QUASI-ISOMETRIC CLASSIFICATION OF GRAPH MANIFOLD GROUPS

JASON A. BEHRSTOCK and WALTER D. NEUMANN

Abstract

We show that the fundamental groups of any two closed irreducible nongeometric graph manifolds are quasi-isometric. We also classify the quasi-isometry types of fundamental groups of graph manifolds with boundary in terms of certain finite two-colored graphs. A corollary is the quasi-isometric classification of Artin groups whose presentation graphs are trees. In particular, any two right-angled Artin groups whose presentation graphs are trees of diameter greater than 2 *are quasi-isometric; further, this quasi-isometry class does not include any other right-angled Artin groups.*

A finitely generated group can be considered geometrically when endowed with a word metric. Up to quasi-isometric equivalence, such metrics are unique. (In this article, only finitely generated groups are considered.) Given a collection of groups $\mathscr{G}$, Gromov [11] proposed the fundamental questions of identifying which groups are quasi-isometric to those in $\mathscr{G}$ (*rigidity*) and which groups in $\mathscr{G}$ are quasi-isometric to each other (*classification*).

In this article, we focus on the classification question for graph manifold groups and right-angled Artin groups.

A compact 3-manifold M is called *geometric* if $M \backslash \partial M$ admits a geometric structure in the sense of Thurston (i.e., a complete locally homogeneous Riemannian metric of finite volume). Thurston's geometrization conjecture (see [29], [17], [21], [23], [22]) provides that every irreducible 3-manifold of zero Euler characteristic (i.e., with boundary consisting only of tori and Klein bottles) admits a decomposition along tori and Klein bottles into geometric pieces, the minimal such decomposition being called the *geometric decomposition*.

There is a considerable literature on quasi-isometric rigidity and classification of 3-manifold groups. The rigidity results can be briefly summarized in the following form.

DUKE MATHEMATICAL JOURNAL
Vol. 141, No. 2, © 2008
Received 18 April 2006. Revision received 29 April 2007.
2000 *Mathematics Subject Classification*. Primary 20F65; Secondary 57N10, 20F36.
Behrstock's work partially supported by National Science Foundation grant DMS-0604524.
Neumann's work partially supported by National Science Foundation grant DMS-0456227.

THEOREM A

If a group G is quasi-isometric to the fundamental group of a 3-manifold M with zero Euler characteristic, then G is weakly commensurable with $\pi_1(M')$ for some such 3-manifold M'. Moreover, M' is closed, respectively, irreducible, respectively, geometric if and only if the same is true of M.*

This quasi-isometric rigidity for 3-manifold groups is the culmination of the work of many authors, key steps being provided by Gromov [10]; Gromov and Sullivan; Cannon and Cooper [5]; Eskin, Fisher, and Whyte [7]; Kapovich and Leeb [13]; Rieffel [24]; and Schwartz [25]. The reducible case reduces to the irreducible case using Papasoglu and Whyte [20, Theorem 0.4]; the irreducible nongeometric case is considered by Kapovich and Leeb [13].

The classification results in the geometric case can be summarized by the following, the first half of which is an easy application of the Milnor-Švarc lemma (see [16], [28]).

THEOREM B

There are exactly seven quasi-isometry classes of fundamental groups of closed geometric 3-manifolds, namely, any such group is quasi-isometric to one of the eight Thurston geometries ($\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{E}^3$, Nil, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{\mathrm{PSL}}$, Sol, $\mathbb{H}^3$), but the two geometries $\mathbb{H}^2 \times \mathbb{E}^1$ and $\widetilde{\mathrm{PSL}}$ are quasi-isometric.

If a geometric manifold M has boundary, then it is either Seifert fibered and its fundamental group is quasi-isometric (indeed, commensurable) with $F_2 \times \mathbb{Z}$ (see [13]), or it is hyperbolic, in which case quasi-isometry also implies commensurability (see [25]).

A *graph manifold* is a 3-manifold that can be decomposed along embedded tori and Klein bottles into finitely many Seifert manifolds; equivalently, these are exactly the manifolds with no hyperbolic pieces in their geometric decomposition. Since the presence of a hyperbolic piece can be quasi-isometrically detected (see [8], [14], [1]), this implies that the class of fundamental groups of graph manifolds is rigid. We answer the classification question for graph manifold groups. Before discussing the general case, we note the answer for closed nongeometric graph manifolds, thereby resolving a question of Kapovich and Leeb [14, Question 1.2].

THEOREM 2.1

Any two closed nongeometric graph manifolds have bi-Lipschitz homeomorphic universal covers. In particular, their fundamental groups are quasi-isometric.

This contrasts with commensurability of closed graph manifolds. In the case where the graph manifold is composed of just two Seifert pieces, there already are infinitely

*Two groups are said to be *weakly commensurable* if they have quotients by finite normal subgroups which have isomorphic finite-index subgroups.

many commensurability classes (they are classified in that case but not in general; see Neumann [18, Theorem B]).

We also classify compact graph manifolds with boundary. To describe this, we need some further terminology. We associate to the geometric decomposition of a nongeometric graph manifold M its *decomposition graph* $\Gamma(M)$, which has a vertex for each Seifert piece and an edge for each decomposing torus or Klein bottle. We color the vertices of $\Gamma(M)$ **b**lack or **w**hite according to whether or not the Seifert piece includes a boundary component of M (**b**ounded or **w**ithout boundary). We call this the *two-colored decomposition graph*. We can similarly associate a two-colored tree to the decomposition of the universal cover $\tilde{M}$ into its fibered pieces. We call this infinite valence two-colored tree $\mathrm{BS}(M)$ since it is the Bass-Serre tree corresponding to the graph of groups decomposition of $\pi_1(M)$.

The Bass-Serre tree $\mathrm{BS}(M)$ can be constructed directly from the decomposition graph $\Gamma = \Gamma(M)$ by first replacing each edge of Γ by a countable infinity of edges with the same endpoints and then taking the universal cover of the result. If two two-colored graphs Γ_1 and Γ_2 lead to isomorphic two-colored trees by this procedure, then we say that Γ_1 and Γ_2 are *bisimilar*. In Section 4, we give a simpler, algorithmically checkable criterion for bisimilarity* and show that each bisimilarity class contains a unique minimal element.

Our classification theorem, which includes the closed case (Theorem 2.1), takes this form.

THEOREM 3.2
If M and M' are nongeometric graph manifolds, then the following are equivalent:
(1) $\tilde{M}$ *and* $\tilde{M}'$ *are bi-Lipschitz homeomorphic;*
(2) $\pi_1(M)$ *and* $\pi_1(M')$ *are quasi-isometric;*
(3) $\mathrm{BS}(M)$ *and* $\mathrm{BS}(M')$ *are isomorphic as two-colored trees; and*
(4) *the minimal two-colored graphs in the bisimilarity classes of the decomposition graphs* $\Gamma(M)$ *and* $\Gamma(M')$ *are isomorphic.*

One can list minimal two-colored graphs of small size, yielding, for instance, the result that there are exactly $2, 6, 26, 199, 2811, 69711, 2921251, 204535126, \ldots$ quasi-isometry classes of fundamental groups of nongeometric graph manifolds which are composed of at most $1, 2, 3, 4, 5, 6, 7, 8, \ldots$ Seifert pieces.

For closed nongeometric graph manifolds, we recover that there is just one quasi-isometry class (see Theorem 2.1): the minimal two-colored graph is a single white vertex with a loop. Similarly, for nongeometric graph manifolds that have boundary

*We thank Ken Shan [27] for pointing out that our equivalence relation is a special case of the computer science concept of *bisimilarity*, which is related to *bisimulation*.

components in every Seifert component, there is just one quasi-isometry class. (The minimal two-colored graph is a single black vertex with a loop.)

For graph manifolds with boundary, the commensurability classification is also rich but not yet well understood. If M consists of two Seifert components glued to each other such that M has boundary components in both Seifert components, one can show that M is commensurable with any other such M, but this already appears to be no longer true in the case of three Seifert components.

We end by giving an application to the quasi-isometric classification of Artin groups. The point is that if the presentation graph is a tree, then the group is a graph-manifold group, and thus our results apply. In particular, answering a question of Mladen Bestvina [2], we obtain the classification of right-angled Artin groups whose presentation graph is a tree. We also show rigidity of such groups among right-angled Artin groups.

We call a right-angled Artin group whose presentation graph is a tree a *right-angled tree group*. If the tree has diameter at most 2, then the group is $\mathbb{Z}$, $\mathbb{Z}^2$ or (free) $\times$ $\mathbb{Z}$. We show that right-angled tree groups with presentation graph of diameter greater than 2 are all quasi-isometric to each other. In fact, we have the following.

THEOREM 5.3
Let G' be any Artin group, and let G be a right-angled tree group whose tree has diameter greater than 2. Then G' is quasi-isometric to G if and only if G' has a presentation graph that is an even-labeled tree of diameter at least 2 satisfying the following:
(i) *all interior edges have label 2, and*
(ii) *if the diameter is 2, then at least one edge has label greater than 2.*
(An "interior edge" is an edge that does not end in a leaf of the tree.)

The commensurability classification of right-angled tree groups is richer: any two such groups whose presentation graphs have diameter 3 are commensurable, but it appears that there are already infinitely many commensurability classes for diameter 4.

Theorem 5.3 also has implications for quasi-isometric rigidity phenomena in relatively hyperbolic groups. For such applications, see Behrstock, Druţu, and Mosher [1], where it is shown that graph manifolds, and thus tree groups, can only quasi-isometrically embed in relatively hyperbolic groups in very constrained ways.

In the course of proving Theorem 5.3, we classify which Artin groups are quasi-isometric to 3-manifold groups. This family of groups coincides with those that Gordon [9] proved to be isomorphic to 3-manifold groups.

1. Quasi-isometry of fattened trees

Let T be a tree all of whose vertices have valence in the interval $[3, K]$ for some K. We fix a positive constant L and assume that T has been given a simplicial metric in which each edge has length between 1 and L. Now, consider a "fattening" of T, where we replace each edge E by a strip isometric to $E \times [-\epsilon, \epsilon]$ for some $\epsilon > 0$ and replace each vertex by a regular polygon around the boundary of which the strips of incoming edges are attached in some given order. Call this object X. Let X_0 be similarly constructed, but starting from the regular 3-valence tree with all edges having length 1, and with $\epsilon = 1/2$.

We first note the following easy lemma, whose proof we omit.

LEMMA 1.1
There exists C, depending only on K, L, ϵ, such that X is C bi-Lipschitz homeomorphic to X_0.

Note that if S is a compact Riemannian surface with boundary having Euler characteristic less than zero, then its universal cover $\tilde{S}$ is bi-Lipschitz homeomorphic to a fattened tree as above and hence to X_0. We can thus use X_0 as a convenient bi-Lipschitz model for any such $\tilde{S}$.

Let X be a manifold as above, bi-Lipschitz equivalent to X_0 (so that X may be a fattened tree or an $\tilde{S}$). Thus, X is a 2-manifold with boundary, and its boundary consists of infinitely many copies of $\mathbb{R}$.

THEOREM 1.2
Let X be as above with a chosen boundary component $\partial_0 X$. Then there exist K and a function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for any K' and any K'-bi-Lipschitz homeomorphism Φ_0 from $\partial_0 X$ to a boundary component $\partial_0 X_0$ of the "standard model" X_0, Φ_0 extends to a $\phi(K')$-bi-Lipschitz homeomorphism $\Phi\colon X \to X_0$ which is K-bi-Lipschitz on every other boundary component.

Proof
If true for some X, then the theorem will be true (with K replaced by KL) for any X' L-bi-Lipschitz homeomorphic to X, so we may assume that X is isometric to our standard model X_0. In this case, we see that K can be arbitrarily close to 1. (With very slightly more effort, one can make $K = 1$.)

We construct the homeomorphism in two steps. The first step is to extend near $\partial_0 X$ and the second to extend over the rest of X.

We consider vertices of the underlying tree adjacent to the boundary component $\partial_0 X$. These have a certain "local density" along $\partial_0 X$ given by the number of them in an interval of a given length, measured with respect to the metric on $\partial_0 X$ which pulls back from $\partial_0 X_0$ by Φ_0. We first describe how to modify this local density using a $(1, L)$-quasi-isometric bi-Lipschitz homeomorphism with $L = O(|\log(D)|)$, where D is the factor by which we want to modify density. We increase density locally by moves on the underlying tree in which we take a vertex along $\partial_0 X$ and a vertex adjacent

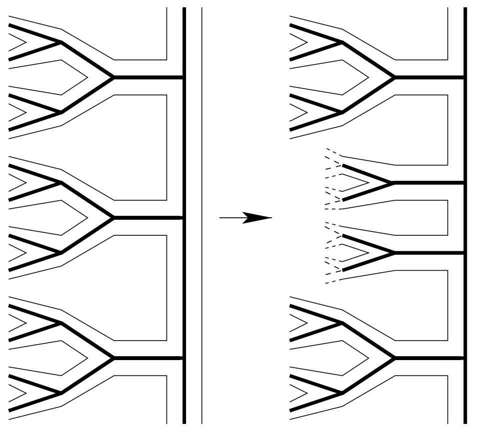


Figure 1. Increasing number of vertices along $\partial_0 X$ by a *depth* 1 *splitting*

to it not along $\partial_0 X$ and collapse the edge between them to give a vertex of valence 4, which we then expand again to two vertices of valence 3, now both along $\partial_0 X$ (see Figure 1). This can be realized by a piecewise-linear homeomorphism. Since it is an isometry outside a bounded set, it has a finite bi-Lipschitz bound, say, k. To increase density along an interval by at most a factor of D, we need to repeat this process at most $\log_2(D)$ times, so we get a bi-Lipschitz homeomorphism whose bi-Lipschitz bound is bounded in terms of D. Similarly, we can decrease density (using the inverse move) by a bi-Lipschitz map whose bi-Lipschitz bound is bounded in terms of D.

One can then apply this process simultaneously on disjoint intervals to change the local density along disjoint intervals. For instance, applying the above doubling procedure to all vertices along $\partial_0 X$ doubles the density; furthermore, since it affects disjoint bounded sets, we still have bi-Lipschitz bound k. Similarly, given two disjoint intervals, one can, for instance, increase the local density by a factor of D on one of the intervals and decrease it on the other by a different factor D'; since the intervals are disjoint, the bi-Lipschitz bound depends only on the largest factor, which is uniformly bounded by K', the bi-Lipschitz bound for Φ_0.

By these means, replacing X with its image under a bi-Lipschitz map with bi-Lipschitz bound bounded in terms of K' and which is an isometry on $\partial_0 X$, we can assure that the number of vertices along $\partial_0 X$ and $\partial_0 X_0$ matches to within a fixed constant over any interval in $\partial_0 X$ and the corresponding image under Φ_0. We now construct a bi-Lipschitz map from this new X to X_0 by first extending Φ_0 to a 1-neighborhood of $\partial_0 X$ and then extending over the rest of X by isometries of the components of the complement of this neighborhood. By composing the two bi-Lipschitz maps, we get a bi-Lipschitz homeomorphism Ψ from the original X which does what we want on $\partial_0 X$ while, on every other boundary component $\partial_i X$, it is an isometry outside an interval of length bounded in terms of K'.

Now, choose arbitrary K greater than 1. On $\partial_i X$, we can find an interval J of length bounded in terms of K' and K which includes the interval J_0 on which our map is not an isometry and whose length increases or decreases under Ψ by a factor of at most K. (Specifically, if the length of J_0 was multiplied by s, choose J of length $\lambda\ell(J_0)$ with $\lambda \geq \max((K - K_s)/(K-1), (s-1)/(K-1))$.) Let Ψ' be the map of J which is a uniform stretch or shrink by the same factor (so that the images of Ψ' and $\Psi|J$ are identical). The following self-map α of a collar neighborhood $J \times [0, \epsilon]$ restricts to $\Psi' \circ \Psi^{-1}$ on the left boundary $J \times \{0\}$ and to the identity on the rest of the boundary:

$$\alpha(x, t) = \frac{\epsilon - t}{\epsilon} \Psi' \circ \Psi^{-1}(x) + \frac{t}{\epsilon} x.$$

This α has bi-Lipschitz constant bounded in terms of the bound on the left boundary and the length of J and hence is bounded in terms of K and K'. By composing Ψ with α on a collar along the given interval, we adjust $\Psi|\partial_i X$ to be a uniform stretch or shrink along this interval. We can do this on each boundary component other than $\partial_0 X_0$. The result is a bi-Lipschitz homeomorphism whose bi-Lipschitz bound L is still bounded in terms of K' and K and which satisfies the conditions of the theorem. □

We now deduce an analogue of Theorem 1.2 in the case where the boundary curves ∂X are each labeled by one from a finite number of colors, C, and the maps are required to be color preserving. We call a labeling a *bounded coloring* if there is a uniform bound such that, given any point in X and any color, there is a boundary component of that color a uniformly bounded distance away. The lift of a coloring on a compact surface yields a bounded coloring. We now fix a bounded coloring on our "standard model" X_0; furthermore, we choose this coloring so that it satisfies the following regularity condition, which is stronger than the above hypothesis: for every point on a boundary component and for every color in C, there is an adjacent boundary component with that color a bounded distance from the given point. Call the relevant bound B.

THEOREM 1.3
Let X be as in Lemma 1.1 with a chosen boundary component $\partial_0 X$, and fix a bounded coloring on the elements of ∂X. Then there exist K and a function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that for any K' and any color-preserving K'-bi-Lipschitz homeomorphism Φ_0 from $\partial_0 X$ to a boundary component $\partial_0 X_0$ of the "standard model" X_0, Φ_0 extends to a $\phi(K')$-bi-Lipschitz homeomorphism $\Phi\colon X \to X_0$ which is K-bi-Lipschitz on every other boundary component and which is a color-preserving map from ∂X to ∂X_0.

Proof
As in the proof of Theorem 1.2, we assume that X is isometric to our standard model X_0. Then we proceed in two steps, first extending near $\partial_0 X$, then extending over the rest of X.

To extend near $\partial_0 X$, we proceed as in the proof of Theorem 1.2, except that now we need to match not only density but colors as well. Instead of using only a *depth 1 splitting* as in Figure 1, one may perform a *depth n splitting* by choosing a vertex at distance n from $\partial_0 X$ and then moving that vertex so that it is adjacent to $\partial_0 X$; this bi-Lipschitz map increases the density of vertices along a given boundary component. Note that a depth n move (and its inverse) can be obtained as a succession of depth 1 moves and their inverses, so the effect of using such moves is only to yield a more concise language. Since the coloring of X is a bounded coloring, from any point on $\partial_0 X$ there is a uniform bound on the distance to a vertex adjacent to a boundary component of any given color. Thus, with a bounded bi-Lipschitz constant, we may alter the density and coloring as needed.

As in the proof of Theorem 1.2, we may extend to a map that does what is required on $\partial_0 X$, is an isometry (but not preserving boundary colors) outside a neighborhood of $\partial_0 X$, and is a K-bi-Lipschitz map on the boundary components other than $\partial_0 X$ with K close to 1.

In step two, a further bi-Lipschitz map is applied that fixes up colors on these remaining boundary components.

Consider a boundary component $\partial_1 X$ adjacent to $\partial_0 X$. We want to make colors correct on boundary components adjacent to $\partial_1 X$. They are already correct on $\partial_0 X$ and the boundary components adjacent on each side of this. Call these $\partial_2 X$ and $\partial_2' X$. As we move along $\partial_1 X$ looking at boundary components, number the boundary components $\partial_0 X, \partial_2 X, \partial_3 X, \ldots$ until we come to a $\partial_{j+1} X$, which is the wrong color. We use splitting moves to bring new boundary components of the desired colors in to be adjacent to $\partial_1 X$ between $\partial_j X$ and $\partial_{j+1} X$. By our regularity assumption on X_0, we need to add at most $2B$ new boundary components before the color of $\partial_{j+1} X$ is needed; thus, we need to perform at most $2B$ splitting moves. Moreover, the bounded coloring hypothesis implies that each of these splitting moves can be chosen to be of a uniformly bounded depth. (Note that the bounded coloring assumption implies that at any point of X and in any direction in the underlying tree, any desired color is a uniformly bounded distance away.) We repeat this process along all of $\partial_1 X$ in both directions to make colors correct. The fact that we do at most $2B$ such moves for each step along $\partial_1 X$ means that we affect the bi-Lipschitz constant along $\partial_1 X$ by at most a factor of $2B+1$. Since bounded depth-splitting moves have compact support and since there are at most $2B$ of these performed between any pair $\partial_j X, \partial_{j+1} X$, we see that the bi-Lipschitz constant needed to fix this part of $\partial_1 X$ is bounded in terms of B and the bounded coloring constant. Since for $i \neq j$ the neighborhoods affected by fixing the part of $\partial_1 X$ between $\partial_j X, \partial_{j+1} X$ are disjoint from those affected by fixing between $\partial_i X, \partial_{i+1} X$, we see that fixing all of $\partial_1 X$ requires a bi-Lipschitz bound depending only on B and on the bounded coloring constant; let us call this bi-Lipschitz bound C.

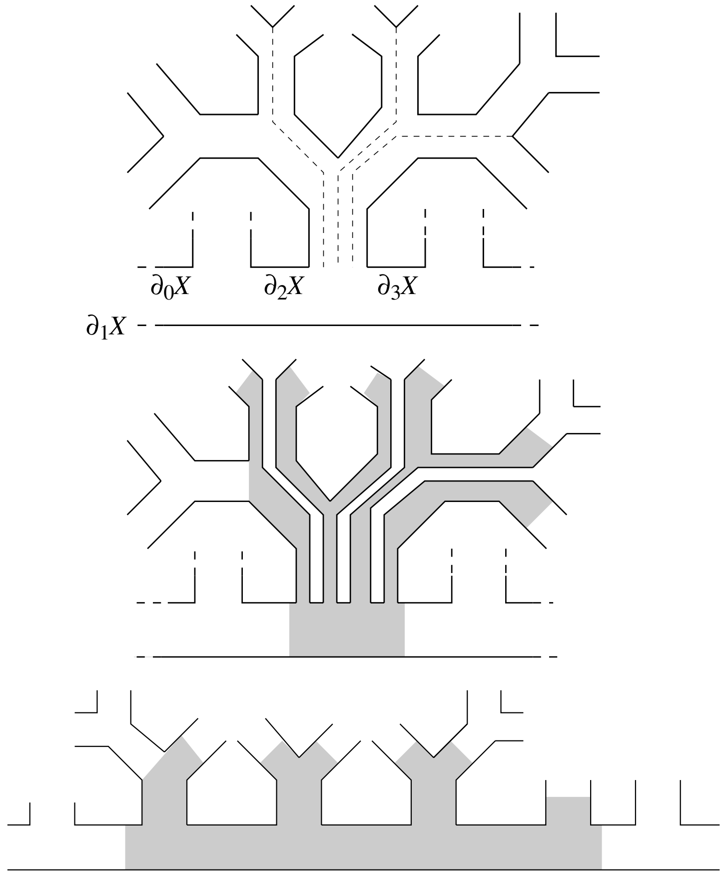


Figure 2. Adjusting colors along $\partial_1 X$. Three depth 3 moves are illustrated. The shaded region shows where the metric is adjusted. The final metric is shown at the bottom. In the first and last pictures, all edges should have the same length since both are isometric to the standard model. (Some distortion was needed to draw them.)

We claim that by repeating this process boundary component by boundary component, one can keep the bi-Lipschitz constant under control and thus prove the theorem. To see this, consider Figure 2, which illustrates a typical set of splitting moves and shows the neighborhoods on which the metric has been altered. Since we can assure an upper bound on the diameter of these neighborhoods, we can bound the bi-Lipschitz constants of the modifications; this is the constant C above. Only a bounded number of the neighborhoods needed for later modifications intersects these neighborhoods, yielding an overall bi-Lipschitz bound that is at most a bounded power of C. □

2. Closed graph manifolds

The following theorem answers [14, Question 1.2], posed by Kapovich and Leeb. It is a special case of Theorem 3.2, but we treat it separately here since its proof is simple and serves as preparation for the general result.

THEOREM 2.1
Any two closed nongeometric graph manifolds have bi-Lipschitz homeomorphic universal covers. In particular, their fundamental groups are quasi-isometric.

Let us begin by recalling the following lemma.

LEMMA 2.2 (Kapovich and Leeb [14, Lemma 2.1]; Neumann [18, Remark, p. 371])
Any nongeometric graph manifold has an orientable finite cover where all Seifert components are circle bundles over orientable surfaces of genus at least 2*. Furthermore, one can arrange the intersection numbers of the fibers of adjacent Seifert components to be* ± 1.

If we replace our graph manifold by a finite cover as in the above lemma, then we have a trivialization of the circle bundle on the boundary of each Seifert piece using the section given by a fiber of a neighboring piece. The fibration of this piece then has a relative Euler number.

LEMMA 2.3 (Kapovich and Leeb [14, Theorem 2.3])
Up to a bi-Lipschitz homeomorphism of the universal cover, we can assume that all of the above relative Euler numbers are zero.

A graph manifold G as in Lemma 2.2 is what Kapovich and Leeb call a "flip-manifold." It is obtained by gluing together finitely many manifolds of the form (surface) $\times S^1$, where the gluing maps along the boundary tori are chosen to flip base and fiber coordinates. We can give it a metric in which every fiber S^1 (and hence every boundary circle of a base surface) has length 1.

A topological model for the universal cover $\tilde{G}$ can be obtained by gluing together infinitely many copies of $X_0 \times \mathbb{R}$ according to a tree, gluing by the *flip map* $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ when gluing boundary components. We call the resulting manifold Y.

We wish to show that $\tilde{G}$ is bi-Lipschitz homeomorphic to Y.

Proof of Theorem 2.1
The universal cover of each Seifert component of G is identified with $\tilde{S}_i \times \mathbb{R}$, where S_i is one of a finite collection of compact surfaces with boundary. Choose a number K sufficiently large so that Theorem 1.2 applies to each of them. Choose a bi-Lipschitz

homeomorphism from one piece $\tilde{S}_i \times \mathbb{R}$ of $\tilde{G}$ to a piece $X_0 \times \mathbb{R}$ of Y, preserving the (surface) $\times\, \mathbb{R}$ product structure. We want to extend to a neighboring piece of $\tilde{G}$. On the common boundary $\mathbb{R} \times \mathbb{R}$, we have a map that is of the form $\phi_1 \times \phi_2$ with ϕ_1 and ϕ_2 both bi-Lipschitz. By Theorem 1.2, we can extend over the neighboring piece by a product map, and on the other boundaries of this piece we then have maps of the form $\phi_1' \times \phi_2$ with ϕ_1' K-bi-Lipschitz. We do this for all neighboring pieces of our starting piece. Because of the flip, when we extend over the next layer we have maps on the outer boundaries that are K-bi-Lipschitz in both base and fiber. We can thus continue extending outward inductively to construct our desired bi-Lipschitz map. □

3. Graph manifolds with boundary

A nongeometric graph manifold M has a minimal decomposition along tori and Klein bottles into geometric (Seifert-fibered) pieces, called the *geometric decomposition*. The cutting surfaces are then π_1-injective. In this decomposition, one cuts along one-sided Klein bottles; this differs from JSJ decomposition, where one cuts along the torus neighborhood boundaries of these Klein bottles (see, e.g., Neumann and Swarup [19, Section 4]).

We associate to this decomposition its *decomposition graph*, which is the graph with a vertex for each Seifert component of M and an edge for each decomposing torus or Klein bottle. If there are no one-sided Klein bottles, then this graph is the graph of the associated graph of groups decomposition of $\pi_1(M)$. (If there are decomposing Klein bottles, the graph of groups has, for each Klein bottle, an edge to a new vertex rather than a loop. This edge corresponds to an amalgamation to a Klein bottle group along a $\mathbb{Z} \times \mathbb{Z}$ and corresponds also to an inversion for the action of $\pi_1(M)$ on the Bass-Serre tree. Using a loop rather than an edge makes the Bass-Serre tree a weak covering of the decomposition graph.)

We color vertices of the decomposition graph **b**lack or **w**hite according to whether the Seifert piece includes a boundary component of M or does not (**b**ounded or **w**ithout boundary).

A second graph that we consider is the two-colored decomposition graph for the decomposition of the universal cover $\tilde{M}$ into its fibered pieces. We denote it $\mathrm{BS}(M)$ and call it the *two-colored Bass-Serre tree* since it is the Bass-Serre tree for our graph of groups decomposition. It can be obtained from the two-colored decomposition graph by replacing each edge with a countable infinity of edges between its endpoints and then taking the universal cover of the resulting graph.

A *weak covering map* from a two-colored graph Γ to a two-colored graph Γ' is a color-preserving graph homomorphism $\phi\colon \Gamma \to \Gamma'$ with the property that for any vertex v of Γ and every edge e' at $\phi(v)$, there is at least one edge e at v mapping to e'. An example of such a map is the map that collapses any multiple edge of Γ to a single edge. Any covering map of nongeometric graph manifolds induces a weak covering map of their two-colored decomposition graphs.

Note that if a weak covering map exists from Γ to Γ', then Γ and Γ' have isomorphic two-colored Bass-Serre trees. The equivalence relation on two-colored graphs generated by the relation of the existence of a weak covering map is called *bisimilarity*. In Section 4, we prove the following.

PROPOSITION 3.1

If we restrict to countable connected graphs, then each equivalence class of two-colored graphs includes two characteristic elements: a unique tree that weakly covers every element in the class (the Bass-Serre tree) and a unique minimal element that is weakly covered by all elements in the class.

For example, if all the vertices of the graph have the same color, then the minimal graph for its bisimilarity class is a single vertex with a loop attached, and the Bass-Serre tree is the single-colored regular tree of countably infinite degree.

Our main theorem is the following.

THEOREM 3.2

If M and M' are nongeometric graph manifolds, then the following are all equivalent:

(1) *$\tilde{M}$ and $\tilde{M}'$ are bi-Lipschitz homeomorphic;*

(2) *$\pi_1(M)$ and $\pi_1(M')$ are quasi-isometric;*

(3) $\mathrm{BS}(M)$ *and* $\mathrm{BS}(M')$ *are isomorphic as two-colored trees; and*

(4) *the minimal two-colored graphs in the bisimilarity classes of the decomposition graphs $\Gamma(M)$ and $\Gamma(M')$ are isomorphic.*

Proof

Clearly, (1) implies (2). The equivalence of (3) and (4) is Proposition 3.1. Kapovich and Leeb [14, Theorem 1.1] proved that any quasi-isometry essentially preserves the geometric decomposition of Haken manifolds and therefore induces an isomorphism between their Bass-Serre trees. To prove the theorem, it remains to show that (3) or (4) implies (1).

Suppose therefore that M and M' are nongeometric graph manifolds that satisfy the equivalent conditions (3) and (4). Let Γ be the minimal graph in the bisimilarity class of $\Gamma(M)$ and $\Gamma(M')$. It suffices to show that each one of $\tilde{M}$ and $\tilde{M}'$ is bi-Lipschitz homeomorphic to the universal cover of some standard graph manifold associated to Γ. There is therefore no loss in assuming that M' *is* such a standard graph manifold; "standard" in that case means that $\Gamma(M') = \Gamma$ and that each loop at a vertex in Γ corresponds to a decomposing Klein bottle (i.e., a boundary torus of the corresponding Seifert fibered piece which is glued to itself by a covering map to the Klein bottle).

Denote by C the set of pairs consisting of a vertex of Γ and an outgoing edge at that vertex. Since the decomposition graphs $\Gamma(M)$, $\Gamma(M')$, $\mathrm{BS}(M)$, and $\mathrm{BS}(M')$ for M, M', $\tilde{M}$, and $\tilde{M}'$ map to Γ, we can label the boundary components of the geometric pieces of these manifolds by elements of C.

Our desired bi-Lipschitz map can now be constructed inductively as in the proof of Theorem 2.1, at each stage of the process having extended over some submanifold Y of $\tilde{G}$. The difference from that proof is that now, when we extend the map from Y over a further fibered piece $X \times \mathbb{R}$, we must make sure that we are mapping boundary components to boundary components with the same C-label. That this can be done is exactly the statement of Theorem 1.3. □

Remark 3.3

With some work, Theorem 3.2 can be generalized to cover many situations outside of the context of 3-manifolds; such a formulation, including applications to Artin groups, will appear in the authors' forthcoming article with Tadeusz Januszkiewicz.

4. Two-colored graphs

Definition 4.1

A *graph* Γ consists of a vertex set $V(\Gamma)$ and an edge set $E(\Gamma)$ with a map $\epsilon\colon E(\Gamma) \to V(\Gamma)^2/C_2$ to the set of unordered pairs of elements of $V(\Gamma)$.

A *two-colored graph* is a graph Γ with a "coloring" $c\colon V(\Gamma) \to \{\mathbf{b}, \mathbf{w}\}$.

A *weak covering* of two-colored graphs is a graph homomorphism $f\colon \Gamma \to \Gamma'$ which respects colors and has the property that for each $v \in V(\Gamma)$ and for each edge $e' \in E(\Gamma')$ at $f(v)$, there exists an $e \in E(\Gamma)$ at v with $f(e) = e'$.

From this point forward, all graphs that we consider are assumed to be connected. It is easy to see that a weak covering is then surjective. The graph-theoretic results are valid for n-color graphs, but we only care about $n = 2$.

Definition 4.2

Two-colored graphs Γ_1, Γ_2 are *bisimilar*, written $\Gamma_1 \sim \Gamma_2$, if Γ_1 and Γ_2 weakly cover some common two-colored graph.

The following proposition implies, among other things, that this definition agrees with our earlier version.

PROPOSITION 4.3

The bisimilarity relation $\sim$ is an equivalence relation. Moreover, each equivalence class has a unique minimal element up to isomorphism.

LEMMA 4.4
If a two-colored graph Γ *weakly covers each of a collection of graphs* $\{\Gamma_i\}$*, then the* Γ_i *all weakly cover some common* Γ'*.*

Proof
The graph homomorphism that restricts to a bijection on the vertex set but identifies multiple edges with the same ends to a single edge is a weak covering. Moreover, if we do this to both graphs Γ and Γ_i of a weak covering $\Gamma \to \Gamma_i$, then we still have a weak covering. So there is no loss in assuming that all of our graphs have no multiple edges. A graph homomorphism $\Gamma \to \Gamma_i$ is then determined by its action on vertices. The induced equivalence relation $\equiv$ on vertices of Γ satisfies this property:

> If $v \equiv v_1$ and if e is an edge with $\epsilon(e) = \{v, v'\}$, then there exists an edge e_1 with $\epsilon(e_1) = \{v_1, v_1'\}$ and $v' \equiv v_1'$.

Conversely, an equivalence relation on vertices of Γ with this property induces a weak covering. Thus, we must just show that if we have several equivalence relations on $V(\Gamma)$ with this property, then the equivalence relation $\equiv$ that they generate still has this property. Suppose that $v \equiv w$ for the generated relation. Then we have $v = v_0 \equiv_1 v_1 \equiv_2 \cdots \equiv_n v_n = w$ for some n, where the equivalence relations $\equiv_i$ are chosen from our given relations. Let e_0 be an edge at $v = v_0$ with the other end at v_0'. Then the above property guarantees inductively that we can find an edge e_i at v_i for $i = 1, 2, \ldots, n$ with the other end at v_i' and with $v_{i-1}' \equiv_i v_i'$. Thus, we find an edge e_n at $w = v_n$ whose other end v_n' satisfies $v_0' \equiv v_n'$. □

Proof of Proposition 4.3
We must show that $\Gamma_1 \sim \Gamma_2 \sim \Gamma_3$ implies $\Gamma_1 \sim \Gamma_3$. Now, Γ_1 and Γ_2 weakly cover a common Γ_{12}, and Γ_2 and Γ_3 weakly cover some Γ_{23}. Lemma 4.4 applied to $\Gamma_2, \{\Gamma_{12}, \Gamma_{23}\}$ gives a graph weakly covered by all three of $\Gamma_1, \Gamma_2, \Gamma_3$, so $\Gamma_1 \sim \Gamma_3$.

The minimal element in a bisimilarity class is found by applying Lemma 4.4 to an element Γ and to the set $\{\Gamma_i\}$ of all two-colored graphs that Γ weakly covers. □

PROPOSITION 4.5
If we restrict to two-colored graphs, all of whose vertices have countable valence (so that the graphs are also countable, by our connectivity assumption), then each bisimilarity class contains a tree T*, unique up to isomorphism, that weakly covers every element of the class. It can be constructed as follows: if* Γ *is in the bisimilarity class, duplicate every edge of* Γ *a countable infinity of times, and then take the universal cover of the result (in the topological sense).*

Note that the uniqueness of T in the above proposition depends on the fact that T is a tree; there are many different two-colored graphs that weakly cover every two-colored graph in a given bisimilarity class.

Proof of Proposition 4.5
Given a two-colored graph Γ, we can construct a tree T as follows. Start with one vertex x, labeled by a vertex v of Γ. Then for each vertex w of Γ connected to v by an edge, add infinitely many edges at x leading to vertices labeled w. Then repeat the process at these new vertices and continue inductively. Finally, forget the Γ-labels on the resulting tree and only retain the corresponding $\{\mathbf{b}, \mathbf{w}\}$-labels.

If Γ weakly covers a graph Γ', then using Γ' instead of Γ to construct the above tree T makes no difference to the inductive construction. Thus, T is an invariant for bisimilarity. It clearly weakly covers the original Γ, and since Γ was arbitrary in the bisimilarity class, we see that T weakly covers anything in the class.

To see uniqueness, suppose that T' is another tree that weakly covers every element of the bisimilarity class. Then T' weakly covers the T constructed above from Γ. Composing with $T \to \Gamma$ gives a weak covering $f : T' \to \Gamma$ for which infinitely many edges at any vertex $v \in V(T')$ lie over each edge at the vertex $f(v) \in V(\Gamma)$. It follows that T' itself can be constructed from Γ as in the first paragraph of this proof, so T' is isomorphic to T. □

Using a computer, we have found (in about five months of processor time) the following.

PROPOSITION 4.6
The number of connected minimal two-colored graphs with n vertices, of which exactly b are black (excluding the two 1*-vertex graphs with no edges) is given by Table 1.*

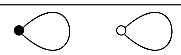
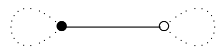
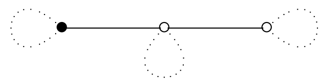
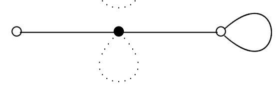
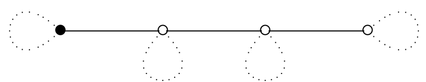
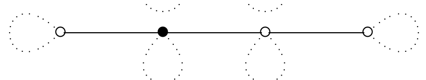
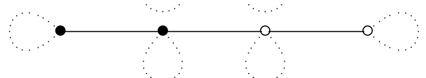
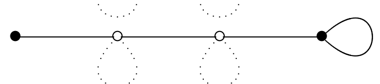
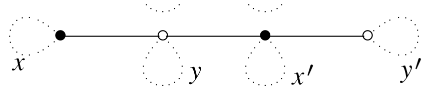
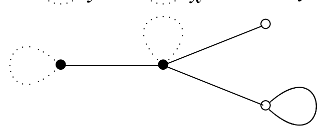
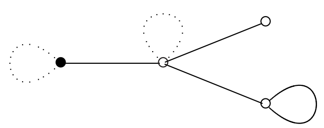
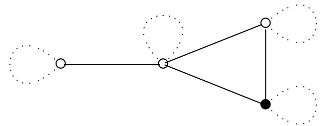
Table 1

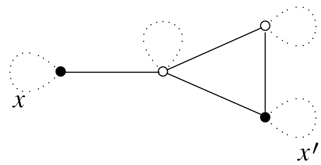
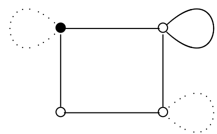
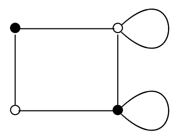
n	$b:0$	1	2	3	4	5	6	7	Total
1	1	1	0	0	0	0	0	0	2
2	0	4	0	0	0	0	0	0	4
3	0	10	10	0	0	0	0	0	20
4	0	56	61	56	0	0	0	0	173
5	0	446	860	860	446	0	0	0	2612
6	0	6140	17084	20452	17084	6140	0	0	66900
7	0	146698	523416	755656	755656	523416	146698	0	2851540
8	0	6007664	25878921	44839104	48162497	44839104	25878921	6007664	201613875

The proposition shows, for example, that there are 199 quasi-isometry classes for nongeometric graph manifolds having four or fewer Seifert pieces ($199 = 2 + 4 + 20 + 173$). In Section 4.1, we list the corresponding 199 graphs. These were found by hand before programming the above count. This gives some confidence that the computer program is correct.

4.1. Enumeration of minimal two-colored graphs up to 4 vertices
We consider only connected graphs, and we omit the two 1-vertex graphs with no edges. In Table 2, "number of graphs $n + n$" means n graphs as drawn and n with **b** and **w** exchanged. Dotted loops in the pictures represent loops that may or may not be present and sometimes carry labels $x, x', \ldots$, referring to the two-element set {present, absent}.

Table 2

Number of vertices		Number of graphs
1		2
2		4
3		$8 + 8$
		$2 + 2$ (total: 20)
4		$16 + 16$
		$16 + 16$
		16
		$4 + 4$
		12 ($x \neq x'$ or $y \neq y'$)
		$4 + 4$
		$4 + 4$
		$16 + 16$

		$8 + 8$ $(x \neq x')$
		$4 + 4$
		1

4.2. Algorithm for finding the minimal two-colored graph

Let Γ be a connected two-colored graph. We wish to construct the minimal two-colored graph Γ_0 for which there is a weak covering $\Gamma \to \Gamma_0$. Note that any coloring $c\colon V(\Gamma) \to C$ of the vertices of Γ induces a graph homomorphism to a graph Γ_c with vertex set C and with an edge connecting the vertices $w_1, w_2 \in C$ if and only if there is some edge connecting a $v_1, v_2 \in V(\Gamma)$, with $c(v_i) = w_i$, $i = 1, 2$.

We start with C containing just our original two colors, which we now call 0, 1, and gradually enlarge C while modifying c until the map $\Gamma \to \Gamma_c$ is a weak covering. For a vertex v, let Adjacent(v) be the set of colors of vertices connected to v by an edge (these may include v itself). We always call our coloring c, even as we modify it:

(1) CurrentColor $= 0$; MaxColor $= 1$;

(2) while CurrentColor $\leq$ MaxColor;

 (a) if there are two vertices v_1, v_2 with $c(v_i) =$ CurrentColor which have different Adjacent(v_i)'s;

 (b) then increment MaxColor and add it to the set C, change the color of each v with $c(v) =$ CurrentColor and Adjacent(v) $=$ Adjacent(v_1) to MaxColor, and then set CurrentColor $= 0$;

 (c) else increment CurrentColor;

 (d) end if;

(3) end while.

We leave it to the reader to verify that this algorithm terminates with $\Gamma \to \Gamma_c$ equal to the weak covering to the minimal two-colored graph. (In step (2b), we can add a new color for each new value of Adjacent(v) with $v \in \{v : c(v) = \text{CurrentColor}\}$ rather than for just one of them; this seems a priori more efficient, but it proved hard to program efficiently.) The algorithm is inspired by Brendan McKay's "nauty" software (see [15]).

Counting the number of minimal two-colored graphs with b black vertices and w white vertices is now easy. We order the vertices $1, \dots, b, \dots, b+w$, and we consider all connected graphs on this vertex set. For each, we check by the above procedure to see if it is minimal; if so, we count it. Finally, we divide our total count by $b!w!$ since each graph has been counted exactly that many times. (A minimal two-colored graph has no automorphisms.)

5. Artin groups

An *Artin group* is a group given by a presentation in the form

$$A = \langle x_1, \dots, x_n \mid (x_i, x_j)_{m_{ij}} = (x_j, x_i)_{m_{ji}} \rangle,$$

where for all $i \neq j$ in $\{1, \dots, n\}$, $m_{ij} = m_{ji} \in \{2, 3, \dots, \infty\}$ with $(x_i, x_j)_{m_{ij}} = x_i x_j x_i \cdots$ (m_{ij} letters) if $m_{ij} < \infty$; when $m_{ij} = \infty$, we do not add a defining relation between x_i and x_j. A concise way to present such a group is as a finite graph labeled by integers greater than 1; such a graph has n vertices, one for each generator, and a pair of vertices are connected by an edge labeled by m_{ij} if $m_{ij} < \infty$.

An important class of Artin groups is the class of *right-angled Artin groups*. These are Artin groups in which each m_{ij} is either 2 or ∞ (i.e., the only defining relations are commutativity relations between pairs of generators). These groups interpolate between the free group on n generators (n vertices and no edges) and $\mathbb{Z}^n$ (the complete graph on n vertices).

We call a presentation tree *big* if it has diameter at least 3 or has diameter 2 and at least one weight on it is greater than 2. An Artin group given by a tree that is not big has infinite center and is virtually (free) $\times\ \mathbb{Z}$. The Artin groups given by presentation trees that are not big thus fall into three quasi-isometry classes ($\mathbb{Z}$, $\mathbb{Z}^2$, $F_2 \times \mathbb{Z}$, where F_2 is the 2-generator free group) and are not quasi-isometric to any Artin group with big presentation trees. (This follows, e.g., from [13].) Therefore, we are only concerned with Artin groups whose presentation trees are big. For right-angled Artin groups, this just indicates that the presentation tree has diameter larger than 2.

We use the term *tree group* to refer to any Artin group whose presentation graph is a big tree. Any right-angled tree group is the fundamental group of a flip graph manifold; this is seen by identifying each diameter 2 region with a (punctured surface) $\times\ \mathbb{S}^1$ and by noting that pairs of such regions are glued together by switching fiber and base directions.

Since any right-angled tree group corresponds to a graph manifold with boundary components in each Seifert piece, Theorem 3.2 yields immediately the following answer to the question posed by Mladen Bestvina [2] about their quasi-isometry classification.

THEOREM 5.1
Any pair of right-angled tree groups is quasi-isometric.

This raises the following natural question.

Question 5.2
When is a finitely generated group G quasi-isometric to a right-angled tree group?

The simple answer is that G must be weakly commensurable with the fundamental group of a nongeometric graph manifold with boundary components in every Seifert component. (This follows from our Theorem 3.2 and from Kapovich and Leeb's quasi-isometric rigidity result for nongeometric 3-manifolds, [13, Theorem 1.2].) But it is natural to ask the question within the class of Artin groups, where this answer is not immediately helpful. We give the following answer, which, in particular, shows that right-angled tree groups are quasi-isometrically rigid in the class of right-angled Artin groups.

THEOREM 5.3
Let G' be any Artin group, and let G be a right-angled tree group. Then G' is quasi-isometric to G if and only if G' has a presentation graph that is a big even-labeled tree with all interior edges labeled 2. *(An* interior edge *is defined as an edge that does not end in a leaf of the tree.)*

We first recall two results relevant to Artin groups given by trees. The first identifies which Artin groups are 3-manifold groups, and the second defines what those 3-manifolds are.

THEOREM 5.4 (Gordon [9, Theorem 1.1])
The following are equivalent for an Artin group A:
(1) *A is virtually a* 3*-manifold group;*
(2) *A is a* 3*-manifold group; and*
(3) *each connected component of its presentation graph is either a tree or a triangle with each edge labeled* 2.

THEOREM 5.5 (see Brunner [4], Hermiller and Meier [12])
The Artin group associated to a weighted tree T is the fundamental group of the complement of the following connected sum of torus links. For each n-weighted edge of T, associate a copy of the $(2, n)$-torus link; if n is even, associate each end of the edge with one of the two components of this link, and if n is odd, associate both ends of the edge with the single component (a $(2, n)$-knot). Now, take the connected sum of all these links, doing the connected sum whenever two edges meet at a vertex, using the associated link components to do the sum.

We note that in Theorem 5.5, the fact that the $(2, n)$-torus knot can be associated with either end of the edge of an odd-weighted edge shows that one can modify the presentation tree without changing the group. This is a geometric version of the "diagram twisting" of [3].

Proof of Theorem 5.3
Let G' be an Artin group that is quasi-isometric to a right-angled tree group. Right-angled tree groups are one-ended; hence G, as well as G', is not freely decomposable. Thus, the presentation graph for G' is connected.

By the quasi-isometric rigidity theorem for 3-manifolds, as stated in the introduction, we know that G' is weakly commensurable to a 3-manifold group.

Unfortunately, it is not yet known if every Artin group is torsion-free. If we knew that G' were torsion-free, then we could argue as follows. First, since G' is torsion-free, it follows that G' is commensurable with a 3-manifold group. Thus, by Theorem 5.4, it *is* a 3-manifold group and is a tree group. By Theorem 3.2, the corresponding graph manifold must have boundary components in every Seifert component. Using Theorem 5.5, it is then easy to see that this gives precisely the class of trees of the theorem. We say more on this in Theorem 5.7.

Since we only know that the quotient of G' by a finite group, rather than G' itself, is commensurable with a 3-manifold group, we cannot use Gordon's result (Theorem 5.4) directly. But we follow its proof.

Gordon rules out the possibility that most Artin groups are fundamental groups of 3-manifolds by proving that they contain finitely generated subgroups that are not finitely presented (i.e., they are not *coherent*). Since Scott [26] proved that 3-manifold groups are coherent, and since coherence is a commensurability invariant, such Artin groups are not 3-manifold groups. Since coherence is also a weak commensurability invariant, this also rules out these Artin groups in our situation.

The remaining Artin groups that Gordon treats with a separate argument are those that include triangles with labels $(2, 3, 5)$ or $(2, 2, m)$. The argument given by Gordon for these cases also applies for weak commensurability. (A simpler argument than Gordon's in the $(2, 2, m)$-case is that A then contains both a $\mathbb{Z}^3$-subgroup and a nonabelian free subgroup, which easily rules out weak commensurability with a 3-manifold group.) □

The above argument leads also to the following generalization of Gordon's theorem.

THEOREM 5.6
An Artin group A is quasi-isometric to a 3-manifold group if and only if it is a 3-manifold group (and is hence as in Theorem 5.4).

Proof

Fix an Artin group A that is quasi-isometric to a 3-manifold group. By Papasoglu and Whyte [20, Theorem 0.4], the reducible case reduces to the irreducible case, so we assume that the Artin group has a connected presentation graph.

The quasi-isometric rigidity theorem for 3-manifolds implies that A is weakly commensurable (or, in some cases, even commensurable) with a 3-manifold group; therefore, as in the previous proof, an easy modification of Gordon's argument applies. □

We can, in fact, more generally describe the quasi-isometry class of any tree group A in terms of Theorem 5.3. That is, we can describe the two-colored decomposition graph for the graph manifold G whose fundamental group is A.

THEOREM 5.7

The colored decomposition graph is obtained from the presentation tree of the Artin group by the following sequence of moves.

(1) *Color all existing vertices black.*

(2) *For each odd-weighted edge, collapse the edge, thus identifying the vertices at its ends, and add a new edge from this vertex to a new leaf that is colored white.*

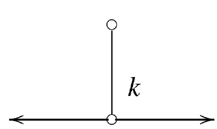
(3) *Remove any* 2*-weighted edge leading to a leaf, along with the leaf; on each* 2*-weighted edge that does not lead to a leaf, simply remove the weight.*

(4) *The only weights now remaining are even weights greater than* 2*. If such a weight is on an edge to a leaf, just remove the weight. If it is on an edge joining two nodes, remove the weight and add a white vertex in the middle of the edge.*

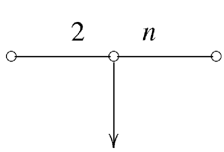
Proof

By Theorem 5.5, our graph manifold G is a link complement. Eisenbud and Neumann [6] classify link complements (in arbitrary homology spheres) in terms of what they call *splice diagrams*. We first recall from [6] how to write down the splice diagram in our special case. The splice diagram for the $(2, n)$-torus link, in which arrowheads correspond to components of the link, is as follows:

$n = 2$



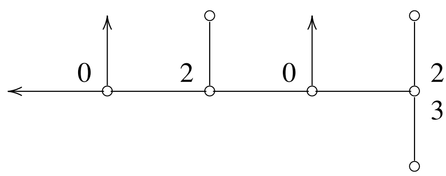
$n = 2k > 2$



$n = 2k + 1 > 2$

(Omitted splice diagram weights are 1.) The splice diagram for a connected sum of two links is obtained by joining the splice diagrams for each link at the arrowheads corresponding to the link components along which the connected sum is performed, changing the merged arrowhead into an ordinary vertex, and adding a new zero-weighted arrow at that vertex. For example, the splice diagram corresponding to the Artin presentation graph

would be



Now the nodes of the splice diagram correspond to Seifert pieces in the geometric decomposition of the graph manifold. Thus, the colored decomposition graph is obtained by taking the full subtree on the nodes of the diagram with nodes that had arrowheads attached colored black and the others colored white. This is as described in the theorem. □

Acknowledgments. We express our appreciation to Mladen Bestvina, Mohamad Hindawi, Tadeusz Januszkiewicz, and Misha Kapovich for useful conversations and to the referees for pertinent comments. We are also grateful to Dylan Thurston for bringing to our attention Brendan McKay's "nauty" software [15], referred to in Section 4.

References

[1] J. BEHRSTOCK, C. DRUŢU, and L. MOSHER, *Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity*, preprint, arXiv:math/0512592v4 [math.GT]

[2] M. BESTVINA, personal communication, Oct. 2005.

[3] N. BRADY, J. P. MCCAMMOND, B. MÜHLHERR, and W. D. NEUMANN, "Rigidity of Coxeter groups and Artin groups" in *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, Israel, 2000)*, Geom. Dedicata **94** (2002), 91 – 109. MR 1950875

[4] A. M. BRUNNER, *Geometric quotients of link groups*, Topology Appl. **48** (1992), 245 – 262. MR 1200426

[5] J. W. CANNON and D. COOPER, *A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three*, Trans. Amer. Math. Soc. **330** (1992), 419 – 431. MR 1036000

[6] D. EISENBUD and W. D. NEUMANN, *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*, Ann. of Math. Stud. **110**, Princeton Univ. Press, Princeton, 1985. MR 0817982

[7] A. ESKIN, D. FISHER, and K. WHYTE, *Quasi-isometries and rigidity of solvable groups*, preprint, arXiv:math/0511647v3 [math.GR]

[8] S. M. GERSTEN, *Divergence in* 3*-manifold groups*, Geom. Funct. Anal. **4** (1994), 633 – 647. MR 1302334

[9] C. MCA. GORDON, *Artin groups,* 3*-manifolds and coherence*, Bol. Soc. Mat. Mexicana (3) **10** (2004), 193 – 198. MR 2199348

[10] M. GROMOV, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53 – 73. MR 0623534

[11] ———, "Asymptotic invariants of infinite groups" in *Geometric Group Theory, Vol. 2 (Sussex, U.K. 1991)* London Math. Soc. Lecture Note Ser. **182**, Cambridge Univ. Press, Cambridge, 1993, 1 – 295. MR 1253544

[12] S. M. HERMILLER and J. MEIER, *Artin groups, rewriting systems and three-manifolds*, J. Pure Appl. Algebra **136** (1999), 141 – 156. MR 1674774

[13] M. KAPOVICH and B. LEEB, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. **128** (1997), 393 – 416. MR 1440310

[14] ———, 3*-manifold groups and nonpositive curvature*, Geom. Funct. Anal. **8** (1998), 841 – 852. MR 1650098

[15] B. D. MCKAY, nauty, http://cs.anu.edu.au/∼bdm/nauty/

[16] J. MILNOR, *A note on curvature and the fundamental group*, J. Differential Geometry **2** (1968), 1 – 7. MR 0232311

[17] J. W. MORGAN and H. BASS, eds., *The Smith Conjecture (New York, 1979)*, Pure Appl. Math. **112**, Academic Press, Orlando, 1984. MR 0758459

[18] W. D. NEUMANN, *Commensurability and virtual fibration for graph manifolds*, Topology **36** (1997), 355 – 378. MR 1415593

[19] W. D. NEUMANN and G. A. SWARUP, *Canonical decompositions of* 3*-manifolds*, Geom. Topol. **1** (1997), 21 – 40. MR 1469066

[20] P. PAPASOGLU and K. WHYTE, *Quasi-isometries between groups with infinitely many ends*, Comment. Math. Helv. **77** (2002), 133 – 144. MR 1898396

[21] G. PERELMAN, *The entropy formula for the Ricci flow and its geometric applications*, preprint, arXiv:math/0211159v1 [math.DG]

[22] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint, arXiv:math/0307245v1 [math.DG]

[23] ———, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math/0303109v1 [math.DG]

[24] E. G. RIEFFEL, *Groups quasi-isometric to* $\mathbf{H}^2 \times \mathbf{R}$, J. London Math. Soc. (2) **64** (2001), 44 – 60. MR 1840770

[25] R. E. SCHWARTZ, *The quasi-isometry classification of rank one lattices*, Inst. Hautes Études Sci. Publ. Math. **82** (1995), 133 – 168. MR 1383215

[26] G. P. SCOTT, *Finitely generated 3-manifold groups are finitely presented*, J. London Math. Soc. (2) **6** (1973), 437 – 440. MR 0380763

[27] K. SHAN, personal communication, March 2006.

[28] A. S. ŠVARC, *A volume invariant of coverings* (in Russian), Dokl. Akad. Nauk. SSSR (N.S.) **105** (1955), 32 – 34. MR 0075634

[29] W. P. THURSTON, *Hyperbolic structures on 3-manifolds, I: Deformation of a cylindrical manifold,* Ann. of Math. (2) **124** (1986), 203 – 246. MR 0855294

Behrstock
Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, USA; jason@math.utah.edu

Neumann
Department of Mathematics, Barnard College, Columbia University, New York, New York 10027, USA; neumann@math.columbia.edu