CENTROIDS AND THE RAPID DECAY PROPERTY IN MAPPING CLASS GROUPS

JASON A. BEHRSTOCK AND YAIR N. MINSKY

ABSTRACT. We study a notion of an equivariant, Lipschitz, permutation-invariant *centroid* for triples of points in mapping class groups $\mathcal{MCG}(S)$, which satisfies a certain polynomial growth bound. A consequence (via work of Druţu-Sapir or Chatterji-Ruane) is the Rapid Decay Property for $\mathcal{MCG}(S)$.

1. Introduction

A finitely generated group has the *Rapid Decay property*[1] if the space of rapidly decreasing functions on G (with respect to every word metric) is inside the reduced C^*–algebra of G (see the end of section 2 for a more detailed definition). Rapid Decay was first introduced for the free group by Haagerup [10]. Jolissaint then formulated this property in its modern form and established it for several classes of groups, including groups of polynomial growth and discrete cocompact subgroups of isometries of hyperbolic space [13]. Jolissaint also showed that many groups, for instance $SL_3(\mathbb{Z})$, fail to have the Rapid Decay property [13]. Rapid Decay was established for Gromov-hyperbolic groups by de la Harpe [8].

Throughout this paper $S = S_{g,p}$ will denote a compact orientable surface with genus g and p punctures. The mapping class group of S, denoted $\mathcal{MCG}(S)$, is the group of isotopy classes of orientation preserving homeomorphisms of S. We will prove:

Theorem 1.1. *$\mathcal{MCG}(S)$ has the Rapid Decay property for every compact orientable surface S.*

The only previously known cases of this theorem were in low complexity when the mapping class group is hyperbolic (these are tori with at most one puncture, or spheres with at most 4 punctures) and for the braid group on four strands, which was recently established by Barré and Pichot [1]. The results in this paper also hold for the braid group on any number of strands. (The case of braid groups follows from the above theorem, since braid groups

2000 *Mathematics Subject Classification.* Primary 20F65; Secondary 20F67, 57M50, 46Lxx, 20F36.

Partially supported by NSF grants DMS-0812513 and DMS-0504019.

[1]This property is also sometimes called *property RD* or the *Haagerup inequality*

are subgroups of mapping class groups of surfaces [12, Theorem 2.7.I] and RD is inherited by subgroups [13, Proposition 2.1.1].)

The Rapid Decay property has several interesting applications. For instance, in order to prove the Novikov Conjecture for hyperbolic groups, Connes-Moscovici [7, Theorem 6.8] showed that if a finitely generated group has the Rapid Decay property and has group cohomology of polynomial growth (property PC), then it satisfies Kasparov's Strong Novikov Conjecture [14]. Accordingly, since any automatic group has property PC [18], Kasparov's Strong Novikov Conjecture follows from the above Theorem 1.1 and Mosher's result that mapping class groups are automatic [19]. The strong Novikov conjecture for $\mathcal{MCG}(S)$ has been previously established by both Hamenstädt [11] and Kida [15].

We prove the Rapid Decay property by appealing to a reduction by Druţu-Sapir (alternatively Chatterji-Ruane) to a geometric condition. Namely, we introduce a notion of *centroids* for unordered triples in the mapping class group which satisfies a certain polynomial growth property. Despite the presence of large quasi-isometrically embedded flat subspaces in the mapping class group, these centroids behave much like centers of triangles in hyperbolic space. Our notion of centroid is provided by the following result which to each unordered triple in the mapping class group gives a Lipschitz assignment of a point, which has the property that it is a centroid in every curve complex projection. We obtain the following:

Theorem 1.2. *For each $S = S_{g,p}$ with $\xi(S) = 3g-3+p \geq 1$ there exists a map $\kappa\colon \mathcal{MCG}(S)^3 \to \mathcal{MCG}(S)$ with the following properties:*

1. *$\kappa(x,y,z)$ is invariant under permutation of the arguments.*
2. *κ is equivariant.*
3. *κ is Lipschitz.*
4. *For any $x, y \in \mathcal{MCG}(S)$ and $r > 0$ we have the following cardinality bound:*
$$\#\left\{\kappa(x,y,z) : d(x,z) \leq r\right\} \leq br^{\xi(S)}$$
where b depends only on S.

These properties, and especially the count provided by part (4), are essentially Druţu and Sapir's condition of *(**)-relative hyperbolicity* with respect to the trivial subgroup [9], and the main theorem of [9] states that this condition implies the Rapid Decay property. Thus to obtain Theorem 1.1 from Theorem 1.2 we appeal to [9], without dealing directly with the Rapid Decay property itself.

Outline of the proof

Let us first recall the situation for a hyperbolic group, G. In this setting, for a triple of points $x, y, z \in G$, one defines a centroid for the triangle with vertices x, y, z to be a point κ with the property that κ is in the δ–neighborhood of any geodesics $[x,y], [y,z], [x,z]$, where δ is the hyperbolicity constant for G considered with some fixed word metric. Thus, if one fixes x and y and allows z to vary in the ball of radius r around x, the corresponding centroid must lie in a δ–neighborhood of the length r initial segment of $[x, y]$. It follows that the number of such centers is linear in r.

When $3g + p - 3 > 1$, then $\mathcal{MCG}(S_{g,p})$ is not hyperbolic. Nonetheless, it has a closely associated space, the complex of curves, $\mathcal{C}(S)$, which is hyperbolic [16]. Moreover, given any subsurface $W \subseteq S$, there is a geometrically defined *projection map*, π_W, from the mapping class group of S to the curve complex of W.

For any $x, y, z \in \mathcal{MCG}(S)$, in Theorem 3.2, we construct a centroid $\kappa(x, y, z)$ with the property that for each $W \subset S$, in the hyperbolic space $\mathcal{C}(W)$ the point $\pi_W(\kappa(x, y, z))$ is a centroid of the triangle with vertices $\pi_W(x)$, $\pi_W(y)$, and $\pi_W(z)$.

Due to the lack of hyperbolicity in $\mathcal{MCG}(S)$, if one were to fix ahead of time a geodesic $[x, y]$, it need not be the case that the center $\kappa(x, y, z)$ is close to $[x, y]$. For this reason, we do not fix a geodesic between x and y, but rather we use the notion of a Σ–hull, as introduced in [3]. The Σ–hull of a finite set is a way of taking the convex hull of these points, in particular, the convex hull of a pair of points is roughly the union of all geodesics between those points.

In analogy to the fact that for any triangle in a Gromov-hyperbolic space any centroid is uniformly close to each of the three geodesics, in Section 4 we show that in $\mathcal{MCG}(S)$, any centroid $\kappa(x, y, z)$ is contained in each Σ–hull between a pair of vertices. This reduces the problem of counting centroids to counting subsets of the Σ–hull, which we also do in this section.

In Section 2, we will review the relevant properties of surfaces, curve complexes, and mapping class groups. In Section 3, we will use properties of curve complexes and Σ–hulls, as developed in [3], to construct the Lipschitz, permutation-invariant centroid map. In Section 4 we will prove the polynomial bound (3), thus completing the proof of Theorem 1.2.

Acknowledgements. The authors would like to thank Indira Chatterji, Cornelia Druţu, and Mark Sapir for useful conversations and for raising our interest in the Rapid Decay property. We would like to thank Ken Shackleton for comments on an earlier draft. Behrstock would also like to thank the Mathematics Departments at Columbia University and Lehman College for their support, as well as his support by an Alfred P. Sloan Fellowship. The authors appreciate the careful readings by the referees.

2. **Background**

We recall first some notation and results that were developed in [16], [17] and [3].

Surfaces and subsurfaces. As above, $S = S_{g,p}$ is an oriented connected surface with genus g and p punctures (or boundary components) and we measure the complexity of this surface by $\xi(S_{g,p}) = 3g - 3 + p$. An essential subsurface $W \subseteq S$ is one whose inclusion is π_1–injective, and which is not peripheral, i.e., not homotopic to the boundary or punctures of S. We also consider disconnected essential subsurfaces, in which each component is essential and no two are isotopic. For such a subsurface X we define another notion of complexity $\xi'(X)$ as follows: $\xi'(X) = \xi(X)$ if X is connected and $\xi(X) \geq 0$, $\xi'(Y) = 1$ if Y is an annulus, and ξ' is additive over components of a disconnected surface. (In [4], $\xi'(S)$ was denoted $r(S)$). It is not hard to check that ξ' is monotonic, i.e., $\xi'(X) \leq \xi'(Y)$ if $X \subseteq Y$ is an essential subsurface. (From now on we implicitly understand subsurfaces to be essential, and defined up to isotopy.)

If W is a subsurface and γ a curve in S we say that *W and γ overlap*, or $W \pitchfork \gamma$, if γ cannot be isotoped outside of W. We say that two surfaces W and V overlap, or $W \pitchfork V$, if neither can be isotoped into the other or into its complement. Equivalently, $W \pitchfork V$ iff $W \pitchfork \partial V$ and $V \pitchfork \partial W$.

See [3, Section 2] for a careful discussion of these and related notions.

Curves and markings. The *curve complex*, $\mathcal{C}(S)$, is a complex whose vertices are essential simple closed curves up to homotopy, and whose k-simplices correspond to $(k+1)$-tuples of disjoint curves. Endow the 1-skeleton $\mathcal{C}_1(S)$ with a path metric giving each edge length 1. With this metric $\mathcal{C}_1(S)$ is a δ–hyperbolic metric space [16]. In our discussion we will often conflate the quasi-isometric spaces $\mathcal{C}_1(S)$ and $\mathcal{C}(S)$

The definition of $\mathcal{C}(W)$ is slightly different for $\xi(W) \leq 1$: If W is a torus with at most one puncture then edges correspond to pairs of curves intersecting once, and if W is a sphere with 4 punctures then edges correspond to pairs of vertices intersecting twice. In all these cases $\mathcal{C}(W)$ is isomorphic to the Farey graph. If W is an annulus embedded in a larger surface S then W admits a natural compactification as an annulus with boundary and $\mathcal{C}(W)$ is the set of homotopy classes of essential arcs in W rel endpoints, with edges corresponding to arcs with disjoint interior. In this case $\mathcal{C}(W)$ admits a quasi-isometry to $\mathbb{Z}$ which takes Dehn twists to translation by 1. See [17] for details.

The *marking graph* $\mathcal{M}(S)$ is a locally finite, connected graph whose vertices are *complete markings* on S and whose edges are *elementary moves.* A complete marking is a system of closed curves consisting of a *base*, which is a maximal simplex in $\mathcal{C}(S)$, together with a choice of transversal curve for each element of the base, satisfying certain minimal intersection properties, see [17, Section 2.5]. For a detailed discussion, and proofs of the properties we will list below, see [17] or [3].

We make $\mathcal{M}(S)$ into a path metric space by again assigning length 1 to edges. We will denote distance in $\mathcal{M}(S)$ as $d_{\mathcal{M}(S)}(\mu, \nu)$, or sometimes just $d(\mu, \nu)$. We will need to use the fact that $\mathcal{MCG}(S)$ isometrically acts properly discontinuously and cocompactly on $\mathcal{M}(S)$, thus any orbit map

$g \mapsto g(\mu_0)$ induces a quasi-isometry from $\mathcal{MCG}(S)$ to $\mathcal{M}(S)$, see [3, Proposition 2.5].

For an annulus $W \subset S$, we identify $\mathcal{M}(W)$ with $\mathbb{Z}$, and map this to $\mathcal{C}(W)$ via the twist-equivariant quasi-isometry mentioned above.

Projections. Given a curve in S that intersects essentially a subsurface W, we can apply a surgery to the intersection to obtain a curve in W. This gives a partially-defined map from $\mathcal{C}(S)$ to $\mathcal{C}(W)$ which we call a *subsurface projection*, and in fact this construction extends to a system of maps of both curve and marking complexes that fit into coarsely commutative diagrams:

$$\begin{tikzcd} \mathcal{M}(S) \arrow[r, "\pi_S"] \arrow[d, "\pi_{\mathcal{M}(W)}"] & \mathcal{C}(S) \arrow[d, "\pi_W"] \\ \mathcal{M}(W) \arrow[r, "\pi_W"] & \mathcal{C}(W) \end{tikzcd} \tag{2.1}$$

Here W is an essential subsurface of S, and we follow the convention of denoting a map π_W if its target is $\mathcal{C}(W)$ and $\pi_{\mathcal{M}(W)}$ if its target is $\mathcal{M}(W)$. The vertical π_W is only partially defined, namely just for those curves in S that intersect W essentially. The horizontal maps take each marking to an (arbitrary) vertex of its base (except in the case an an annulus). By "coarsely commutative" we mean that the diagram commutes up to errors bounded by a constant depending only on the topological type of S. We will also use the fact that $\pi_{\mathcal{M}(W)}$ is coarse-Lipschitz (i.e., Lipschitz up to additive error), with constants depending on the topological type of S. Similarly, π_W is coarse Lipschitz in a restricted sense: if $a, b \in \mathcal{C}(S)$ both intersect W and $d_S(a,b) \le 1$ then $d_W(\pi_W(a), \pi_W(b)) \le 3$. (Again, see [17] and [3] for details.)

Quasidistance formula. In [17, Theorem 6.12] an approximation formula for distances in $\mathcal{M}(S)$ is obtained. To state this, define the *threshold function* $\{\!\{x\}\!\}_A$ to be x if $x \ge A$ and 0 otherwise. We define $x \approx y$ to mean $x \le ay + b$ and $y \le ax + b$, where in the sequel a and b will typically be constants depending only on the topological type of S or on previously chosen constants.

For $\mu, \mu' \in \mathcal{M}(S)$ and $W \subseteq S$, we define the abbreviation

$$d_W(\mu, \mu') = d_{\mathcal{C}_1(W)}(\pi_W(\mu), \pi_W(\mu')).$$

We then have:

Theorem 2.1. (Quasidistance formula). *There exists a constant $A_0 \ge 0$ depending only on the topology of S such that for each $A \ge A_0$, and for any $\mu, \mu' \in \mathcal{M}(S)$ we have*

$$d_{\mathcal{M}(S)}(\mu, \mu') \approx \sum_{Y \subseteq S} \{\!\{d_Y(\mu, \mu')\}\!\}_A$$

and the constants of approximation depend only on A and on the topological type of S.

As a corollary of this, we note:

Corollary 2.2. *For any r there exists t such that for any $\mu,\nu \in \mathcal{M}(S)$, if $d_W(\mu,\nu) \le r$ for all $W \subseteq S$, then $d_{\mathcal{M}(S)}(\mu,\nu) \le t$.*

Product regions. If $W = W_1 \cup \cdots \cup W_k$ is a disconnected surface with components W_i we define $\mathcal{M}(W)$ to be

$$\mathcal{M}(W) = \mathcal{M}(W_1) \times \cdots \times \mathcal{M}(W_k)$$

metrized with the ℓ^1 sum of the metrics on the factors. As a matter of convention we allow $\mathcal{M}(W)$ to refer to a disconnected surface, but only consider $\mathcal{C}(W)$ when W is connected. Such products occur when considering certain regions in $\mathcal{M}(S)$:

If Δ is a curve system in S, let $\mathcal{Q}(\Delta)$ denote the set of markings containing Δ in their base. This set admits a natural product structure, described in [4] and [3]: Let $\sigma(\Delta)$ denote the set of components of $S \setminus \Delta$ that are not 3-holed spheres, together with the annuli whose cores are components of Δ. The following lemma is a consequence of the quasidistance formula, Theorem 2.1:

Lemma 2.3. *Given a curve system Δ, there is a quasi-isometry*

$$\mathcal{Q}(\Delta) \to \prod_{W \in \sigma(\Delta)} \mathcal{M}(W)$$

with constants depending only on the topological type of S, which is given by the product of projection maps, $\prod_{W\in\sigma(\Delta)} \pi_{\mathcal{M}(W)}$.

As a special case, if $U \subset S$ is a (possibly disconnected) surface, let U^c be the surface consisting of all components of $\sigma(\partial U)$ which are not components of U. Note that $\xi'(U) + \xi'(U^c) = \xi'(S) = \xi(S)$. Lemma 2.3 gives a quasi-isometry

$$\mathcal{Q}(\partial U) \to \mathcal{M}(U) \times \mathcal{M}(U^c).$$

Projection bounds. The projections π_W satisfy a number of useful inequalities. One, from [2, Theorem 4.3], is:

Lemma 2.4. *There exists a universal constant m_0 such that for any marking $\mu \in \mathcal{M}(S)$ and subsurfaces $V \pitchfork W$,*

$$\min\left(d_W(\mu,\partial V), d_V(\mu,\partial W)\right) < m_0.$$

The *geodesic projection lemma* [17] states:

Lemma 2.5. *Let Y be a connected essential subsurface of S satisfying $\xi(Y) \ne 3$ and let g be a geodesic segment in $\mathcal{C}(S)$ for which $Y \pitchfork v$ for every vertex v of g. Then*

$$\mathrm{diam}_Y(g) \le B,$$

where B is a constant depending only on $\xi(S)$.

The following generalization of the geodesic projection lemma is proven in [3, Lemma 5.5]:

Lemma 2.6. *Let $V, W \subseteq S$ be essential subsurfaces such that $W \pitchfork \partial V$. Let g be a geodesic in $\mathcal{C}(W)$. If*

$$d_W(g, \partial V) > m_1$$

then

$$\mathrm{diam}_V(g) \leq m_2.$$

Where the constants m_1, m_2 depend only on S.

Partial orders. The inequalities of Lemmas 2.4 and 2.5 can be interpreted as describing a family of partial orders for connected subsurfaces that are "between" pairs of markings in $\mathcal{M}(S)$. Given $x, y \in \mathcal{M}(S)$, and a constant $c > 0$, define for a natural number k

$$\mathcal{F}_k(x, y) = \{U \subsetneq S : d_U(x, y) > kc\}.$$

Note that this is (for appropriate threshold) the set of proper (connected) subsurfaces participating in the quasidistance formula for $d(x, y)$; in particular it is finite. Define also a family of relations $\prec_k$ on proper connected subsurfaces of S, by saying that $V \prec_k W$ if and only if $V \pitchfork W$, and

$$d_V(x, \partial W) > kc.$$

Note that $\prec_k$ depends on x, not y, so we assume throughout an *ordered* pair (x, y). In [3, Lemma 4.5] we show that

Lemma 2.7. *There exists c_0 such that, if $c > c_0$ in the above definitions, then for $k > 2$ the relation $\prec_{k-1}$ is a partial order on $\mathcal{F}_k(x, y)$ for any $x, y \in \mathcal{M}(S)$. Moreover if $V, W \in \mathcal{F}_k(x, y)$ and $V \pitchfork W$ then V and W are $\prec_{k-1}$–ordered.*

Moreover, it will be useful to see that the relation $\prec_{k-1}$ can be characterized in a few ways:

Lemma 2.8. *Let $V, W \in \mathcal{F}_k(x, y)$, $W \pitchfork V$, and take c to be any sufficently large real number. The following are equivalent:*

1. *$W \prec_{k-1} V$*
2. *$d_W(x, \partial V) > (k-1)c$,*
3. *$d_W(y, \partial V) \leq c$,*
4. *$d_V(x, \partial W) \leq c$,*
5. *$d_V(y, \partial W) > (k-1)c$.*

A related fact is that if $V \pitchfork \partial W$, $d_V(x, \partial W) > (k+1)c$ and $W \in \mathcal{F}_2(x, y)$, then $V \in \mathcal{F}_k(x, y)$.

These partial orders are closely related to the "time-order" that appears in [17]. In [3] these facts are established using just the projection inequalities, as an extended exercise in the triangle inequality.

Consistency Theorem. Consider the combined projection map

$$\Pi : \mathcal{M}(S) \to \prod_{W \subseteq S} \mathcal{C}(W),$$

$\Pi(\mu) = (\pi_W(\mu))_W$, where W varies over essential subsurfaces of S and π_W denotes the subsurface projection map $\mathcal{M}(S) \to \mathcal{C}(W)$. We say that an element $x = (x_W) \in \prod_W \mathcal{C}(W)$ is *D–close to the image of* Π if there exists $\mu \in \mathcal{M}(S)$ such that $d_W(x_W, \mu) < D$ for all $W \subseteq S$. The following *Consistency Theorem*, from [3, Theorem 4.3], gives a coarse characterization of the image of Π.

Theorem 2.9. (Consistency Theorem). *Given $c_1, c_2 > 0$ there exists D such that any point $(x_W)_W \in \prod_W \mathcal{C}(W)$ satisfying the following two conditions is D–close to the image of* Π.

C1: *For any $U \subset V \subseteq S$, if $d_V(\partial U, x_V) \geq c_1$ then*

$$d_U(x_U, x_V) < c_2.$$

C2: *For any $U, V \subset S$ with $U \pitchfork V$,*

$$\min\left(d_U(x_U, \partial V), d_V(x_V, \partial U)\right) < c_2.$$

Conversely given D there exist c_1, c_2 so that if (x_W) is D–close to the image of Π *then it satisfies conditions C1-2.*

Note that the converse direction of the theorem includes Lemma 2.4 as condition C2 in the case $D = 0$.

Σ–hulls. If $x, y \in \mathcal{M}(S)$ and W is a connected subsurface, let $[x, y]_W$ be a geodesic in $\mathcal{C}(S)$ connecting $\pi_W(x)$ to $\pi_W(y)$ (this may not be unique but we can make an arbitrary choice — all of them are δ–close to each other by hyperbolicity). For a finite set $A \subset \mathcal{M}(S)$, define $\mathrm{hull}_W(A)$ to be the union of $[a, b]_W$ over $a, b \in A$. For any fixed $\epsilon > 0$, we define a Σ–hull of A to be a set of the following form:

$$\Sigma_\epsilon(A) = \{\mu \in \mathcal{M}(S) : \forall W \subseteq S,\ d_W(\mu, \mathrm{hull}_W(A)) \leq \epsilon\}.$$

If A is a pair $\{x, y\}$ we also write $\Sigma_\epsilon(A) = \Sigma_\epsilon(x, y)$. In [3, Lemma 5.4] we study these sets, and in particular prove the following:

Lemma 2.10. *Given ϵ and n there exists b such that*

$$\mathrm{diam}(\Sigma_\epsilon(A)) \leq b(\mathrm{diam}(A) + 1)$$

for any $A \subset \mathcal{M}(S)$ of cardinality n.

Tight geodesics, footprints and hierarchies. A geodesic in $\mathcal{C}(S)$ is a sequence of vertices $\{v_i\}$ such that $d(v_i, v_j) = |j-i|$. In [17] this is generalized a bit to sequences of *simplices*, i.e., disjoint curve systems $\{w_i\}$, such that $d(v_i, v_j) = |j - i|$ for any $v_i \in w_i$, $v_j \in w_j$, and $i \neq j$. For a (generalized) geodesic $g = \{w_i\}$ in $\mathcal{C}(V)$, and any subsurface $U \subset V$, we define the *footprint* $\phi_g(U)$ to be the set of simplices w_i disjoint from U. By the triangle inequality $\mathrm{diam}_V(\phi_g(U)) \leq 2$. A condition called *tightness* is formulated in

[17] which has the following property: for a tight geodesic, all nonempty footprints are *contiguous* intervals of one, two, or three simplices (leaving out the possibility of two simplices at distance 2, with their midpoint not included). This is the basic definition that leads to the notion of a *hierarchy of tight geodesics* between any two $x, y \in \mathcal{M}(S)$. A hierarchy, H, consists of a particular collection of tight geodesics k, each in $\mathcal{C}(W)$ for a subsurface $W \subseteq S$ known as the support of k. We will only need a few basic facts about hierarchies, to be used in the proof of Lemma 4.5; these facts are all from [17].

Lemma 2.11. *If $x, y \in \mathcal{M}(S)$ and $H = H(x, y)$ is a hierarchy of tight geodesics, then*

1. *H contains a tight geodesic $[x, y]_S$ with support S and endpoints $\pi_S(x)$ and $\pi_S(y)$.*
2. *If h is a tight geodesic in H and U is its support, then the endpoints of h are within uniform distance $m_3 = m_3(S)$ in $\mathcal{C}(U)$ from $\pi_U(x)$ and $\pi_U(y)$.*
3. *If h is a tight geodesic in H and $U \subsetneq S$ is its support, then there exists k in H with support W, and a simplex w in $\phi_k(U)$, such that U is either a component of $W \setminus w$, or an annulus whose core is a component of w.*
4. *A subsurface W can be the support of at most one geodesic in H, which we denote $h_{x,y,W}$.*
5. *For a uniform $m_4 = m_4(S)$, all connected subsurfaces W satisfying $d_W(x, y) > m_4$ are domains of geodesics in H.*

Rapid Decay. Although in the text we do not work directly with the rapid decay property, for the benefit of the reader who (like the authors) is not an analyst, we briefly discuss the formulation of the rapid decay property and related notions. For further details see [20] or [6].

Given a finitely generated group G, we consider its action by left-translation on $l^2(G)$, the square-summable $\mathbb{C}$-valued functions on G. This action extends by linearity to an action of the group algebra $\mathbb{C}G$, and indeed $\mathbb{C}G$ is just the subset of $l^2(G)$ consisting of functions with finite support, and the action is nothing more than convolution, i.e., $f * g(z) = \sum_{x \in G} f(x) g(x^{-1} z)$.

This gives us an embedding of $\mathbb{C}G$ into the bounded operators on $l^2(G)$, indeed for $f \in \mathbb{C}G$ and $h \in l^2(G)$ we have $||f * h||_2 \leq ||f||_1 ||h||_2$ by Young's inequality. The reduced C^*-algebra of G, denoted $C^*_r(G)$, is the closure of $\mathbb{C}G$ in the operator norm.

On the other hand, $\mathbb{C}G$ embeds in the normed spaces $H^s(G) = \{h : ||h||_{2,s} < \infty\}$, where

$$||h||_{2,s} = (\sum_{x \in G} ((1 + |x|)^s h(x))^2)^{1/2}$$

for $s > 0$, and $| \cdot |$ denotes word length in G. The intersection $H^\infty(G) = \cap_s H^s(G)$ is the space of *rapidly decreasing functions* on G. (Note that H^s

and H^∞ are invariant, up to bounded change of norm, under change of generators).

Recall that, in the abelian setting (e.g., $G = \mathbb{Z}^n$), functions of rapid decrease in G Fourier-transform to smooth functions on the Pontryagin dual $\widehat{G}$ (e.g., $\widehat{\mathbb{Z}^n} = T^n$). In the nonabelian setting, there is no Pontryagin dual so H^∞ acts as a substitute for the algebra of smooth functions (see Connes-Moscovici [7]).

We say that G has the *rapid decay property* if the embedding of $\mathbb{C}G$ into $C_r^*(G)$ extends continuously to an embedding of $H^\infty(G)$.

This condition boils down (see [5, 9]) to a polynomial convolution norm bound of the following form: there exists a polynomial $P(s)$ such that, if f is supported in a ball of radius s in G, then

$$||f * g||_2 \leq P(s)||f||_2||g||_2. \tag{2.2}$$

Here one can start to see at least the relevance of the centroid condition and its bound. Indeed, in the sum $f * g(z) = \sum_x f(x)g(x^{-1}z)$, x can be restricted to the ball of radius s. Now we can rearrange this as a sum over the centroids $t = \kappa(1, x, z)$, and the number of such t is polynomial in s by Theorem 1.2. This observation plays a role in the proofs of (2.2) in both Druţu-Sapir [9] and Chatterji-Ruane [5].

3. Centroids

In a δ–hyperbolic metric space X, define a ρ*–centroid* of a triple of points $A = \{a_1, a_2, a_3\}$ to be a point x which is within ρ of each of the geodesics $[a_i, a_j]$ (if geodesics are not unique make an arbitrary choice). Hyperbolicity implies that ρ–centroids always exist for a uniform ρ (depending on δ), and indeed this condition is equivalent to hyperbolicity. The next lemma states a few more facts that we need; the proof, which is an exercise, is left out.

Lemma 3.1. *Let X be a δ–hyperbolic geodesic metric space. There exist $\delta_0, L > 0$ and a function D, depending only on δ, such that*

1. *Every triple has a ρ–centroid if $\rho \geq \delta_0$.*
2. *The diameter of the set of ρ–centroids of any triple is at most $D(\rho)$.*
3. *The map taking a triple to the set of its ρ–centroids is L–coarse-Lipschitz in the Hausdorff metric.*

In this section we will utilize this idea to give a "centroid" map for $\mathcal{M}(S)$ satisfying the first three properties of Theorem 1.2. Using the quasi-isometry between $\mathcal{MCG}(S)$ and $\mathcal{M}(S)$, the same construction can be carried out in $\mathcal{MCG}(S)$; we are abusing notation slightly by using the same notation, κ, for both centroid maps.

Theorem 3.2. *There exists $\epsilon, \rho > 0$ and a map $\kappa\colon \mathcal{M}(S)^3 \to \mathcal{M}(S)$ with the following properties:*

1. *$\kappa(a, b, c)$ is invariant under permutation of the arguments.*
2. *$\kappa(ga, gb, gc) = g\kappa(a, b, c)$ for any $g \in \mathcal{MCG}(S)$.*

(3) *κ is Lipschitz.*
(4) *$\kappa(a,b,c) \in \Sigma_\epsilon(a,b,c)$.*
(5) *For each $W \subseteq S$, $\pi_W(\kappa(a,b,c))$ is a ρ–centroid of the triangle, in $\mathcal{C}(W)$, with vertices $\pi_W(a)$, $\pi_W(b)$, and $\pi_W(c)$.*

Proof. Let δ be a hyperbolicity constant for $\mathcal{C}(W)$ for all $W \subseteq S$, let δ_0 be the constant given in Lemma 3.1, and fix $\epsilon > \delta_0$. For each $\mathcal{MCG}$–orbit of triple (a,b,c) choose a representative $A = \{a_1, a_2, a_3\} \subset \mathcal{M}(S)$. For any $W \subseteq S$, the triangle $\cup_{i,j}[\pi_W(a_i), \pi_W(a_j)]$ (which is coarsely equal to $\mathrm{hull}_W(A)$) has an ϵ–centroid. Choose such an ϵ–centroid and call it x_W. We will show that $(x_W) \in \prod_W \mathcal{C}(W)$ satisfies conditions C1-2 of the Consistency Theorem, for suitable c_1, c_2.

Consider $V, W \subset S$ satisfying $\partial V \pitchfork W$. Suppose that $d_W(x_W, \partial V) > D(\max(m_0, \epsilon))$, where $D(\cdot)$ is the function given by Lemma 3.1. Then by part (2) of Lemma 3.1, $\pi_W(\partial V)$ is not an m_0–centroid for $\pi_W(A)$, and so for at least one leg g of $\mathrm{hull}_W(A)$, $d_W(\partial V, g) > m_0$. Suppose without loss of generality that $g = [\pi_W(a_1), \pi_W(a_2)]$.

By Lemma 2.6, this gives us an upper bound $\mathrm{diam}_V(g) \leq m_2$. In other words $\pi_V(a_1)$ and $\pi_V(a_2)$ are close together and hence the centroid x_V is close to both.

Since x_W is within ϵ of g in $\mathcal{C}(W)$, and $d_W(\partial V, g) > \epsilon$, we may connect x_W to g by a path of length at most ϵ consisting of curves that all intersect V, and so the Lipschitz property of π_V gives an upper bound on $d_V(x_W, g)$. From this and the previous paragraph, we conclude that there is a bound of the form

$$d_V(x_V, x_W) < m_3$$

for suitable uniform m_3.

When $V \subset W$, this establishes C1.

When $V \not\subset W$, i.e., $V \pitchfork W$, since we have assumed $d_W(x_W, \partial V)$ is large, the direction of the Consistency Theorem given by Lemma 2.4 yields a bound on $d_V(x_W, \partial W)$. Since we already have a bound on $d_V(x_V, x_W)$ by the above, this in turn bounds $d_V(x_V, \partial W)$, and thus establishes C2.

Having established C1 and C2, we can apply the Consistency Theorem to conclude that there exists $\mu \in \mathcal{M}(S)$ with $d_W(\mu, x_W)$ uniformly bounded for all W. Let this μ be $\kappa(A)$. (Note that, by the quasidistance formula, μ is determined up to bounded error.) For any triple $A' = (a,b,c)$ satisfying $gA = A'$ define $\kappa(A') = g\kappa(A)$.

By construction, κ satisfies conditions (2) and (5) of the theorem.

Condition (1) is evident since the construction depended on the unordered set A. Condition (3), the Lipschitz property, follows immediately from the quasidistance formula and the coarse-Lipschitz property of hyperbolic centroids (part (3) of Lemma 3.1); note that in this case we obtain a Lipschitz map and not just a coarse-Lipschitz one, since there is a lower bound on the distance in $\mathcal{M}(S)$ between pairs of distinct points. Condition (4) follows from (5) and the definition of Σ_ϵ. $\square$

4. **Polynomial bounds**

It remains to prove that the map κ of Theorem 3.2 satisfies the polynomial bound of part (4) of Theorem 1.2. That is, letting

$$K(x,y,r) = \big\{\kappa(x,y,z) : z \in \mathcal{N}_r(x)\big\}$$

(where $\mathcal{N}_r(Y)$ denotes a neighborhood of $Y \subset \mathcal{M}(S)$ of radius r) we need to find a polynomial in r, independent of x and y, which bounds $\#K(x,y,r)$. Throughout this section, r will denote a fixed positive real number and the points x, z will satisfy $z \in \mathcal{N}_r(x)$.

4.1. **Reduction to Σ–hulls**

We first reduce the problem to that of counting the number of elements in a suitable Σ–hull.

If $\mu = \kappa(x,y,z)$, then by definition of κ and of hyperbolic centroids, for each $W \subseteq S$ the set $\pi_W(\mu)$ is in a uniformly bounded neighborhood of $[x,y]_W$. It follows that

$$K(x,y,r) \subset \Sigma_{\epsilon'}(x,y)$$

for suitable ϵ' (depending on ϵ and the hyperbolicity constant). Moreover, $\mu \in \Sigma_{\epsilon'}(x,z)$ by the same argument, so $d(x,\mu) \le \operatorname{diam}(\Sigma_{\epsilon'}(x,z)) \le b(d(x,z)+1)$ (by Lemma 2.10) and thus $\mu \in \mathcal{N}_{b(r+1)}(x)$. Hence (changing variable names) it suffices to give a polynomial bound in r, depending on ϵ but not on (x,y), on the cardinality of

$$A(x,y,r) = \Sigma_\epsilon(x,y) \cap \mathcal{N}_r(x).$$

The following lemma indicates that $A(x,y,r)$ is contained in a Σ–hull that gives a good estimate on its size:

Lemma 4.1. *Given ϵ there exists ϵ' such that, for each $x, y \in \mathcal{M}(S)$ and each $r > 1$, there exists $q \in \Sigma_{\epsilon'}(x,y)$ for which*

$$\Sigma_\epsilon(x,y) \cap \mathcal{N}_r(x) \subseteq \Sigma_{\epsilon'}(x,q)$$

and satisfying $d(x,q) < ar$, with b depending only on ϵ and $\xi(S)$.

Proof. For each $W \subseteq S$, by the definition of Σ–hulls, we have that $\pi_W(A(x,y,r))$ is contained in the ϵ–neighborhood of a $\mathcal{C}(W)$–geodesic $[x,y]_W$ between a point of $\pi_W(x)$ and a point of $\pi_W(y)$. Let m_W denote the vertex of $[x,y]_W$ which is within ϵ of $\pi_W(A(x,y,r))$ and is farthest from $\pi_W(x)$.

Finding a marking using consistency. We now claim that the tuple, $(m_W)_{W\subseteq S}$, satisfies the consistency conditions of Theorem 2.9 and thus gives rise to a marking, which we will call q.

For convenience we note that we can simultaneously establish both C1 and C2 by showing the following: there exists a uniform constant such that for any $U, V \subseteq S$ such that $\partial U \pitchfork V$, either $d_V(m_V, \partial U)$ or $d_U(m_U, \partial V \cup m_V)$ is bounded by this constant. Here by $\partial V \cup m_V$ we mean the union as curve systems, or equivalently the join as simplices in $\mathcal{C}(S)$.

Thus, let U and V be such that $\partial U \pitchfork V$ and

$$d_V(m_V, \partial U) > 2(\epsilon + 2). \tag{4.1}$$

Let $\mu \in A(x, y, r)$ be such that $\pi_V(\mu)$ is within ϵ of m_V, and let $\nu \in A(x, y, r)$ be such that $\pi_U(\nu)$ is within ϵ of m_U. Since $d_V(\partial U, m_V) > 2\epsilon + 2$, there is a $\mathcal{C}(V)$–path from $\pi_V(\mu)$ to m_V consisting of curves that intersect U, so by the Lipschitz property of π_U we have a bound

$$d_U(m_V, \mu) < b_1$$

for some uniform b_1. In fact, since m_V and ∂V are disjoint we may instead write

$$d_U(m_V \cup \partial V, \mu) < b_1. \tag{4.2}$$

Since m_V lies on $[x, y]_V$, the bound (4.1) also implies, by the triangle inequality, that $\pi_V(\partial U)$ cannot be within $\epsilon + 2$ of both $[x, m_V]_V$ and $[m_V, y]_V$. This gives us two cases which we treat separately.

Case a: Suppose $\pi_V(\partial U)$ is more than $\epsilon + 2$ from $[x, m_V]_V$.

Let $\sigma \in A(x, y, r)$, and let $t \in [x, y]_V$ be within ϵ of $\pi_V(\sigma)$. By definition of m_V, t must be in $[x, m_V]$. Hence, again by the Lipschitz property of π_U, we have

$$d_U(\sigma, t) \leq b_1.$$

The bounded geodesic projection lemma implies that

$$d_U(x, t) \leq B,$$

so we have a bound on $d_U(x, \sigma)$. Applying this to ν, which was chosen to satisfy $d_U(\nu, m_U) < \epsilon$, we get

$$d_U(x, m_U) \leq b_2$$

for suitable b_2. By construction of m_U, the above bound implies $d_U(x, \mu) \leq b_2$ as well. Hence, by the triangle inequality, for suitable b_3 we have:

$$d_U(m_U, \mu) \leq b_3.$$

Hence by (4.2) we conclude there exists a uniform constant, b_4, satisfying:

$$d_U(m_U, m_V \cup \partial V) \leq b_4$$

which is what we wanted to show.

Case b: Suppose $\pi_V(\partial U)$ is more than $\epsilon + 2$ from $[m_V, y]_V$.

Then, by the geodesic projection lemma,

$$d_U(m_V, y) \leq B,$$

and applying (4.2) again we have

$$d_U(\mu, y) \leq b_5.$$

Let $t \in [x, y]_U$ be within ϵ of $\pi_U(\mu)$ — then we have

$$d_U(t, y) \leq b_5 + \epsilon.$$

By definition, m_U is in $[t,y]_U$, so by the triangle inequality

$$d_U(m_U, m_V \cup \partial V) \leq b_6,$$

and again we are done.

Having established C1-2 for (m_W), the Consistency Theorem gives us $q \in \mathcal{M}(S)$ such that

$$d_W(q, m_W) < b_7 \tag{4.3}$$

for a uniform b_7. By definition of m_W, we have that $\pi_W(A(x,y,r))$ is within ϵ of $[x, m_W]_W$ for each W, and hence by hyperbolicity of $\mathcal{C}(W)$ this set is within a suitable ϵ' of $[x,q]_W$. In other words,

$$A(x,y,r) \subset \Sigma_{\epsilon'}(x,q).$$

Bounding $d(x,q)$. It remains to check that $d_{\mathcal{M}(S)}(x,q) < br$, for a uniform b.

Fix $c > \max\{A_0, m_0 + b_7 + \epsilon + 3\}$, where A_0, m_0, and b_7 are the constants of Theorem 2.1, Lemma 2.4, and Equation (4.3). Recall from Section 2 the set of subsurfaces $\mathcal{F}_3(x,q)$, with its partial ordering $\prec_2$, where $U \in \mathcal{F}_3(x,q)$ iff $d_U(x,q) > 3c$. Using $3c$ as a threshold in the quasidistance formula, we have

$$d(x,q) \approx \sum_{W \in \mathcal{F}_3(x,q)} d_W(x,q),$$

which we will use to get an upper bound on $d(x,q)$. Let $\mathcal{U}$ be the set of maximal elements of $\mathcal{F}_3(x,q)$ with respect to the partial order $\prec_2$. Since overlapping subsurfaces are $\prec_2$–ordered (Lemma 2.7), the elements U_i of $\mathcal{U}$ are either disjoint or nested, which means there are at most $2\xi(S)$ of them (the bound of $2\xi(S)$ is an easy exercise, but all that matters is that there is some universal constant).

Let $u_i \in A(x,y,r)$ be a marking such that $d_{U_i}(u_i, m_{U_i}) < \epsilon$ (this exists by definition of m_{U_i}). Moreover, by definition of q we know that $d_{U_i}(q, m_{U_i})$ is uniformly bounded by b_7, hence we have

$$d_{U_i}(q, u_i) < b_7 + \epsilon. \tag{4.4}$$

We claim that for each $W \in \mathcal{F}_3(x,q)$, there exists at least one of the u_i which satisfies

$$d_W(q, u_i) < b \tag{4.5}$$

for some uniform constant b. For W an element of $\mathcal{U}$ we have just established this. For any other W, we must have $W \prec_2 U_i$ for some $U_i \in \mathcal{U}$, so $W \pitchfork U_i$ and so we have (from Lemma 2.8) the inequalities

$$d_W(q, \partial U_i) < c \tag{4.6}$$

and

$$d_{U_i}(x, \partial W) < c. \tag{4.7}$$

Now from (4.4) and the fact that $d_{U_i}(x, q) > 3c$, the triangle inequality yields $d_{U_i}(x, u_i) > 3c - 3 - b_7 - \epsilon > 2c$, where the 3 is being subtracted because $\operatorname{diam}_{U_i}(u_i) \le 3$. Together with (4.7) we then have

$$d_{U_i}(\partial W, u_i) > c.$$

Hence by Lemma 2.4,

$$d_W(\partial U_i, u_i) < c_2.$$

Now with (4.6), we get an inequality of the form

$$d_W(q, u_i) < b$$

for a uniform b. This establishes (4.5), which proves the claim.

Now this means that

$$d_W(x, q) \le d_W(x, u_i) + b.$$

In the quasidistance formula (Theorem 2.1), we may choose any sufficiently large threshold at a cost of changing the approximation constants. Hence we may choose a threshold greater than $2b$, and then for every term $d_W(x, q)$ larger than the threshold we have

$$\frac{1}{2} d_W(x, q) \le d_W(x, u_i).$$

It then follows that every term in the quasidistance formula for $d(x, q)$ appears, with a multiplicative error, in the quasidistance formula for one of the $d(x, u_i)$. This means

$$d(x, q) \le \sum_i a' d(x, u_i) + b' \le 2\xi(S)(a'r + b')$$

where a', b' come from Theorem 2.1. A bound of the form $d(x, q) < ar$ follows since $r > 1$. □

4.2. **Polynomial bound on Σ–hulls**

Now that our set $K(x, y, r)$ is known to be contained in a Σ–hull of comparable diameter, it suffices to obtain an appropriate polynomial bound on the Σ–hull of two points.

Theorem 4.2. *Given $\epsilon > 0$, there is a constant $c = c(\epsilon, S)$, such that, for any $x, y \in \mathcal{M}(S)$,*

$$\#\Sigma_\epsilon(x, y) \le c d(x, y)^{\xi(S)}.$$

Note by Lemma 2.10 that $d(x, y)$ is interchangeable, up to bounded factor, with $\operatorname{diam}(\Sigma_\epsilon(x, y))$, and we will freely make use of that.

Proof. The idea of the proof is to (coarsely) cover $\Sigma_\epsilon(x, y)$ by sets for which the desired inequality holds by induction, and the sum of whose diameters is bounded by the diameter of $\Sigma_\epsilon(x, y)$.

Given x and y, we will construct a finite collection $\Gamma = \Gamma_{x,y}$ of finite subsets of $\mathcal{M}(S)$, and a map $\gamma \colon \Sigma_\epsilon(x, y) \to \Gamma$. The proof will follow from three lemmas about this construction. The first one states that the bound of the theorem holds for the sets in Γ:

Lemma 4.3. (Inductive Bound Lemma). *For each $\mathcal{G} \in \Gamma_{x,y}$,*

$$\#\mathcal{G} \leq b_1 \operatorname{diam}(\mathcal{G})^{\xi(S)}.$$

where b_1 depends only on S.

The second lemma implies that the collection Γ coarsely covers $\Sigma_\epsilon(x,y)$:

Lemma 4.4. (Covering Lemma). *For each $\mu \in \Sigma_\epsilon(x,y)$,*

$$\mu \in \mathcal{N}_{b_2}(\gamma(\mu)).$$

where b_2 depends only on S.

Finally, we will bound the sum of the diameters of the sets in Γ:

Lemma 4.5. (Diameter sum bound). *For a constant b_3 depending only on S,*

$$\sum_{\mathcal{G}\in\Gamma_{x,y}} \operatorname{diam}(\mathcal{G}) \leq b_3 \operatorname{diam}(\Sigma_\epsilon(x,y)) \tag{4.8}$$

The proof of Theorem 4.2 is then an inductive argument. First we note that the statement of the theorem can be made not just for S but for any connected subsurface W of S, and that in order to correctly account for annuli we should write the inequality as

$$\#\Sigma_{\epsilon,W}(x,y) \leq c(\operatorname{diam}(\Sigma_{\epsilon,W}(x,y)))^{\xi'(W)} \tag{4.9}$$

where $x, y \in \mathcal{M}(W)$ and $\Sigma_{\epsilon,W}$ denotes the Σ-hull within $\mathcal{M}(W)$. Recall that $\xi'(W) = \xi(W)$ for connected non-annular W, as in Section 2. For an annulus W, $\xi'(W) = 1$, $\mathcal{M}(W)$ is $\mathbb{Z}$, and $\Sigma_{\epsilon,W}(x,y)$ is just the interval between x and y. So this establishes the base case of (4.9).

We will establish the three lemmas for complexity $\xi(S)$, where Lemma 4.3 in particular will rely on assuming (4.9) for all smaller complexities. Once that is done, for uniform constants c_i we have

$$\begin{aligned}
\#\Sigma_\epsilon(x,y) &\leq c_1 \sum_{\mathcal{G}\in\Gamma} \#\mathcal{G} \\
&\leq c_2 \sum_{\mathcal{G}\in\Gamma} \operatorname{diam}(\mathcal{G})^{\xi(S)} \\
&\leq c_2 \left(\sum_{\mathcal{G}\in\Gamma} \operatorname{diam}(\mathcal{G})\right)^{\xi(S)} \\
&\leq c_3 \operatorname{diam}(\Sigma_\epsilon(x,y))^{\xi(S)}.
\end{aligned}$$

where the first line follows from Lemma 4.4, together with a bound on the cardinality of b_2–balls in $\mathcal{M}(S)$; the second line follows from Lemma 4.3; the third from arithmetic; and the last from Lemma 4.5.

This gives the proof of Theorem 4.2, modulo the construction of Γ and proofs of the three lemmas.

Construction of the cover. Let U be a (possibly disconnected) nonempty essential subsurface, with its components denoted $U_1, \dots, U_k$. Recall from Section 2 that $\mathcal{M}(U) = \prod_i \mathcal{M}(U_i)$, and define

$$\Sigma_{\epsilon,U}(x,y) = \prod_i \Sigma_{\epsilon,U_i}(\pi_{\mathcal{M}(U_i)}(x), \pi_{\mathcal{M}(U_i)}(y)).$$

Now recalling from Lemma 2.3 that $\mathcal{Q}(\partial U)$ is identified quasi-isometrically with $\mathcal{M}(U) \times \mathcal{M}(U^c)$, define $\mathcal{G}(U,x,y) \subset \mathcal{Q}(\partial U)$ to be the set:

$$\mathcal{G}(U,x,y) = \Sigma_{\epsilon,U}(x,y) \times \{\pi_{\mathcal{M}(U^c)}(y)\}.$$

We also need a degenerate form of this: if p is a simplex in $\mathcal{C}(S)$, Lemma 2.3 tells us that $\mathcal{Q}(p)$ is quasi-isometrically identified with $\prod_{W \in \sigma(p)} \mathcal{M}(W)$, where $\sigma(p)$ is the decomposition associated to p. Let y_p denote the point corresponding to the tuple $(\pi_{\mathcal{M}(W)}(y))_{W \in \sigma(p)}$, and let y'_p be an additional point at distance 1 from y_p. We define

$$\mathcal{G}(p,y) = \{y_p, y'_p\}.$$

The second point is included for the technical purpose of making $\operatorname{diam}(\mathcal{G}(p,y))$ equal to 1 instead of 0. Note that $\mathcal{G}(p,y)$ is *not* the same as $\mathcal{G}(U,x,y)$ even when U is a union of annuli with cores comprising p.

Now we will construct our particular collection Γ of sets of this type, together with the map $\gamma\colon \Sigma_\epsilon(x,y) \to \Gamma$. Let $\mu \in \Sigma_\epsilon(x,y)$, and let $\mathcal{F} = \mathcal{F}_3(\mu, y)$ be as in §2, taking $c = \epsilon + B + \frac{m_4}{2} + c_0$ where B, c_0, and m_4 are the constants from Lemmas 2.5, 2.7, and 2.11. Lemma 2.7 says that the relation $\prec_2$ is a partial order on $\mathcal{F}$. Assuming $\mathcal{F} \neq \emptyset$, among all $\prec_2$–minimal subsurfaces, consider the set U of those that are maximal with respect to inclusion. Then U is a union of disjoint essential subsurfaces.

Fix a constant $a > 2\epsilon$. Suppose that $d_{\mathcal{C}_1(S)}(\mu, \partial U) \leq a$. Then we let $\gamma(\mu) = \mathcal{G}(U,x,y)$.

Suppose that $d_{\mathcal{C}_1(S)}(\mu, \partial U) > a$. Let $q(\mu)$ be the nearest point to μ on the tight geodesic $[x,y]_S = h_{x,y,S}$ (see §2 and Lemma 2.11.) In particular $d_S(\mu, q) \leq \epsilon$ since $\mu \in \Sigma_\epsilon(x,y)$. Let $p(\mu)$ be a simplex along the segment $[q, \pi_S(y)] \subset [x,y]_S$ which is at distance $a/2$ from q. Let $\gamma(\mu) = \mathcal{G}(p,y)$.

If $\mathcal{F} = \emptyset$, define q and p as above, unless $d(q,y) < a/2$ in which case let p be the last simplex of $h_{x,y,S}$ (i.e., a subset of y). Again let $\gamma(\mu) = \mathcal{G}(p,y)$.

We let Γ be the set of all $\gamma(\mu)$ thus obtained.

Proof of the Inductive Bound Lemma. We now prove Lemma 4.3, relating cardinality to diameter for $\mathcal{G} \in \Gamma$. In this proof we will assume by induction that Theorem 4.2, or rather its subsurface version (4.9), holds for all connected proper subsurfaces of S.

If $\mathcal{G} = \mathcal{G}(p,y)$ then $\#\mathcal{G} = 2$ and $\operatorname{diam}(\mathcal{G}) = 1$, so we are done. Now consider $\mathcal{G} = \mathcal{G}(U,x,y)$. In each component U_i of U we have

$$\#\Sigma_{\epsilon,U_i}(x,y) \leq C \operatorname{diam}(\Sigma_{\epsilon,U_i}(x,y))^{\xi'(U_i)}$$

by (4.9). Now since $\mathcal{G}(U,x,y)$ is a product of such sets over the components of U and ξ' is additive, the bound follows with exponent $\xi'(U)$. Since $\xi'(U) \leq \xi'(S) = \xi(S)$, Lemma 4.3 follows.

Proof of the Covering Lemma. We next prove Lemma 4.4, which says that each $\mu \in \Sigma_\epsilon(x,y)$ is uniformly close to $\gamma(\mu)$.

We estimate $d(\mu, \mathcal{G}(U,x,y))$ via the following lemma:

Lemma 4.6. *Let $\mu \in \Sigma_\epsilon(x,y)$ and U a (possibly disconnected) subsurface.*

$$d(\mu, \mathcal{G}(U,x,y)) \approx \sum_{W \subseteq U^c} \{\!\{d_W(\mu,y)\}\!\}_L + \sum_{W \pitchfork \partial U} \{\!\{d_W(\mu,\partial U)\}\!\}_L$$

for a uniform choice of constants.

Proof. For any $\mu \in \mathcal{M}(S)$, define $\tau(\mu) \in \mathcal{G}(U,x,y)$ as follows. In view of Lemma 2.3, to describe $\tau \in \mathcal{Q}(\partial U)$ we must simply give its restrictions $\pi_{\mathcal{M}(V)}(\tau)$ to each component V of U and of U^c.

Hence, for each component U_i of U, let $\pi_{\mathcal{M}(U_i)}(\tau) \equiv \pi_{\mathcal{M}(U_i)}(\mu)$. For each component V of U^c, let $\pi_{\mathcal{M}(V)}(\tau) \equiv \pi_{\mathcal{M}(V)}(y)$.

If $\mu \in \Sigma_\epsilon(x,y)$ then $\pi_W(\mu)$ is within ϵ of $[x,y]_W$ for each W, and hence for $W \subset U_i$ the same is true (perhaps with a change of ϵ) for $\pi_W(\tau)$, since $\pi_{\mathcal{M}(U_i)}$ and π_W (coarsely) commute.

It follows that $\pi_{\mathcal{M}(U_i)}(\tau) \in \Sigma_{\epsilon',U_i}(x,y)$. Thus, since $\epsilon' - \epsilon$ is uniformly bounded, the quasidistance formula yields that $\tau(\mu)$ is a uniformly bounded distance from $\mathcal{G}(U,x,y)$; for the coarse measurements we make below it is no loss of generality to assume $\tau(\mu) \in \mathcal{G}(U,x,y)$.

Now the quasidistance formula gives

$$d(\mu, \tau(\mu)) \approx \sum_W \{\!\{d_W(\mu,\tau)\}\!\}_L$$

but we notice that, for all $W \subset U_i$, the corresponding terms are uniformly bounded. Thus by choosing L sufficiently large those terms disappear (at the expense of changing the constants implicit in the "$\approx$").

If $W \subseteq U^c$ then $d_W(\mu,\tau)$ is estimated by $d_W(\mu,y)$ up to bounded error, so again possibly choosing L larger we can replace one by the other at a bounded cost in the constants. Finally, if $W \pitchfork \partial U$ then, since τ contains ∂U, we can replace those terms by $d_W(\mu,\partial U)$. (See [3] for other examples of this type of argument).

This gives an upper bound for $d(\mu,\mathcal{G}(U,x,y))$ of exactly the type in the lemma. To get the lower bound we observe that for *any* point in $\mathcal{G}(U,x,y)$ the terms of the given type must appear in its quasidistance formula. □

Consider the case that $\gamma(\mu) = \mathcal{G}(U,x,y)$. To bound $d(\mu,\gamma(\mu))$ we must control both types of terms that appear in Lemma 4.6.

In the case $W \pitchfork \partial U$, notice that if $d_W(\mu,\partial U)$ is sufficiently large, then by Lemma 2.8 it follows that $W \in \mathcal{F} = \mathcal{F}_3(\mu,y)$ and W precedes U in the $\prec_2$–order, which contradicts the minimality of U. For the second, we see that if $d_W(\mu, y)$ is sufficiently large then again W would belong to $\mathcal{F}$, and since it is disjoint from U, there would be a $\prec_2$–minimal element disjoint from U, which again contradicts the choice of U.

This uniformly bounds all the terms in Lemma 4.6, and hence gives a uniform bound on $d(\mu, \gamma(\mu))$. (Again, this is done by increasing the threshold past the uniform bound so that all the terms disappear; all that is left is the additive error in "$\approx$".)

Now consider the case that $\gamma(\mu) = \mathcal{G}(p, y)$ with p a simplex in $[x, y]_S$. Recall this means that $\mathcal{F}$ is either empty, or its $\prec_2$–minimal elements are at $\mathcal{C}(S)$–distance at least a from μ.

In fact a bit more is true. If $W \in \mathcal{F}_3(\mu, y)$ then $d_W(x, y) > 3c - \epsilon$, because $\mu \in \Sigma_\epsilon(x, y)$ and hence $\pi_W(\mu)$ is ϵ–close to $[x, y]_W$. Since $3c - \epsilon > 2c > B$, the geodesic projection lemma (2.5) implies some of the simplices of $[x, y]_S$ are disjoint from W — in other words the *footprint*, denoted $\phi_{[x,y]_S}(W)$ as in §2, is nonempty.

Let $W, V \in \mathcal{F}$, $W \pitchfork V$, and suppose that $\phi_{[x,y]_S}(W)$ is disjoint from and to the *right* of $\phi_{[x,y]_S}(V)$ (i.e., $\phi_{[x,y]_S}(W)$ lies on $[x, y]_S$ closer to y then $\phi_{[x,y]_S}(V)$). We claim this implies $V \prec_2 W$. Indeed, letting $t \in \phi_{[x,y]_S}(V)$, the segment $[x, t]_S$ consists of curves intersecting W, and by Lemma 2.5 $d_W(x, t) \leq B$. Since t and ∂V are disjoint, $d_W(x, \partial V) \leq B < 2c$, so $W \not\prec_2 V$. Since by Lemma 2.7 they are $\prec_2$–ordered, $V \prec_2 W$. Equivalently, we can say that if $W \prec_2 V$, then $\phi_{[x,y]}(V)$ either intersects or is to the right of $\phi_{[x,y]}(W)$. (These are essentially variations on arguments in [17].)

Now, if U is $\prec_2$–minimal in $\mathcal{F}$, then ∂U is at least a from μ; hence its footprint is at least $a - \epsilon$ either to the left or to the right of $q(\mu)$ (recall $d_S(\mu, q) \leq \epsilon$). If it were on the left, since $a > 2\epsilon$, using the Lipschitz property of π_U as earlier, we would find that $d_U(\mu, q) \leq 3\epsilon$. Lemma 2.5 would give us $d_U(q, p) < B$, but this bounds $d_U(\mu, p)$ and contradicts $U \in \mathcal{F}$. We conclude the footprint is to the right of q.

By the previous paragraph on ordering, we conclude since U is $\prec_2$–minimal that *all* elements of $\mathcal{F}$ have footprints at least distance $a - \epsilon$ to the right of q, and in fact to the right of p since $d_S(p, q) \leq a/2$.

To get a bound on $d(\mu, \mathcal{G}(p, y))$ we must again bound the terms from the quasidistance formula, i.e., $d_W(\mu, \mathcal{G}(p, y))$ for $W \subseteq S$.

Let $W \subset S$ be a proper connected subsurface. If W is disjoint from p, then ∂W is within $\mathcal{C}(S)$–distance $a/2 + 1$ from q, and hence no more than a from μ. It follows that W is not in $\mathcal{F}$, and hence $d_W(\mu, y)$ is bounded. By construction, the projection of $\mathcal{G}(p, y)$ to W is the projection of y, so this gives us the desired bound.

If W intersects p, we must bound $d_W(\mu, p)$. We claim that if $d_W(\mu, p)$ is sufficiently large, then $\phi_{[x,y]_S}(W)$ is nonempty. For if it were empty, Lemma 2.5 would bound $\mathrm{diam}_W([x, y]_S)$, and since $\pi_W(\mu)$ is within ϵ of $[x, y]_W$, this would bound $d_W(\mu, p)$ as well. Hence we may assume $\phi_{[x,y]_S}(W)$ is nonempty, and so lies either to the right or the left of p.

If it is on the left, then by the previous discussion W cannot be in $\mathcal{F}$, and so $d_W(\mu, y)$ is bounded. Moreover since the footprint is outside of $[p, y]_S$, Lemma 2.5 gives a bound on $d_W(\mu, p)$ as well.

If it is on the right, then its distance from q is at least $a/2$, and so we have a bound on $d_W(\mu, q)$, by the Lipschitz property of π_W, and on $d_W(q, p)$, by Lemma 2.5. Hence we obtain a bound on $d_W(\mu, p)$.

The only case left is that $W = S$. However, we have already noted that $d_S(\mu, q) \leq \epsilon$ and hence $d_S(\mu, p) \leq \epsilon + a/2$.

This completes the proof of Lemma 4.4.

Diameter sum bound. Our final step is to prove Lemma 4.5, bounding the diameter sum over Γ.

First, consider the members of Γ of the form $\mathcal{G}(U, x, y)$. Each of these has diameter comparable with $d_{\mathcal{M}(U)}(\pi_{\mathcal{M}(U)}(x), \pi_{\mathcal{M}(U)}(y))$, which by the quasidistance formula is estimated by

$$\sum_{V \subseteq U} \{\!\{d_V(x, y)\}\!\}_A$$

for any sufficiently large A. Let us choose $A > 2c$. So the sum over all possible $\mathcal{G}(U, x, y)$ should be comparable to the sum of $\{\!\{d_V(x, y)\}\!\}_A$ over all proper subsurfaces V in S, provided we show that each V occurs in a bounded number of U's.

Fix $V \subsetneq S$. Consider $\mu \in \Sigma_\epsilon(x, y)$ such that $\gamma(\mu) = \mathcal{G}(U, x, y)$ with $V \subset U$, and let U_1 be a component of U. In particular, $d_{\mathcal{C}_1(S)}(\partial U_1, \partial V) \leq 1$.

We will control the number of possible such U_1's using a hierarchy $H = H(x, y)$. Since $U_1 \in \mathcal{F}$ we have $d_{U_1}(\mu, y) > 3c$, and hence $d_{U_1}(x, y) > 3c - \epsilon > 2c$ which by Lemma 2.11 implies U_1 is a domain in H. Let W be any other domain of a tight geodesic $k = h_{x,y,W}$ in H such that $U_1 \subset W$. We claim:

Lemma 4.7. *The footprint $\phi_k(U_1)$ is a uniformly bounded distance in $\mathcal{C}_1(W)$ from one of the endpoints of k.*

Proof. Since the endpoints of k are within m_3 of $\pi_W(x)$ and $\pi_W(y)$ respectively, and $\phi_k(U_1)$ is within 1 of ∂U_1, it suffices to bound $d_W(\partial U_1, x)$ or $d_W(\partial U_1, y)$.

Suppose first that $d_W(\mu, y) \leq 3c$. We claim in this case that $d_W(\partial U_1, y)$ is at most $3c + 2$. For if not, then Lemma 2.5 can be applied to the geodesic segment from $\pi_W(y)$ to $\pi_W(\mu)$, yielding $d_{U_1}(\mu, y) \leq B < 3c$, a contradiction with the definition of U_1. Thus we are done in this case.

Suppose now that $d_W(\mu, y) > 3c$, so that W is in $\mathcal{F}_3(\mu, y)$. It cannot be $\prec_2$–minimal, because if it were then U_1 would not have been chosen (U_1 is inclusion-maximal among $\prec_2$–minimal elements). Hence there is some $W' \prec_2 W$. If $W' \pitchfork U_1$, then they are $\prec_2$–ordered; but since U_1 is $\prec_2$–minimal, we get $U_1 \prec_2 W'$, and thus $U_1 \prec_2 W$ which contradicts $U_1 \subset W$. Hence W' and U_1 have disjoint boundaries, so $d_W(\partial U_1, \partial W') \leq 3$ by the coarse Lipschitz property of π_W. Now, since $W' \prec_2 W$, we have by Lemma

2.8 that $d_W(x, \partial W')$ is bounded, and so we get a bound on $d_W(x, \partial U_1)$. This completes the proof. $\square$

We now bound the number of possible U_1's, by induction. Using part (3) of Lemma 2.11, every U_1 is contained in a chain $U_1 = W_0 \subset W_1 \subset \cdots \subset W_s = S$ such that each W_i supports a geodesic in H, each W_i for $i < s$ is a component domain (complementary component or annulus) of a simplex in $h_{x,y,W_{i+1}}$, and this simplex (being in the footprint) is a bounded distance from one of the endpoints of $h_{x,y,W_{i+1}}$ for $i+1 < s$. For $i = s-1$, the footprint is on $h_{x,y,W_s} = [x,y]_S$, and here it is constrained to a bounded interval by the inequality $d_S(\partial V, \partial U_1) \le 1$ (remember that we have fixed V).

Hence, starting with $W_s = S$ and working backwards, for each W_i there is a uniformly bounded number of choices for W_{i-1}. We conclude that there is a uniformly bounded number of choices for $U_1 = W_0$.

So now we have a uniform bound on the number of different $\mathcal{G}(U,x,y)$'s in Γ for which U contains a fixed subsurface V. We conclude that

$$\sum_{\mathcal{G}(U,x,y)\in\Gamma} \operatorname{diam}(\mathcal{G}(U,x,y)) \le N \sum_{V \subsetneq S} \{\!\{ d_V(x,y) \}\!\}_A$$

for uniform N.

What remains in the left-hand-side of inequality (4.8) is the sum over the $\mathcal{G}(p,y) \in \Gamma$ where $p \in [x,y]_S$. Since for these, by definition, the diameters are all 1, this sum satisfies

$$\sum_{\mathcal{G}(p,y)\in\Gamma} \operatorname{diam}(\mathcal{G}(p,y)) \le d_S(x,y) + 1$$

Putting these together we have

$$\sum_{\mathcal{G}\in\Gamma} \operatorname{diam}(\mathcal{G}) \le N' \sum_{V\subseteq S} \{\!\{ d_V(x,y) \}\!\}_A \le N''\ d(x,y).$$

which establishes Lemma 4.5, and so completes the proof of Theorem 4.2. $\square$

4.3. **Proofs of the Main Theorems**

To wrap up the proof of Theorem 1.2: Using the quasi-isometric orbit map $\mathcal{MCG}(S) \to \mathcal{M}(S)$ we can convert the centroid map κ of Theorem 3.2 to a centroid map, which we also call κ, defined for $\mathcal{MCG}(S)$. The first three properties of Theorem 3.2 clearly imply the first three properties of Theorem 1.2.

We showed that $K(x,y,r)$ is contained in $\Sigma_{\epsilon'}(x,y)$ and in $\mathcal{N}_{b_1 r}(x)$, for uniform ϵ' and b_1. Lemma 4.1 then implies that there exists $q \in \Sigma_\epsilon(x,y)$ such that $d(x,q) \le b_2 r$, and $K(x,y,r)$ is contained in $\Sigma_{\epsilon'}(x,q)$. Finally, Theorem 4.2 gives us a bound for $\#\Sigma_{\epsilon'}(x,q)$ which is polynomial (of degree $\xi(S)$) in $d(x,q)$. This gives the desired bound for $\#K(x,y,r)$.

Because the centroid for $\mathcal{MCG}(S)$ is obtained from the centroid for $\mathcal{M}(S)$ via a quasi-isometry and the volume of all r-balls in $\mathcal{MCG}(S)$ is uniformly bounded for any fixed r, the polynomial bound on $\#K(x,y,r)$ yields the fourth property of Theorem 1.2.

As mentioned in the introduction, Theorem 1.1 now follows by applying a result of Druţu-Sapir [9, Theorem 3.1 and Remark 3.5]. They showed that the Rapid Decay property holds for groups which are $(**)$–*relatively hyperbolic with respect to the trivial group.* This property is said to hold when the following are satisfied: there exists a function $T\colon G\times G\to G$ and a polynomial $Q(r)$ satisfying:

(1) $T(g,h)=T(h,g)$
(2) $T(h^{-1},h^{-1}g)=h^{-1}T(h,g)$
(3) If $g\in G$ and $r\in\mathbb{N}$, then $\#\{T(g,h) : |h|=r\}<Q(r)$.

For the mapping class group, we define

$$T(g,h)=\kappa(1,g,h)\colon \mathcal{MCG}(S)\times\mathcal{MCG}(S)\to\mathcal{MCG}(S)$$

where κ is the centroid map as given by Theorem 1.2. The first condition above follows from the property that κ is invariant under permutation of its arguments (Theorem 1.2 part (1)). The second condition holds since $\kappa(1,h^{-1},h^{-1}g) = h^{-1}\kappa(h,1,g) = h^{-1}\kappa(1,h,g)$, where the first equality is from the equivariance of κ (Theorem 1.2 part (2)). The third condition follows from our cardinality bound on centroids (Theorem 1.2 part (4)).

Alternatively, Chatterji-Ruane in [5, Proposition 1.7] give a similar criterion that implies Rapid Decay. Their condition involves an equivariant family of subsets $S(x,y)\subset G$, where $x,y\in G$, satisfying a number of conditions, in particular

$$S(x,y)\cap S(y,z)\cap S(x,z)\neq\emptyset$$

and

$$\#S(x,y)\leq P(d(x,y))$$

for a polynomial P. It is not hard to see that our Σ-hulls give such a family, i.e., $S(x,y)\equiv\Sigma_\epsilon(x,y)$, and that the existence of centroids gives the non-empty triple intersection property.

References

[1] S. Barré and M. Pichot, *The 4-string Braid group B_4 has property RD and exponential mesoscopic rank*, Preprint, arXiv:0809.0645.
[2] J. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space*, Geometry & Topology **10** (2006), 2001–2056.
[3] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, *Geometry and rigidity of mapping class groups*, Preprint. arXiv:0801.2006, 2008.
[4] J. Behrstock and Y. Minsky, *Dimension and rank for mapping class groups*, Annals of Mathematics **167** (2008), 1055–1077.
[5] I. Chatterji, I. and K. Ruane, *Some geometric groups with rapid decay*, Geom. Funct. Anal. **15** (2005), 311–339.
[6] A. Connes. *Noncommutative geometry.* Academic Press, 1994.

[7] Alain Connes and Henri Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990), no. 3, 345–388.
[8] Pierre de la Harpe, *Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 14, 771–774.
[9] Cornelia Druţu and Mark Sapir, *Relatively hyperbolic groups with rapid decay property*, Int. Math. Res. Not. (2005), no. 19, 1181–1194.
[10] Uffe Haagerup, *An example of a nonnuclear C^*-algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), no. 3, 279–293.
[11] U. Hamenstädt, *Geometry of the mapping class groups I: Boundary amenability*, Preprint, arXiv:math.GR/0510116.
[12] N. V. Ivanov. *Mapping class groups.* In R. Daverman and R. Sher, editor, *Handbook of Geometric Topology.* Elsevier, 2001, 523–633.
[13] Paul Jolissaint, *Rapidly decreasing functions in reduced C^*-algebras of groups*, Trans. Amer. Math. Soc. **317** (1990), no. 1, 167–196.
[14] G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201.
[15] Yoshikata Kida, *The mapping class group from the viewpoint of measure equivalence theory*, Mem. Amer. Math. Soc., To appear. arXiv:math.GR/0512230.
[16] H. Masur and Y. Minsky, *Geometry of the complex of curves, I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.
[17] ———, *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.
[18] Ralf Meyer, *Combable groups have group cohomology of polynomial growth*, Q. J. Math. **57** (2006), no. 2, 241–261.
[19] L. Mosher, *Mapping class groups are automatic*, Ann. of Math. **142** (1995), 303–384.
[20] Alain Valette. *Introduction to the Baum-Connes conjecture.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.

Lehman College, CUNY
E-mail address: `jason.behrstock@lehman.cuny.edu`

Yale University
E-mail address: `yair.minsky@yale.edu`