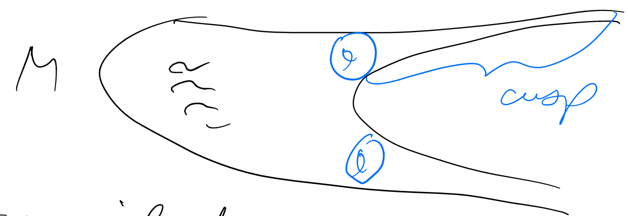
Infinitely many virtual geometric triangulations

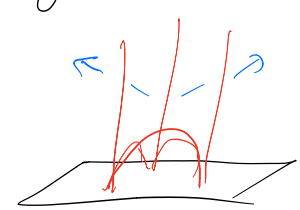
CUNY G&T Seminar, 4/21/20

Joint w/ Neil Hoffman

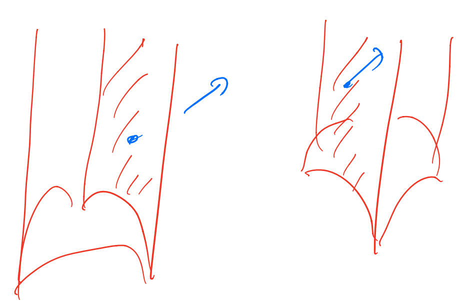
Setting: • hyperbolic 3-mflds, $M = \mathbb{H}^3/\Gamma$,
• cusped manifolds: noncompact, finite volume, orientable



Def A geometric ideal tetrahedron $T \subset \mathbb{H}^3$ is the convex hull of 4 non-coplanar points on $\partial\mathbb{H}^3$.
Note: T, ∂T inherits orientation from $\mathbb{H}^3$ (outward normals on ∂T).



Def A geometric ideal triangulation of M is a decomp. into geom ideal tetrahedra, glued by orient-reversing isometries along their faces.



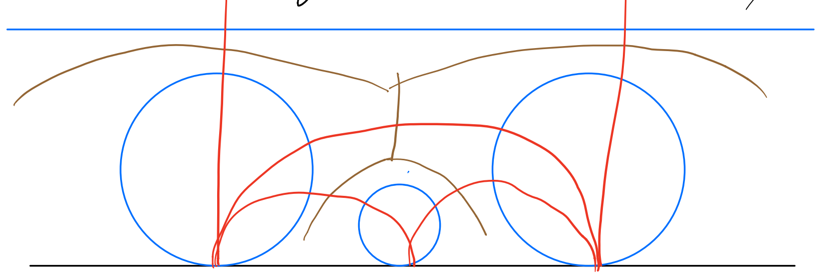
Conjecture (Thurston ~1980, Petronio 1990s)
Every cusped hyperbolic M^3 has a geometric ideal triangulation.

Why care: Thurston's Dehn surgery theorem is way easier to prove if M has geom triang.

Evidence
① Millions of examples

② [Epstein-Penner, 1988] Every cusped M^3 has a geometric ideal polyhedral decomp.

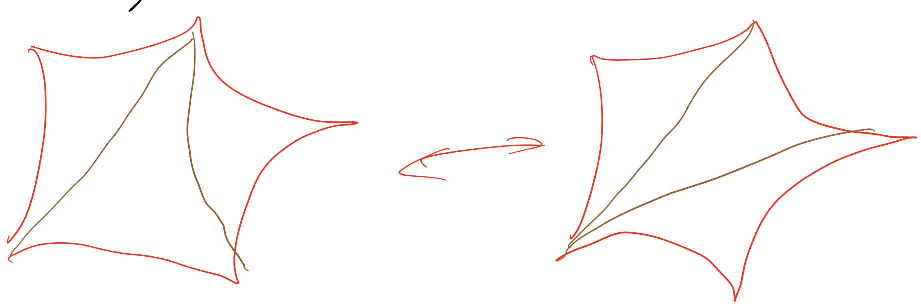
idea: • pack H^3 by horoballs (preimages of cusps)
• assign each horoball a "basin of attraction", get tiling of H^3.
• dual is polyhedral decomp, generically a triangulation



③ [Guéritaud-Schleimer, 2008] For generic Dehn fillings of generic multi-cusped M, the EP decomp is a triangulation

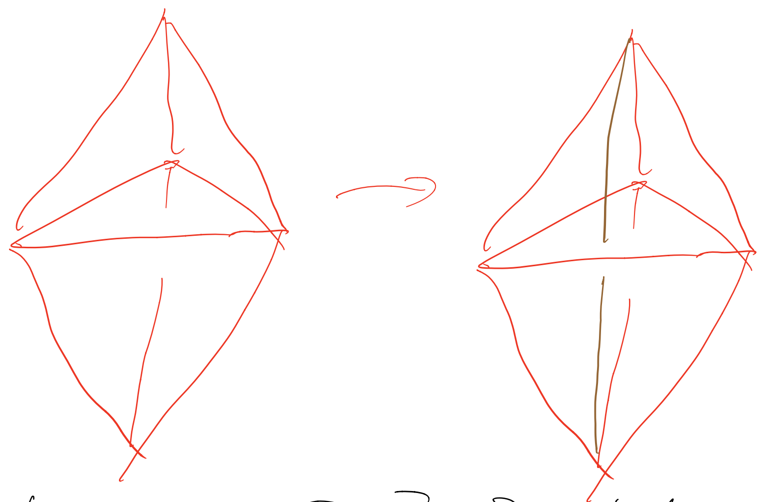
(4) [Luo-Schleimer-Tillmann, 2008]
Every cusped M has a finite cover $\hat{M}$ whose EP decomp can be subdivided into geometric polyhedra

What can go wrong: subdividing polyhedra introduces diagonals in faces. They might be inconsistent.



- find a cover $\hat{M}$ where every polyhedron P has all vertices at distinct cusps of $\hat{M}$. [peripheral separability]
- order cusps of $\hat{M}$ $\rightsquigarrow$ order vertices of every polyhedron.
- cone to smallest vertex.

(5) [Dadd-Duan, 2015] $M = S^3 -$ 
has infinitely many geometric triangulations, related by 2-3 moves



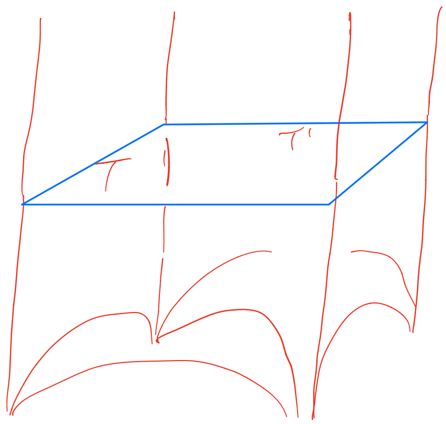
2 tetra ⟶ 3 tetra.

as long as the bipyramid is convex, this operation can be done geometrically.

<u>Theorem</u> (F–Hoffman, 2020) Every cusped hyperbolic 3-manifold M has a finite cover $\hat{M}$ admitting infinitely many distinct geometric triangulations $T_1, T_2, \ldots,$ where T_{i+1} is obtained from T_i by a 2–3 move.

<u>Proof outline</u>, backwards

<u>Step 2</u>: Find a cover $\hat{M} \to M$ with a geometric triangulation T & distinguished cusp $\hat{A}$, such that there are only 2 tetrahedra T & T' poking into $\hat{A}$, one ideal vertex each.



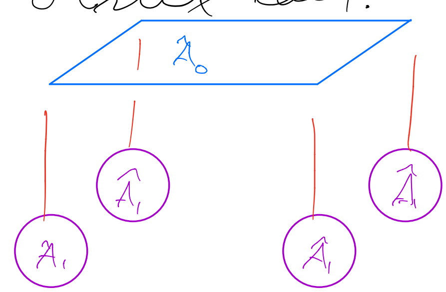
T glued to T' along 3 faces. $\Rightarrow$ 3 possible bipyramids to use in 2-3 move $\Rightarrow$ geometric 2-3 move always possible

[Reg. covers of this picture also fine]

Get J by subdividing Epstein-Penner decomp, following LST. [peripheral separability]

<u>Step 1</u> Find a cover $\hat{M}' \to M$ containing a collection of cusps $\hat{A} = \hat{A}_0, \dots, \hat{A}_n$ s.t. resulting Epstein-Penner decomp has only 1 or 2 cells poking into $\hat{A}$, one ideal vertex each.

<u>Suffices</u> to have unique shortest path from $\hat{A}$ to $\cup_{j=1}^{n} \hat{A}_j$



Build this cover using <u>double coset separability</u>.
If $G \cong \mathbb{Z}^2$, $H \cong \mathbb{Z}^2$ are peripheral subgroups of $\pi_1 M$, and $f \in \pi_1 M$ is an element, want to <u>separate</u> GfH from 1. find $\varphi : \pi_1 M \to F$, finite gp. s.t. $GfH \cap \ker(\varphi) = \emptyset$, i.e. $\varphi(GfH) \cap \varphi(1) = \emptyset$.