending only on g(S) such that

$$\Delta(M_\phi) \leq C\ell_{Tr}(\phi) + D.$$

Proof. Give S a 1-vertex triangulation $T \in \mathrm{Tr}(S)$: $4g - 2$ triangles.
Start with triangulation $S \times I$:



3(4g-2) tetrahedra

Let γ be path in $\mathrm{Tr}(S)$ from T to $\phi(T)$. Each step: layer tetrahedron.

realise by layering:

Proof of upper bound, continued

After $\ell_{\mathrm{Tr}}(\phi)$ steps:

Have triangulation of $S \times I$ with

- $S \times \{0\}$ triangulated by T,
- $S \times \{1\}$ triangulated by $\phi(T)$.

Glue to triangulate M_ϕ.

$$\Delta(M) \le \underbrace{\ell_{\mathrm{Tr}}(\phi)}_{\text{layering}} + \underbrace{3(4g-2)}_{\text{base}}.$$

Part V: Proof ideas for lower bound

Sp

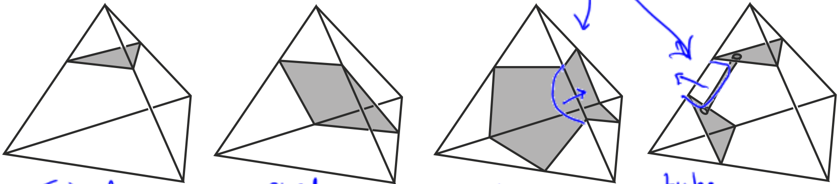
Idea: Suppose M_ϕ is triangulated with $\Delta(M_\phi)$ tetrahedra.

$\exists$ copy of S in *normal form*. Cut along it to get $S \times I$.

Normal form: intersects tetrahedra in triangles & quads.

$\exists$ copy of S in *almost normal form*.

Triangle, quad or at most one



$\exists$ well-understood ways of moving from almost normal to normal.

<u>Goal:</u> Bound moves to sweep spine from bottom to top:

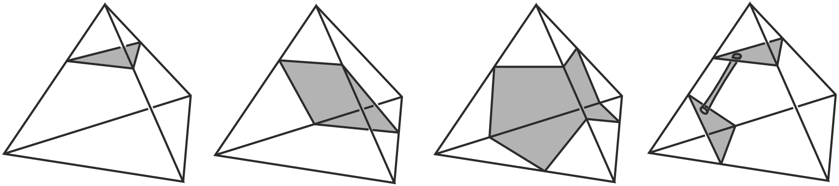
$$\ell_{\mathrm{Sp}}(\phi) \leq A\Delta(M_\phi) + B$$

Part V: Proof ideas for lower bound

Idea: Suppose M_ϕ is triangulated with $\Delta(M_\phi)$ tetrahedra.

$\exists$ copy of S in *normal form*. Cut along it to get $S \times I$.

$\exists$ copy of S in *almost normal form*.



$\exists$ well-understood ways of moving from almost normal to normal.

Goal: Bound moves to sweep spine from bottom to top:

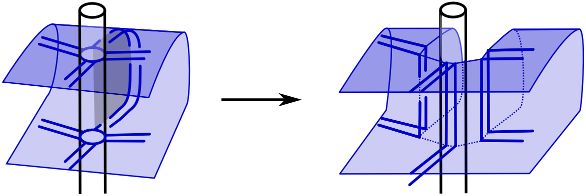
$$\ell_{\mathrm{Sp}}(\phi) \leq A\Delta(M_\phi) + B$$

(This isn't going to work.)

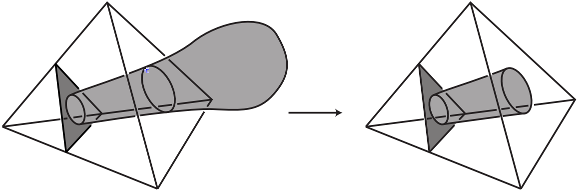
Moves between almost normal, normal

- Face compression:

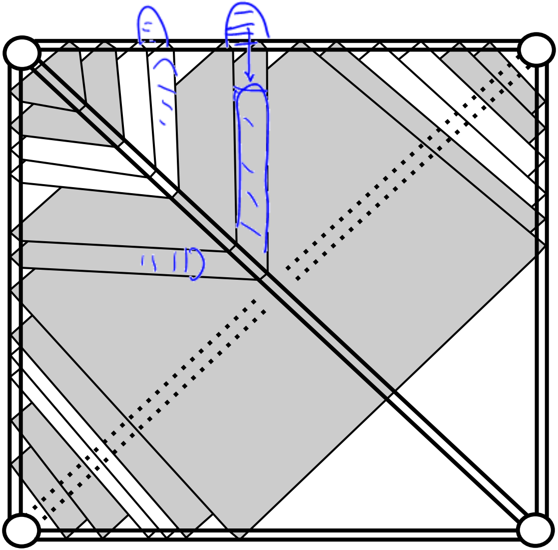
In 3D:



- Compression isotopy:



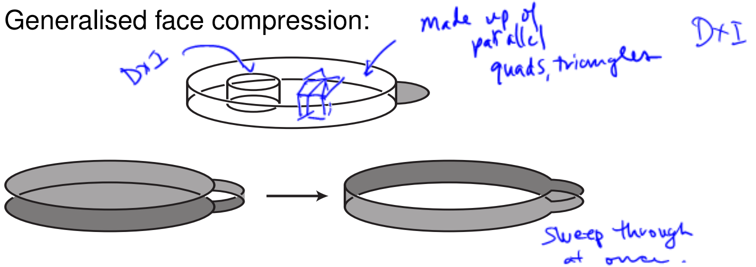
Problem: Parallelity bundles



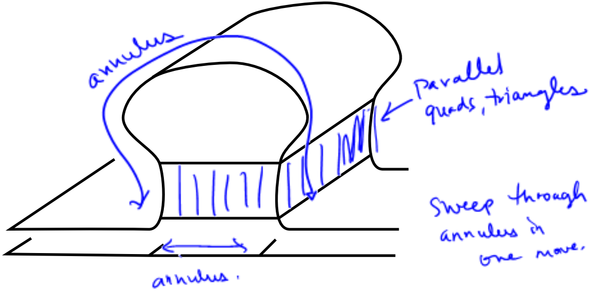
No bound on # triangles & squares that are parallel.

Fix: More drastic simplifications

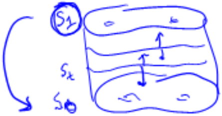
- Generalised face compression:



- Annular simplification:



Finishing up



Idea:

1. Start with M_ϕ. Cut along least weight normal surface S to obtain $S \times I$. Pick spine $s_0 \in S \times \{0\}$.
2. Find surfaces interpolating between $S \times \{0\}$ and $S \times \{1\}$, differing by *generalised isotopy* moves.
3. Bound number of steps in $\mathrm{Sp}(S)$ required to transfer s_0 through interpolating surfaces to $S \times \{1\}$. Bound of form

$$\text{steps} \leq A_0 \Delta(M) + B_0.$$

4. Bound steps to transfer spine s_1 in $S \times \{1\}$ to $\phi(s_0)$, of form

$$\text{steps} \leq A_1 \Delta(M) + B_1.$$

5. Consequence:

$$\ell_{\mathrm{Sp}}(\phi) \leq A\Delta(M) + B.$$

Summary

$$\underbrace{\frac{1}{A}\ell_{\mathrm{Sp}}(\phi) - B}_{} \leq \Delta(M_\phi) \leq \underbrace{\ell_{\mathrm{Tr}}(\phi) + 3(4g-2)}_{}$$

Thus

$\Delta(M_\phi)$ and $\ell_{MCG}(\phi)$ are within bounded ratios of each other.