GLOBAL STRUCTURAL PROPERTIES OF RANDOM GRAPHS

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Abstract. We study two global structural properties of a graph Γ, denoted $\mathcal{AS}$ and $\mathcal{CFS}$, which arise in a natural way from geometric group theory. We study these properties in the Erdős–Rényi random graph model $\mathcal{G}(n,p)$, proving the existence of a sharp threshold for a random graph to have the $\mathcal{AS}$ property asymptotically almost surely, and giving fairly tight bounds for the corresponding threshold for the $\mathcal{CFS}$ property.

As an application of our results, we show that for any constant p and any $\Gamma \in \mathcal{G}(n,p)$, the right-angled Coxeter group W_Γ asymptotically almost surely has quadratic divergence and thickness of order 1, generalizing and strengthening a result of Behrstock–Hagen–Sisto [8]. Indeed, we show that at a large range of densities a random right-angled Coxeter group has quadratic divergence.

Introduction

In this article, we consider two properties of graphs motivated by geometric group theory. We show that these properties are typically present in random graphs. We repay the debt to geometric group theory by applying our (purely graph-theoretic) results to the large-scale geometry of Coxeter groups.

Random graphs. Let $\mathcal{G}(n,p)$ be the random graph model on n vertices obtained by including each edge independently at random with probability $p = p(n)$. The parameter p is often referred to as the *density* of $\mathcal{G}(n,p)$. The model $\mathcal{G}(n,p)$ was introduced by Gilbert [23], and the resulting random graphs are usually referred to as the "Erdős–Rényi random graphs" in honor of Erdős and Rényi's seminal contributions to the field, and we follow this convention. We say that a property $\mathcal{P}$ holds *asymptotically almost surely* (a.a.s.) in $\mathcal{G}(n,p)$ if for $\Gamma \in \mathcal{G}(n,p)$ we have $\mathbb{P}(\Gamma \in \mathcal{P}) \to 1$ as $n \to \infty$. In this paper we will be interested in proving that certain global properties hold a.a.s. in $\mathcal{G}(n,p)$ both for a wide range of probabilities $p = p(n)$.

A graph property is *(monotone) increasing* if it is closed under the addition of edges. A paradigm in the theory of random graphs is that global increasing graph properties exhibit *sharp thresholds* in $\mathcal{G}(n,p)$: for many global increasing properties $\mathcal{P}$, there is a *critical density* $p_c = p_c(n)$ such that for any fixed $\epsilon > 0$ if $p < (1-\epsilon)p_c$ then a.a.s. $\mathcal{P}$ does not hold in $\mathcal{G}(n,p)$, while if $p > (1+\epsilon)p_c$ then a.a.s. $\mathcal{P}$ holds in $\mathcal{G}(n,p)$. A quintessential example is the following classical result of Erdős and Rényi which provides a sharp threshold for connectedness:

Theorem (Erdős-Rényi; [21])**.** *There is a sharp threshold for connectivity of a random graph with critical density* $p_c(n) = \frac{\log(n)}{n}$.

The local structure of the Erdős–Rényi random graph is well understood, largely due to the assumption of independence between the edges. For example, Erdős–Rényi and others have

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obtained threshold densities for the existence of certain subgraphs in a random graph (see e.g. [21, Theorem 1, Corollaries 1–5]). In earlier applications of random graphs to geometric group theory, this feature of the model was successfully exploited in order to analyze the geometry of right-angled Artin and Coxeter groups presented by random graphs; this is notable, for example, in the work of Charney and Farber [14]. In particular, the presence of an induced square implies non-hyperbolicity of the associated right-angled Coxeter group [14, 21, 31].

In this paper, we take a more global approach. Earlier work established a correspondence between some fundamental geometric properties of right-angled Coxeter groups and large-scale structural properties of the presentation graph, rather than local properties such as the presence or absence of certain specified subgraphs. The simplest of these properties is the property of being the join of two subgraphs that are not cliques.

One large scale graph property relevant in the present context is a property studied in [8] which, roughly, says that the graph is constructed in a particular organized, inductive way from joins. In this paper, we discuss a refined version of this property, $\mathcal{CFS}$, which is a slightly-modified version of a property introduced by Dani–Thomas [17]. We also study a stronger property, $\mathcal{AS}$, and show it is generic in random graphs for a large range of $p(n)$, up $1-\omega(n^{-2})$.

$\mathcal{AS}$ graphs. The first class of graphs we study is the class of *augmented suspensions*, which we denote $\mathcal{AS}$. A graph is an augmented suspension if it contains an induced subgraph which is a suspension (see Section 1 for a precise definition of this term), and any vertex which is not in that suspension is connected by edges to at least two nonadjacent vertices of the suspension.

Theorems 3.4 and 3.5 (Sharp Threshold for $\mathcal{AS}$). *Let $\epsilon > 0$ be fixed. If $p = p(n)$ satisfies $p \geq (1+\epsilon)\left(\frac{\log n}{n}\right)^{\frac{1}{3}}$ and $(1-p)n^2 \to \infty$, then $\Gamma \in \mathcal{G}(n, p(n))$ is a.a.s. in $\mathcal{AS}$. On the other hand, if $p \leq (1-\epsilon)\left(\frac{\log n}{n}\right)^{\frac{1}{3}}$, then $\Gamma \in \mathcal{G}(n,p)$ a.a.s. does not lie in $\mathcal{AS}$.*

Intriguingly, Kahle proved that a function similar to the critical density in Theorem 3.4 is the threshold for a random simplicial complex to have vanishing second rational cohomology [28].

Remark (Behaviour near $p = 1$)**.** Note that property $\mathcal{AS}$ is not monotone increasing, since it requires the presence of a number of non-edges. In particular, complete graphs are not in $\mathcal{AS}$. Thus unlike the global properties typically studied in the theory of random graphs, $\mathcal{AS}$ will cease to hold a.a.s. when the density p is very close to 1. In fact, [8, Theorem 3.9] shows that if $p(n) = 1 - \Omega\left(\frac{1}{n^2}\right)$, then a.a.s. Γ is either a clique or a clique minus a fixed number of edges whose endpoints are all disjoint. Thus, with positive probability, $\Gamma \in \mathcal{AS}$. However, [14, Theorem 1] shows that if $(1-p)n^2 \to 0$ then Γ is asymptotically almost surely a clique, and hence not in $\mathcal{AS}$.

$\mathcal{CFS}$ graphs. The second family of graphs, which we call *$\mathcal{CFS}$ graphs* ("Constructed From Squares"), arise naturally in geometric group theory in the context of the large–scale geometry of right–angled Coxeter groups, as we explain below and in Section 2. A special case of these graphs was introduced by Dani–Thomas to study divergence in triangle-free right-angled Coxeter groups [17]. The graphs we study are intimately related to a property called *thickness*, a feature of many key examples in geometric group theory and low dimensional topology that is closely related to divergence, relative hyperbolicity, and a number of other topics. This property is, in essence, a connectivity property because it relies on a space being "connected" through sequences of "large" subspaces. Roughly speaking, a graph is $\mathcal{CFS}$ if it can be built inductively by chaining (induced) squares together in such a way that each square overlaps with one of the previous squares along a diagonal (see Section 1 for a precise definition). We explain in the next section how this class of graphs generalizes $\mathcal{AS}$. Our next result about genericity of $\mathcal{CFS}$ combines with

Proposition 2.1 below to significantly strengthen [8, Theorem VI]. This result is an immediate consequence of Theorems 4.1 and 4.7, which, in fact, establish slightly more precise, but less concise, bounds.

Theorems 4.1 and 4.7 *Suppose* $(1-p)n^2 \to \infty$ *and let* $\epsilon > 0$. *Then* $\Gamma \in \mathcal{G}(n,p)$ *is a.a.s. in* $\mathcal{CFS}$ *whenever* $p(n) > n^{-\frac{1}{2}+\epsilon}$. *Conversely,* $\Gamma \in \mathcal{G}(n,p)$ *is a.a.s. not in* $\mathcal{CFS}$ *whenever* $p(n) < n^{-\frac{1}{2}-\epsilon}$.

We actually show, in Theorem 4.1, that at densities above $5\sqrt{\frac{\log n}{n}}$, with $(1-p)n^2 \to \infty$, the random graph is a.a.s. in $\mathcal{CFS}$, while in Theorem 4.7 we show a random graph a.a.s. not in $\mathcal{CFS}$ at densities below $\frac{1}{\sqrt{n}\log n}$.

Theorem 4.1 applies to graphs in a range strictly larger than that in which Theorem 3.4 holds (though our proof of Theorem 4.1 relies on Theorem 3.4 to deal with the large p case). Theorem 4.1 combines with Theorem 3.5 to show that, for densities between $(\log n/n)^{\frac{1}{2}}$ and $(\log n/n)^{\frac{1}{3}}$, a random graph is asymptotically almost surely in $\mathcal{CFS}$ but not in $\mathcal{AS}$. We also note that Babson–Hoffman–Kahle [3] proved that a function of order $n^{-\frac{1}{2}}$ appears as the threshold for simple-connectivity in the Linial–Meshulam model for random 2–complexes [29]. It would be interesting to understand whether there is a connection between genericity of the $\mathcal{CFS}$ property and the topology of random 2-complexes.

Unlike our results for the $\mathcal{AS}$ property, we do not establish a sharp threshold for the $\mathcal{CFS}$ property. In fact, we believe that neither the upper nor lower bounds, given in Theorem 4.1 and Theorem 4.7, for the critical density around which $\mathcal{CFS}$ goes from a.a.s. not holding to a.a.s. holding are sharp. Indeed, we believe that there is a sharp threshold for the $\mathcal{CFS}$ property located at $p_c(n) = \theta(n^{-\frac{1}{2}})$. This conjecture is linked to the emergence of a giant component in the "square graph" of Γ (see the next section for a definition of the square graph and the heuristic discussion after the proof of Theorem 4.7).

Applications to geometric group theory. Our interest in the structure of random graphs was sparked largely by questions about the large-scale geometry of *right-angled Coxeter groups*. Coxeter groups were first introduced in [15] as a generalization of reflection groups, i.e., discrete groups generated by a specified set of reflections in Euclidean space. A reflection group is *right-angled* if the reflection loci intersect at right angles. An abstract right-angled Coxeter group generalizes this situation: it is defined by a group presentation in which the generators are involutions and the relations are obtained by declaring some pairs of generators to commute. Right-angled Coxeter groups (and more general Coxeter groups) play an important role in geometric group theory and are closely-related to some of that field's most fundamental objects, e.g. CAT(0) cube complexes [18, 33, 27] and (right-angled) Bruhat-Tits buildings (see e.g. [18]).

A right-angled Coxeter group is determined by a unique finite simplicial *presentation graph*: the vertices correspond to the involutions generating the group, and the edges encode the pairs of generators that commute. In fact, the presentation graph uniquely determines the right-angled Coxeter group [32]. In this paper, as an application of our results on random graphs, we continue the project of understanding large-scale geometric features of right-angled Coxeter groups in terms of the combinatorics of the presentation graph, begun in [8, 14, 17]. Specifically, we study right-angled Coxeter groups defined by random presentation graphs, focusing on the prevalence of two important geometric properties: *relative hyperbolicity* and *thickness*.

Relative hyperbolicity, in the sense introduced by Gromov and equivalently formulated by many others [24, 22, 10, 35], when it holds, is a powerful tool for studying groups. On the other hand, thickness of a finitely-generated group (more generally, a metric space) is a property introduced by Behrstock–Druţu–Mosher in [6] as a geometric obstruction to relative hyperbolicity and has a number of powerful geometric applications. For example, thickness gives bounds on divergence (an important quasi-isometry invariant of a metric space) in many different groups and spaces [5, 7, 11, 17, 37].

Thickness is an inductive property: in the present context, a finitely generated group G is *thick of order* 0 if and only if it decomposes as the direct product of two infinite subgroups. The group G is thick of order n if there exists a finite collection $\mathcal{H}$ of undistorted subgroups of G, each thick of order $n-1$, whose union generates a finite-index subgroup of G and which has the following "chaining" property: for each $g, g' \in G$, one can construct a sequence $g \in g_1H_1, g_2H_2, \ldots, g_kH_k \ni g'$ of cosets, with each $H_i \in \mathcal{H}$, so that consecutive cosets have infinite coarse intersection. Many of the best-known groups studied by geometric group theorists are thick, and indeed thick of order 1: one-ended right-angled Artin groups, mapping class groups of surfaces, outer automorphism groups of free groups, fundamental groups of 3–dimensional graph manifolds, etc. [6].

The class of Coxeter groups contains many examples of hyperbolic and relatively hyperbolic groups. There is a criterion for hyperbolicity purely in terms of the presentation graph due to Moussong [31] and an algebraic criterion for relative hyperbolicity due to Caprace [13]. The class of Coxeter groups includes examples which are non-relatively hyperbolic, for instance, those constructed by Davis–Januszkiewicz [19] and, also, ones studied by Dani–Thomas [17]. In fact, in [8], this is taken further: it is shown that every Coxeter group is actually either thick, or hyperbolic relative to a canonical collection of thick Coxeter subgroups. Further, there is a simple, structural condition on the presentation graph, checkable in polynomial time, which characterizes thickness. This result is needed to deduce the applications below from our graph theoretic results.

Charney and Farber initiated the study of *random graph products* (including right-angled Artin and Coxeter groups) using the Erdős-Rényi model of random graphs [14]. The structure of the group cohomology of random graph products was obtained in [20]. In [8], various results are proved about which random graphs have the thickness property discussed above, leading to the conclusion that, at certain low densities, random right-angled Coxeter groups are relatively hyperbolic (and thus not thick), while at higher densities, random right-angled Coxeter groups are thick. In this paper, we improve significantly on one of the latter results, and also prove something considerably more refined: we isolate not just thickness of random right-angled Coxeter groups, but thickness of a specified order, namely 1:

Corollary 2.2 (Random Coxeter groups are thick of order 1.) *There exists a constant $C > 0$ such that if $p\colon \mathbb{N} \to (0,1)$ satisfies $\left(\frac{C\log n}{n}\right)^{\frac{1}{2}} \le p(n) \le 1 - \frac{(1+\epsilon)\log n}{n}$ for some $\epsilon > 0$, then the random right-angled Coxeter group $W_{G_{n,p}}$ is asymptotically almost surely thick of order exactly* 1*, and in particular has quadratic divergence.*

Corollary 2.2 significantly improves on Theorem 3.10 of [8], as discussed in Section 2. This theorem follows from Theorems 4.1 and 3.4, the latter being needed to treat the case of large $p(n)$, including the interesting special case in which p is constant.

Remark 0.1. We note that characterizations of thickness of right-angled Coxeter groups in terms of the structure of the presentation graph appear to generalize readily to graph products of arbitrary finite groups and, probably, via the action on a cube complex constructed by Ruane and Witzel in [36], to arbitrary graph products of finitely generated abelian groups, using appropriate modifications of the results in [8].

Organization of the paper. In Section 1, we give the formal definitions of $\mathcal{AS}$ and $\mathcal{CFS}$ and introduce various other graph-theoretic notions we will need. In Section 2, we discuss the applications of our random graph results to geometric group theory, in particular to right-angled Coxeter groups and more general graph products. In Section 3, we obtain a sharp threshold result for $\mathcal{AS}$ graphs. Section 4 is devoted to $\mathcal{CFS}$ graphs. Finally Section 5 contains some simulations of random graphs with density near the threshold for $\mathcal{AS}$ and $\mathcal{CFS}$.

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1. Definitions

Convention 1.1. A *graph* is a pair of finite sets $\Gamma = (V, E)$, where $V = V(\Gamma)$ is a set of *vertices*, and $E = E(\Gamma)$ is a collection of pairs of distinct elements of V, which constitute the set of *edges* of G. A *subgraph* of Γ is a graph Γ' with $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$; Γ' is said to be an *induced* subgraph of Γ if $E(\Gamma')$ consists exactly of those edges from $E(\Gamma)$ whose vertices lie in $V(\Gamma')$. In this paper we focus on induced subgraphs, and we generally write "subgraph" to mean "induced subgraph". In particular we often identify a subgraph with the set of vertices inducing it, and we write $|\Gamma|$ for the *order* of Γ, that is, the number of vertices it contains. A *clique* of size t is a complete graph on $t \geq 0$ vertices. Note this includes the degenerate case of the empty graph on $t = 0$ vertices.

Definition 1.2 (Link, join)**.** Given a graph Γ, the *link* of a vertex $v \in \Gamma$, denoted $\mathrm{Lk}_\Gamma(v)$, is the subgraph spanned by the set of vertices adjacent to v. Given graphs A, B, the *join* $A \star B$ is the graph formed from $A \sqcup B$ by joining each vertex of A to each vertex of B by an edge. A *suspension* is a join where one of the factors A, B is the graph consisting of two vertices and no edges.

We now describe a family of graphs, denoted $\mathcal{CFS}$, which satisfy the global structural property that they are "constructed from squares."

Definition 1.3 ($\mathcal{CFS}$)**.** Given a graph Γ, let $\square(\Gamma)$ be the auxiliary graph whose vertices are the induced 4–cycles from Γ, with two distinct 4–cycles joined by an edge in $\square(\Gamma)$ if and only if they intersect in a pair of non-adjacent vertices of Γ (i.e., in a diagonal). We refer to $\square(\Gamma)$ as the *square-graph* of Γ. A graph Γ belongs to $\mathcal{CFS}$ if $\Gamma = \Gamma' \star K$, where K is a (possibly empty) clique and Γ' is a non-empty subgraph such that $\square(\Gamma')$ has a connected component C such that the union of the 4–cycles from C covers all of $V(\Gamma')$. Given a vertex $F \in \square(\Gamma)$, we refer to the vertices in the 4–cycle in Γ associated to F as the *support* of F.

Remark 1.4. Dani–Thomas introduced *component with full support* graphs in [17], a subclass of the class of triangle-free graphs. We note that each component with full support graph is constructed from squares, but the converse is not true. Indeed, since we do not require our graphs to be triangle-free, our definition necessarily only counts *induced* 4–cycles and allows them to intersect in more ways than in [17]. This distinction is relevant to the application to Coxeter groups, which we discuss in Section 2.

Definition 1.5 (Augmented suspension)**.** The graph Γ is an *augmented suspension* if it contains an induced subgraph $B = \{w, w'\} \star \Gamma'$, where w, w' are nonadjacent and Γ' is not a clique, satisfying the additional property that if $v \in \Gamma - B$, then $\mathrm{Lk}_\Gamma(v) \cap \Gamma'$ is not a clique. Let $\mathcal{AS}$ denote the class of augmented suspensions. Figure 1 shows a graph in $\mathcal{AS}$.

Remark 1.6. Neither the $\mathcal{CFS}$ nor the $\mathcal{AS}$ properties introduced above are monotone with respect to the addition of edges. This stands in contrast to the most commonly studied global properties of random graphs.

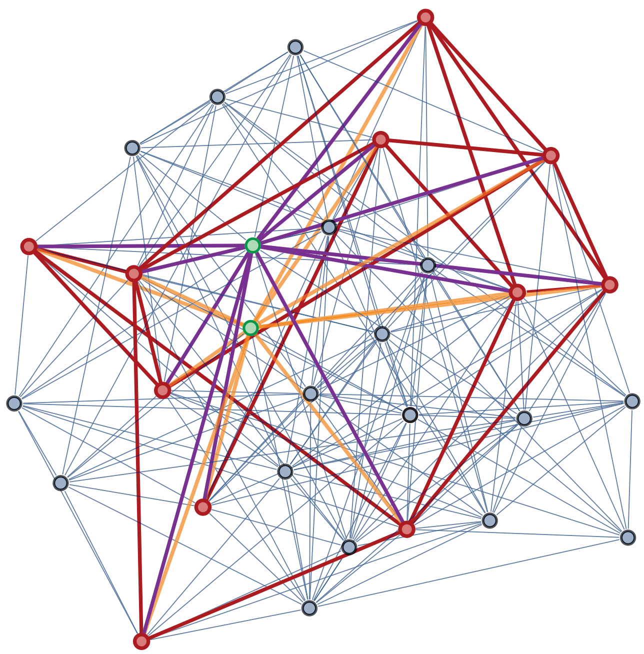

FIGURE 1. A graph in $\mathcal{AS}$. A block exhibiting inclusion in $\mathcal{AS}$ is shown in bold; the two (left-centrally located) *ends* of the bold *block* are highlighted.

Definition 1.7 (Block, core, ends)**.** A *block* in Γ is a subgraph of the form $B(w, w') = \{w, w'\}\star\Gamma'$ where $\{w, w'\}$ is a pair of non-adjacent vertices and $\Gamma' \subset \Gamma$ is a subgraph of Γ induced by a set of vertices adjacent to both w and w'. A block is *maximal* if $V(\Gamma') = \mathrm{Lk}_\Gamma(w) \cap \mathrm{Lk}_\Gamma(w')$. Given a block $B = B(w, w')$, we refer to the non-adjacent vertices w, w' as the *ends of* B, denoted $\mathrm{end}(B)$, and the vertices of Γ' as the *core of* B, denoted $\mathrm{core}(B)$.

Note that $\mathcal{AS} \subsetneq \mathcal{CFS}$, indeed Theorem 4.1 and Theorem 3.5 show that there must exist graphs in $\mathcal{CFS}$ that are not in $\mathcal{AS}$. Here we explain how any graph in $\mathcal{AS}$ is in $\mathcal{CFS}$.

Lemma 1.8. *Let Γ be a graph in $\mathcal{AS}$. Then $\Gamma \in \mathcal{CFS}$.*

Proof. Let $B(w, w') = \{w, w'\}\star\Gamma'$ be a maximal block in Γ witnessing $\Gamma \in \mathcal{AS}$. Write $\Gamma' = A\star D$, where D is the collection of all vertices of Γ' which are adjacent to every other vertex of Γ'. Note that D induces a clique in Γ. By definition of the $\mathcal{AS}$ property Γ' is not a clique, whence A contains at least one pair of non-adjacent vertices. Furthermore by the definition of D, for every vertex $a \in A$ there exists $a' \in A$ with $\{a, a'\}$ non-adjacent. The 4–cycles induced by $\{w, w'a, a'\}$ for non-adjacent pairs a, a' from A are connected in $\square(\Gamma)$. Denote the component of $\square(\Gamma)$ containing them by C.

Consider now a vertex $v \in \Gamma - B(w, w')$. Since $B(w, w')$ is maximal, we have that at least one of w, w' is not adjacent to v — without loss of generality, let us assume that it is w. By the $\mathcal{AS}$

property, v must be adjacent to a pair a, a' of non-adjacent vertices from A. Then $\{w, a, a', v\}$ induces a 4–cycle, which is adjacent to $\{w, w', a, a'\} \in \square(\Gamma)$ and hence lies in C.

Finally consider a vertex $d \in D$. If v is adjacent to all vertices of Γ, then Γ is the join of a graph with a clique containing d, and we can ignore d with respect to establishing the $\mathcal{CFS}$ property. Otherwise d is not adjacent to some $v \in \Gamma - B(w, w')$. By the $\mathcal{AS}$ property, v is connected by edges to a pair of non-adjacent vertices $\{a, a'\}$ from A. Thus $\{d, a, a', v\}$ induces a 4–cycle. Since (as established above) there is some 4–cycle in C containing $\{a, a', v\}$, we have that $\{d, a, a', v\} \in C$ as well. Thus $\Gamma = \Gamma'' \star K$, where K is a clique and $V(\Gamma'')$ is covered by the union of the 4–cycles in C, so that $\Gamma \in \mathcal{CFS}$ as claimed. □

2. Geometry of right-angled Coxeter groups

If Γ is a finite simplicial graph, the *right-angled Coxeter group W_Γ presented by* Γ is the group defined by the presentation

$$\left\langle \mathbf{Vert}(\Gamma) \mid \{w^2, uvu^{-1}v^{-1} : u, v, w \in \mathbf{Vert}(\Gamma), \{u, v\} \in \mathbf{Edge}(\Gamma)\right\rangle .$$

A result of Mühlherr [32] shows that the correspondence $\Gamma \leftrightarrow W_\Gamma$ is bijective. We can thus speak of "the random right-angled Coxeter group" — it is the right-angled Coxeter group presented by the random graph. (We emphasize that the above presentation provides the *definition* of a right-angled Coxeter group: this definition abstracts the notion of a reflection group – a subgroup of a linear group generated by reflections – but infinite Coxeter groups need not admit representations as reflection groups.)

Recent papers have discussed the geometry of Coxeter groups, especially relative hyperbolicity and closely-related quasi-isometry invariants like divergence and thickness, cf. [8, 13, 17]. In particular, Dani–Thomas introduced a property they call *having a component of full support* for triangle-free graphs (which is exactly the triangle-free version of $\mathcal{CFS}$) and they prove that under the assumption Γ is triangle-free, W_Γ is thick of order at most 1 if and only if it has quadratic divergence if and only if Γ is in $\mathcal{CFS}$, see [17, Theorem 1.1 and Remark 4.8]. Since the densities where random graphs are triangle-free are also square-free (and thus not $\mathcal{CFS}$ — in fact, they are disconnected!), we need the following slight generalization of the result of Dani–Thomas:

Proposition 2.1. *Let Γ be a finite simplicial graph. If Γ is in $\mathcal{CFS}$ and Γ does not decompose as a nontrivial join, then W_Γ is thick of order exactly* 1.

Proof. Theorem II of [8] shows immediately that, if $\Gamma \in \mathcal{CFS}$, then W_Γ is thick, being formed by a series of *thick unions* of 4–cycles; since each 4–cycle is a join, it follows that Γ is thick of order at most 1. On the other hand, [8, Proposition 2.11] shows that W_Γ is thick of order at least 1 provided Γ is not a join. □

Our results about random graphs yield:

Corollary 2.2. *There exists $k > 0$ so that if $p\colon \mathbb{N} \to (0, 1)$ and $\epsilon > 0$ are such that $\sqrt{\frac{k \log n}{n}} \le p(n) \le 1 - \dfrac{(1-\epsilon)\log n}{n}$ for all sufficiently large n, then for $\Gamma \in \mathcal{G}(n, p)$ the group W_Γ is asymptotically almost surely thick of order exactly* 1 *and hence has quadratic divergence.*

Proof. Theorem 4.1 shows that any such Γ is asymptotically almost surely in $\mathcal{CFS}$, whence W_Γ is thick of order at most 1. We emphasize that to apply this result for sufficiently large functions $p(n)$ the proof of Theorem 4.1 requires an application of Theorem 3.4 to establish that Γ is a.a.s. in $\mathcal{AS}$ and hence in $\mathcal{CFS}$ by Lemma 1.8.

By Proposition 2.1, to show that the order of thickness is exactly one, it remains to rule out the possibility that Γ decomposes as a nontrivial join. However, this occurs if and only if the complement graph is disconnected, which asymptotically almost surely does not occur whenever $p(n) \leq 1 - \frac{(1-\epsilon)\log n}{n}$, by the sharp threshold for connectivity of $\mathcal{G}(n, 1-p)$ established by Erdős and Rényi in [21]. Since this holds for $p(n)$ by assumption, we conclude that asymptotically almost surely, W_Γ is thick of order at least 1.

Since W_Γ is CAT(0) and thick of order exactly 1, the consequence about divergence now follows from [5]. □

This corollary significantly generalizes Theorem 3.10 of [8], which established that, if $\Gamma \in \mathcal{G}(n, \frac{1}{2})$, then W_Γ is asymptotically almost surely thick. Theorem 3.10 of [8] does not provide effective bounds on the order of thickness and its proof is significantly more complicated than the proof of Corollary 2.2 given above — indeed, it required several days of computation (using 2013 hardware) to establish the base case of an inductive argument.

Remark 2.3 (Higher-order thickness)**.** A lower bound of $p(n) = n^{-\frac{5}{6}}$ for membership in a larger class of graphs whose corresponding Coxeter groups are thick can be found in [8, Theorem 3.4]. In fact, this argument can be adapted to give a simple proof that a.a.s. thickness does not occur at densities below $n^{-\frac{3}{4}}$. The correct threshold for a.a.s. thickness is, however, unknown.

Remark 2.4 (Random graph products versus random presentations)**.** Corollary 2.2 and Remark 2.3 show that the random graph model for producing random right-angled Coxeter groups generates groups with radically different geometric properties. This is in direct contrast to other methods of producing random groups, most notably Gromov's random presentation model [25, 26] where, depending on the density of relators, groups are either almost surely hyperbolic or finite (with order at most 2). This contrast speaks to the merits of considering a random right-angled Coxeter group as a natural place to study random groups. For instance, Calegari–Wilton recently showed that in the Gromov model a random group contains many subgroups which are isomorphic to the fundamental group of a compact hyperbolic 3–manifold [12]; does the random right-angled Coxeter group also contain such subgroups?

Right-angled Coxeter groups, and indeed thick ones, are closely related to Gromov's random groups in another way. When the parameter for a Gromov random groups is $< \frac{1}{6}$ such a group is word-hyperbolic [25] and acts properly and cocompactly on a CAT(0) cube complex [34]. Hence the Gromov random group virtually embeds in a right-angled Artin group [1]. Moreover, at such parameters such a random group is one-ended [16], whence the associated right-angled Artin group is as well. By [4] this right-angled Artin group is thick of order 1. Since any right-angled Artin group is commensurable with a right-angled Coxeter group [19], one obtains a thick of order 1 right-angled Coxeter group containing the randomly presented group.

3. Genericity of $\mathcal{AS}$

We will use the following standard Chernoff bounds, see e.g. [2, Theorems A.1.11 and A.1.13]:

Lemma 3.1 (Chernoff bounds)**.** *Let $X_1, \dots, X_n$ be independent identically distributed random variables taking values in $\{0,1\}$, let X be their sum, and let $\mu = \mathbb{E}[X]$. Then for any $\delta \in (0, 2/3)$*

$$\mathbb{P}\left(|X - \mu| \geq \delta\mu\right) \leq 2e^{-\frac{\delta^2\mu}{3}}.$$

Corollary 3.2. *Let $\varepsilon, \delta > 0$ be fixed.*

(i) If $p(n) \geq \left(\frac{(6+\varepsilon)\log n}{\delta^2 n}\right)^{1/2}$, then a.a.s. for all pairs of distinct vertices $\{x,y\}$ in $\Gamma \in \mathcal{G}(n,p)$ we have $\big||\operatorname{Lk}_\Gamma(x) \cap \operatorname{Lk}_\Gamma(y)| - p^2(n-2)\big| < \delta p^2(n-2)$.

(ii) If $p(n) \geq \left(\frac{(9+\varepsilon)\log n}{\delta^2 n}\right)^{1/3}$, then a.a.s. for all triples of distinct vertices $\{x,y,z\}$ in $\Gamma \in \mathcal{G}(n,p)$ we have $\big||\operatorname{Lk}_\Gamma(x) \cap \operatorname{Lk}_\Gamma(y) \cap \operatorname{Lk}_\Gamma(z)| - p^3(n-3)\big| < \delta p^3(n-3)$.

Proof. For (i), let $\{x, y\}$ be any pair of distinct vertices. For each vertex $v \in \Gamma - \{x, y\}$, set X_v to be the indicator function of the event that $v \in \mathrm{Lk}_\Gamma(x) \cap \mathrm{Lk}_\Gamma(y)$, and set $X = \sum_v X_v$ to be the size of $\mathrm{Lk}_\Gamma(x) \cap \mathrm{Lk}_\Gamma(y)$. We have $\mathbb{E}X = p^2(n-2)$ and so by the Chernoff bounds above, $\mathbb{P}\left(|X - p^2(n-2)| \geq \delta p^2(n-2)\right) \leq 2e^{-\frac{\delta^2 p^2 (n-2)}{3}}$. Applying Markov's inequality, the probability that there exists *some* "bad pair" $\{x, y\}$ in Γ for which $|\mathrm{Lk}_\Gamma(x) \cap \mathrm{Lk}_\Gamma(y)|$ deviates from its expected value by more than $\delta p^2(n-2)$ is at most

$$\binom{n}{2} 2e^{-\frac{\delta^2 p^2 (n-2)}{3}} = o(1),$$

provided $\delta^2 p^2 n \geq (6 + \varepsilon) \log n$ and $\varepsilon, \delta > 0$ are fixed. Thus for this range of $p = p(n)$, a.a.s. no such bad pair exists. The proof of (ii) is nearly identical. □

Lemma 3.3. *(i) Suppose $1 - p \geq \frac{\log n}{2n}$. Then asymptotically almost surely, the order of a largest clique in $\Gamma \in \mathcal{G}(n, p)$ is $o(n)$.*

(ii) Let η be fixed with $0 < \eta < 1$. Suppose $1 - p \geq \eta$. Then asymptotically almost surely, the order of a largest clique in $\Gamma \in \mathcal{G}(n, p)$ is $O(\log n)$.

Proof. For (i), set $r = \alpha n$, for some α bounded away from 0. Write $H(\alpha) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1-\alpha}$. Using the standard entropy bound $\binom{n}{\alpha n} \leq e^{H(\alpha) n}$ and our assumption for $(1 - p)$, we see that the expected number of r-cliques in Γ is

$$\binom{n}{r} p^{\binom{r}{2}} \leq e^{H(\alpha) n} e^{\log(1-(1-p))\left(\frac{\alpha^2 n^2}{2} + O(n)\right)} \leq \exp\left(-\frac{\alpha^2}{2} n \log n + O(n)\right) = o(1).$$

Thus by Markov's inequality, a.a.s. Γ does not contain a clique of size r, and the order of a largest clique in Γ is $o(n)$.

The proof of (ii) is similar: suppose $1 - p > \eta$. Then for any $r \leq n$,

$$\binom{n}{r} p^{\binom{r}{2}} < n^r (1-\eta)^{r(r-1)/2} = \exp\left(r\left(\log n - \frac{r-1}{2} \log \frac{1}{1-\eta}\right)\right),$$

which for $\eta > 0$ fixed and $r - 1 > \frac{2}{\log(1/(1-\eta))}(1 + \log n)$ is as most $n^{-\frac{2}{\log(1/(1-\eta))}} = o(1)$. We may thus conclude as above that a.a.s. a largest clique in Γ has order $O(\log n)$. □

Theorem 3.4 (Genericity of $\mathcal{AS}$)**.** *Suppose $p(n) \geq (1 + \epsilon)\left(\frac{\log n}{n}\right)^{\frac{1}{3}}$ for some $\epsilon > 0$ and $(1 - p)n^2 \to \infty$. Then, a.a.s. $\Gamma \in \mathcal{G}(n, p)$ is in $\mathcal{AS}$.*

Proof. Let $\delta > 0$ be a small constant to be specified later (the choice of δ will depend on ϵ). By Corollary 3.2 (i) for $p(n)$ in the range we are considering, a.a.s. all joint links have size at least $(1 - \delta)p^2(n - 2)$. Denote this event by $\mathcal{E}_1$. We henceforth condition on $\mathcal{E}_1$ occurring (not this only affects the values of probabilities by an additive factor of $\mathbb{P}(\mathcal{E}_1^c) = O(n^{-\varepsilon}) = o(1)$). With probability $1 - p^{\binom{n}{2}} = 1 - o(1)$, Γ is not a clique, whence there there exist non-adjacent vertices in Γ. We henceforth assume $\Gamma \neq K_n$, and choose $v_1, v_2 \in \Gamma$ which are not adjacent. Let B be the maximal block associated with the pair (v_1, v_2). We separate the range of p into three.

Case 1: p is "far" from both the threshold and 1. Let $\alpha > 0$ be fixed, and suppose $\alpha n^{-1/4} \leq p \leq 1 - \frac{\log n}{2n}$. Let $\mathcal{E}_2$ be the event that for every vertex $v \in \Gamma - B$ the set $\mathrm{Lk}_\Gamma(v) \cap B$ has size at least $\frac{1}{2}p^3(n - 3)$. By Corollary 3.2, a.a.s. event $\mathcal{E}_2$ occurs, i.e., all vertices in $\Gamma - B$ have this property.

We claim that a.a.s. there is no clique of order at least $\frac{1}{2}p^3(n-3)$ in Γ. Indeed, if $p < 1 - \eta$ for some fixed $\eta > 0$, then by Lemma 3.3 part (ii), a largest clique in Γ has order $O(\log n) = o(p^3 n)$. On the other hand, if $1 - \eta < p \leq 1 - \frac{\log n}{2n}$, then by Lemma 3.3 part (i), a largest clique in Γ

has order $o(n) = o(p^3 n)$. Thus in either case a.a.s. for *every* $v \in \Gamma - B$, $\mathrm{Lk}_\Gamma(v) \cap B$ is not a clique and hence $v \in \overline{B}$, so that a.a.s. $\overline{B} = \Gamma$, and $\Gamma \in \mathcal{AS}$ as required.

Case 2: p is "close" to the threshold. Suppose that $(1+\epsilon)\left(\frac{\log n}{n}\right)^{\frac{1}{3}} \leq p(n)$ and $np^4 \to 0$. Let $|B| = m+2$. By our conditioning, we have $(1-\delta)(n-2)p^2 \leq m \leq (1+\delta)(n-2)p^2$. The probability that a given vertex $v \in \Gamma$ is not in $\overline{B}$ is given by:

$$\mathbb{P}(v \notin \overline{B} | \{|B| = m\}) = (1-p)^m + mp(1-p)^{m+1} + \sum_{r=2}^{m} \binom{m}{r} p^r (1-p)^{m-r} p^{\binom{r}{2}}. \tag{1}$$

In this equation, the first two terms come from the case where v is connected to 0 and 1 vertex in $B \setminus \{v_1, v_2\}$ respectively, while the third term comes from the case where the link of v in $B \setminus \{v_1, v_2\}$ is a clique on $r \geq 2$ vertices. As we shall see, in the case $np^4 \to 0$ which we are considering, the contribution from the first two terms dominates. Let us estimate their order:

$$\begin{aligned}(1-p)^m + mp(1-p)^{m-1} = \left(1 + \frac{mp}{1-p}\right)(1-p)^m &\leq \left(1 + \frac{mp}{1-p}\right) e^{-mp} \\ &\leq \left(1 + \frac{(1-\delta)(1+\epsilon)^3 \log n}{1-p}\right) n^{-(1-\delta)(1+\epsilon)^3}.\end{aligned}$$

Taking $\delta < 1 - \frac{1}{(1+\epsilon)^3}$ this expression is $o(n^{-1})$.

We now treat the sum making up the remaining terms in Equation 1. To do so, we will analyze the quotient of successive terms in the sum. Fixing $2 \leq r \leq m-1$ we see:

$$\frac{\binom{m}{r+1} p^{r+1} (1-p)^{m-r-1} p^{\binom{r+1}{2}}}{\binom{m}{r} p^r (1-p)^{m-r} p^{\binom{r}{2}}} = \frac{m-r-1}{r+1} \cdot \frac{p^{r+1}}{1-p} \leq mp^{r+1} \leq mp^3.$$

Since $np^4 \to 0$ (by assumption), this also tends to zero as $n \to \infty$. The quotients of successive terms in the sum thus tend to zero uniformly as $n \to \infty$, and we may bound the sum by a geometric series:

$$\sum_{r=2}^{m} \binom{m}{r} p^r (1-p)^{m-r} p^{\binom{r}{2}} \leq \binom{m}{2} p^3 (1-p)^{m-2} \sum_{i=0}^{m-2} (mp^3)^i \leq \left(\frac{1}{2} + o(1)\right) m^2 p^3 (1-p)^{m-2}.$$

Now, $m^2 p^3 (1-p)^{m-2} = \frac{mp^2}{1-p} \cdot mp(1-p)^{m-1}$. The second factor in this expression was already shown to be $o(n^{-1})$, while $mp^2 \leq (1+\delta)np^4 \to 0$ by assumption, so the total contribution of the sum is $o(n^{-1})$. Thus for any value of m between $(1-\delta)p^2(n-2)$ and $(1+\delta)p^2(n-2)$, the right hand side of Equation 1 is $o(n^{-1})$, and we conclude:

$$\mathbb{P}(v \notin \overline{B} | \mathcal{E}_1) \leq o(n^{-1}).$$

Thus, by Markov's inequality,

$$\mathbb{P}(\overline{B} = \Gamma) \geq \mathbb{P}(\mathcal{E}_1)\left(1 - \sum_{v} \mathbb{P}(v \notin \overline{B} | \mathcal{E}_1)\right) = 1 - o(1),$$

establishing that a.a.s. $\Gamma \in \mathcal{AS}$, as claimed.

Case 3: p is "close" to 1. Suppose $n^{-2} \ll (1-p) \leq \frac{\log n}{2n}$. Consider the complement of Γ, $\Gamma^c \in \mathcal{G}(n, 1-p)$. In the range of the parameter Γ^c a.a.s. has at least two connected components that contain at least two vertices. In particular, taking complements, we see that Γ is a.a.s. a join of two subgraphs, neither of which is a clique. It is a simple exercise to see that such as graph is in $\mathcal{AS}$, thus a.a.s. $\Gamma \in \mathcal{AS}$. □

As we now show, the bound obtained in the above theorem is actually a sharp threshold. Analogous to the classical proof of the connectivity threshold [21], we consider vertices which are "isolated" from a block to prove that graphs below the threshold strongly fail to be in $\mathcal{AS}$.

Theorem 3.5. *If $p \leq (1-\epsilon)\left(\frac{\log n}{n}\right)^{\frac{1}{3}}$ for some $\epsilon > 0$, then $\Gamma \in \mathcal{G}(n,p)$ is asymptotically almost surely not in $\mathcal{AS}$.*

Proof. We will show that, for p as hypothesized, every block has a vertex "isolated" from it. Explicitly, let $\Gamma \in \mathcal{G}(n,p)$ and consider $B = B_{v,w} = \mathrm{Lk}(v) \cap \mathrm{Lk}(w) \cup \{v,w\}$. Let $X(v,w)$ be the event that every vertex of $\Gamma - B$ is connected by an edge to some vertex of B. Clearly $\Gamma \in \mathcal{AS}$ only if the event $X(v,w)$ occurs for some pair of non-adjacent vertices $\{v,w\}$. Set $X = \bigcup_{\{v,w\}} X(v,w)$. Note that X is a monotone event, closed under the addition of edges, so that the probability it occurs in $\Gamma \in \mathcal{G}(n,p)$ is a non-decreasing function of p. We now show that when $p = (1-\epsilon)(\log n/n)^{\frac{1}{3}}$, a.a.s. X does not occur, completing the proof.

Consider a pair of vertices $\{v,w\}$, and set $k = |B_{v,w}|$. Conditional on $B_{v,w}$ having this size and using the standard inequality $(1-x) \leq e^{-x}$, we have that

$$\mathbb{P}(X(v,w)) = (1-(1-p)^k)^{n-k} \leq e^{-(n-k)(1-p)^k}.$$

Now, the value of k is concentrated around its mean: by Corollary 3.2, for any fixed $\delta > 0$ and all $\{v,w\}$, the order of $B_{v,w}$ is a.a.s. at most $(1+\delta)np^2$. Conditioning on this event $\mathcal{E}$, we have that for any pair of vertices v, w,

$$\mathbb{P}(X(v,w)|\mathcal{E}) \leq \max_{k \leq (1+\delta)np^2} e^{-(n-k)(1-p)^k} = e^{-(n-(1+\delta)np^2)(1-p)^{(1+\delta)np^2}}.$$

Now $(1-p)^{(1+\delta)np^2} = e^{(1+\delta)np^2 \log(1-p)}$ and by Taylor's theorem $\log(1-p) = -p + O(p^2)$, so that:

$$\mathbb{P}(X(v,w)|\mathcal{E}) \leq e^{-n(1+O(p^2))e^{-(1+\delta)np^3(1+O(p))}} = e^{-n^{1-(1+\delta)(1-\epsilon)^3+o(1)}}.$$

Choosing $\delta < \frac{1}{(1-\epsilon)^3} - 1$, the expression above is $o(n^{-2})$. Thus

$$\begin{aligned}\mathbb{P}(X) &\leq \mathbb{P}(\mathcal{E}^c) + \sum_{\{v,w\}} \mathbb{P}(X(v,w)|\mathcal{E}) \\ &= o(1) + \binom{n}{2} o(n^{-2}) = o(1).\end{aligned}$$

Thus a.a.s. the monotone event X does not occur in $\Gamma \in \mathcal{G}(n,p)$ for $p = (1-\epsilon)(\log n/n)^{\frac{1}{3}}$, and hence a.a.s. the property $\mathcal{AS}$ does not hold for $\Gamma \in \mathcal{G}(n,p)$ and $p(n) \leq (1-\epsilon)(\log n/n)^{\frac{1}{3}}$. ☐

4. Genericity of $\mathcal{CFS}$

The two main results in this section are upper and lower bounds for inclusion in $\mathcal{CFS}$. These results are established in Theorem 4.1 and Theorem 4.7.

Theorem 4.1. *If $p\colon \mathbb{N} \to (0,1)$ satisfies $(1-p)n^2 \to \infty$ and $p(n) \geq 5\sqrt{\frac{\log n}{n}}$ for all sufficiently large n, then a.a.s. $\Gamma \in \mathcal{G}(n,p)$ lies in $\mathcal{CFS}$.*

The proof of Theorem 4.1 divides naturally into two ranges. First of all for large p, namely for $p(n) \geq 2(\log n/n)^{\frac{1}{3}}$, we appeal to Theorem 3.4 to show that a.a.s. a random graph $\Gamma \in \mathcal{G}(n,p)$ is in $\mathcal{AS}$ and hence, by Lemma 1.8, in $\mathcal{CFS}$. In light of our proof of Theorem 3.4, we may think of this as the case when we can "beam up" every vertex of the graph Γ to a single block $B_{x,y}$

in an appropriate way, and thus obtain a connected component of $\square(\Gamma)$ whose support is all of $V(\Gamma)$

Secondly we have the case of "small p" where

$$5\sqrt{\frac{\log n}{n}} \leq p(n) \leq 2\left(\frac{\log n}{n}\right)^{\frac{1}{3}},$$

which is the focus of the remainder of the proof. Here we construct a path of length of order $n/\log n$ in $\square(\Gamma)$ onto which every vertex $v \in V(\Gamma)$ can be "beamed up" by adding a 4–cycle whose support contains v.

This is done in the following manner: we start with an arbitrary pair of non-adjacent vertices contained in a block B_0. We then pick an arbitrary pair of non-adjacent vertices in the block B_0 and let B_1 denote the intersection of the block they define with $V(\Gamma) \setminus B_0$. We repeat this procedure, to obtain a chain of blocks $B_0, B_1, B_2, \ldots, B_t$, with $t = O(n/\log n)$, whose union contains a positive proportion of $V(\Gamma)$, and which all belong to the same connected component C of $\square(\Gamma)$. This common component C is then large enough that every remaining vertex of $V(\Gamma)$ can be attached to it. The main challenge is showing that our process of recording which vertices are included in the support of a component of the square graph does not die out or slow down too much, i.e., that the block sizes $|B_i|$ remains relatively large at every stage of the process and that none of the B_i form a clique.

Having described our strategy, we now fill in the details, beginning with the following upper bound on the probability of $\Gamma \in \mathcal{G}(n,p)$ containing a copy of K_{10}, the complete graph on 10 vertices. The following lemma is a variant of [21, Corollary 4]:

Lemma 4.2. *Let $\Gamma \in \mathcal{G}(n,p)$. If $p = o(n^{-\frac{1}{4}})$, then the probability that $\Gamma \in \mathcal{G}(n,p)$ contains a clique with at least 10 vertices is at most $o(n^{-\frac{5}{4}})$.*

Proof. The expected number of copies of K_{10} in Γ is

$$\binom{n}{10} p^{\binom{10}{2}} \leq n^{10} p^{45} = o(n^{-5/4}).$$

The statement of the lemma then follows from Markov's inequality. □

Proof of Theorem 4.1. As remarked above, Theorem 3.4 proves Theorem 4.1 for "large" p, so we only need to deal with the case where

$$5\sqrt{\frac{\log n}{n}} \leq p(n) \leq 2\left(\frac{\log n}{n}\right)^{\frac{1}{3}}.$$

We iteratively build a chain of blocks, as follows. Let $\{x_0, y_0\}$ be a pair of non-adjacent vertices in Γ, if such a pair exists, and an arbitrary pair of vertices if not. Let B_0 be the block with ends $\{x_0, y_0\}$.

Now assume we have already constructed the blocks $B_0, \ldots, B_i$, for $i \geq 0$. Let $C_i = \bigcup_i B_i$ (for convenience we let $C_{-1} = \emptyset$). We will terminate the process and set $t = i$ if any of the three following conditions occur: $|\mathrm{core}(B_i)| \leq 6\log n$ or $i \geq n/6\log n$ or $|V(\Gamma) \setminus C_i| \leq n/2$. Otherwise, we let $\{x_{i+1}, y_{i+1}\}$ be a pair of non-adjacent vertices in $\mathrm{core}(B_i)$, if such a pair exists, and an arbitrary pair of vertices from $\mathrm{core}(B_i)$ otherwise. Let B_{i+1} denote the intersection of the block whose ends are $\{x_{i+1}, y_{i+1}\}$ and the set $(V(\Gamma) \setminus (C_i)) \cup \{x_{i+1}, y_{i+1}\}$. Repeat.

Eventually this process must terminate, resulting in a chain of blocks $B_0, B_1, \ldots, B_t$. We claim that a.a.s. both of the following hold for every i satisfying $0 \leq i \leq t$:

(i) $|\mathrm{core}(B_i)| > 6\log n$; and
(ii) $\{x_i, y_i\}$ is a non-edge in Γ.

Part (i) follows from the Chernoff bound given in Lemma 3.1: for each $i \geq -1$ the set $V(\Gamma) \setminus C_i$ contains at least $n/2$ vertices by construction. For each vertex $v \in V(\Gamma) \setminus C_i$, let X_v be the indicator function of the event that v is adjacent to both of $\{x_{i+1}, y_{i+1}\}$. The random variables (X_v) are independent identically distributed Bernoulli random variables with mean p. Their sum $X = \sum_v X_v$ is exactly the size of the core of B_{i+1}, and its expectation is at least $p^2 n/2$. Applying Lemma 3.1, we get that

$$\begin{aligned}\mathbb{P}(X < 6\log n) &\leq \mathbb{P}(X \leq \frac{1}{2}\mathbb{E}X) \\ &\leq 2e^{-\left(\frac{1}{2}\right)^2 \frac{25\log n}{6n}} = 2e^{-\frac{25}{24}\log n}.\end{aligned}$$

Thus the probability that $|\text{core}(B_i)| < 6\log n$ for some i with $0 \leq i \leq t$ is at most:

$$t2e^{-\frac{25}{24}\log n} \leq \frac{4n}{5\log n}2e^{-\frac{25}{24}\log n} = o(1).$$

Part (ii) is a trivial consequence of part (i) and Lemma 4.2: a.a.s. $\text{core}(B_i)$ has size at least $6\log n$ for every i with $0 \leq i \leq t$, and a.a.s. Γ contains no clique on $10 < \log n$ vertices, so that a.a.s. at each stage of the process we could choose an non-adjacent pair $\{x_i, y_i\}$.

From now on we assume that both (i) and (ii) occur, and that Γ contains no clique of size 10. In addition, we assume that $|\text{core}(B_0)| < 8n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}$, which occurs a.a.s. by the Chernoff bound. Since $\text{core}(B_i) \geq 6\log n$ for every i, we must have that by time $0 < t \leq n/6\log n$ the process will have terminated with $C_t = \bigcup_{i=0}^t B_i$ supported on at least half of the vertices of $V(\Gamma)$.

Lemma 4.3. *Either one of the assumptions above fails or there exists a connected component F of $\square(\Gamma)$ such that:*

(i) for every i with $0 \leq i \leq t$ and every pair of non-adjacent vertices $\{v, v'\} \in B_i$, there is a vertex in F whose support in Γ contains the pair $\{v, v'\}$; and

(ii) the support in Γ of the 4–cycles corresponding to vertices of F contains all of C_t with the exception of at most 9 vertices of $\text{core}(B_0)$*; moreover, these exceptional vertices are each adjacent to all the vertices of* $\text{core}(B_0)$.

Proof. By assumption the ends $\{x_0, y_0\}$ of B_0 are non-adjacent. Thus, every pair of non-adjacent vertices $\{v, v'\}$ in $\text{core}(B_0)$ induces a 4–cycle in Γ when taken together with $\{x_0, y_0\}$, and all of these squares clearly lie in the same component F of $\square(\Gamma)$. Repeating the argument with the non-adjacent pair $\{x_1, y_1\} \in \text{core}(B_0)$ and the block B_1, and then the non-adjacent pair $\{x_2, y_2\} \in \text{core}(B_1)$ and the block B_2, and so on, we see that there is a connected component F in $\square(\Gamma)$ such that for every $0 \leq i \leq t$, every pair of non adjacent vertices $\{v, v'\} \in B_i$ lies in a 4–cycle corresponding to a vertex of F. This establishes (i).

We now show that the support of F contains all of C_t except possibly some vertices in B_0. We already established that every pair $\{x_i, y_i\}$ is in the support of some vertex of F. Suppose $v \in \text{core}(B_i)$ for some $i > 0$. By construction, v is not adjacent to at least one of $\{x_{i-1}, y_{i-1}\}$, say x_{i-1}. Thus, $\{x_{i-1}, x_i, y_i, v\}$ induces a 4–cycle which contains v and is associated to a vertex of F. Finally, suppose $v \in \text{core}(B_0)$. By (i), v fails to be in the support of F only if v is adjacent to all other vertices of $\text{core}(B_0)$. Since, by assumption, Γ does not contain any clique of size 10, there are at most 9 vertices not in the support of F, proving (ii). □

Lemma 4.3, shows that a.a.s. we have a "large" component F in $\square(\Gamma)$ whose support contains "many" pairs of non-adjacent vertices. In the last part of the proof, we use these pairs to prove that the remaining vertices of $V(\Gamma)$ are also supported on our connected component.

For each i satisfying $0 \leq i \leq t$, consider a a maximal collection, M_i, of pairwise-disjoint pairs of vertices in $\text{core}(B_i) \setminus \{x_{i+1}, y_{i+1}\}$. Set $M = \bigcup_i M_i$, and let M' be the subset of M consisting of pairs, $\{v, v'\}$, for which v and v' are not adjacent in Γ. We have

$$|M| = \sum_{i=1}^{t} \left(\left\lfloor \frac{1}{2} |\mathrm{core}(B_i)| \right\rfloor - 1 \right) \geq \frac{|C_t|}{2} - 2t \geq \frac{n}{4}(1 - o(1)).$$

The expected size of M' is thus $(1-p)n(\frac{1}{4} - o(1)) = \frac{n}{4}(1 - o(1))$, and by the Chernoff bound from Lemma 3.1 we have

$$\mathbb{P}(|M'| \leq \frac{n}{5}) \leq 2e^{-\left(\frac{1}{5}+o(1)\right)^2 \frac{(1-p)n}{12}} \leq e^{-\left(\frac{1}{300}+o(1)\right)n},$$

which is $o(1)$. Thus a.a.s. M' contains at least $n/5$ pairs, and by Lemma 4.3 each of these lies in some 4–cycle of F. We now show that we can "beam up" every vertex not yet supported on F by a 4–cycle using a pair from M'. By construction we have at most $n/2$ unsupported vertices from $V(\Gamma) \setminus C_t$ and at most 9 unsupported vertices from $\mathrm{core}(B_0)$.

Assume that $|M'| \geq n/5$. Fix a vertex $w \in V(\Gamma) \setminus C_t$. For each pair $\{v, v'\} \in M'$, let $X_{v,v'}$ be the event that w is adjacent to both v and v'. We now observe that if $X_{v,v'}$ occurs for some pair $\{v, v'\} \in M' \cap \mathrm{core}(B_i)$, then w is supported on F. By construction, w is not adjacent to at least one of $\{x_i, y_i\}$, let us say without loss of generality x_i. Hence, $\{x_i, v, v', w\}$ is an induced 4–cycle in Γ which contains w and which corresponds to a vertex of F.

The probability that $X_{v,v'}$ fails to happen for every pair $\{v, v'\} \in M'$ is exactly

$$(1-p^2)^{|M'|} \leq (1-p^2)^{n/5} \leq e^{-\frac{p^2 n}{5}} = e^{-5 \log n}.$$

Thus the expected number of vertices $w \in V(\Gamma) \setminus C_t$ which fail to be in the support of F is at most $\frac{n}{2} e^{-5 \log n} = o(1)$, whence by Markov's inequality a.a.s. no such bad vertex w exists.

Finally, we deal with the possible 9 left-over vertices $b_1, b_2, \ldots b_9$ from $\mathrm{core}(B_0)$ we have not yet supported. We observe that since $\mathrm{core}(B_0)$ contains at most $8n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}$ vertices (as we are assuming and as occurs a.a.s. , see the discussion before Lemma 4.3), we do not stop the process with B_0, $\mathrm{core}(B_1)$ is non-empty and contains at least $6 \log n$ vertices. As stated in Lemma 4.3, each unsupported vertex b_i is adjacent to all other vertices in $\mathrm{core}(B_0)$, and in particular to both of $\{x_1, y_1\}$. If b_i fails to be adjacent to some vertex $v \in \mathrm{core}(B_1)$, then the set $\{b_i, x_1, y_1, v\}$ induces a 4–cycle corresponding to a vertex of F and whose support contains b_i. The probability that there is some b_i not supported in this way is at most

$$9\mathbb{P}(b_i \text{ adjacent to all of } \mathrm{core}(B_1)) = 9p^{6 \log n} = o(1).$$

Thus a.a.s. we can "beam up" each of the vertices $b_1, \ldots b_9$ to F using a vertex $v \in \mathrm{core}(B_1)$, and the support of the component F in the square graph $\square(\Gamma)$ contains all vertices of Γ. This shows that a.a.s. $\Gamma \in \mathcal{CFS}$, and concludes the proof of the theorem. □

Remark 4.4. The constant 5 in Theorem 4.1 is not optimal, and indeed it is not hard to improve on it slightly, albeit at the expense of some tedious calculations. We do not try to obtain a better constant, as we believe that the order of the upper bound we have obtained is not sharp. We conjecture that the actual threshold for $\mathcal{CFS}$ occurs when $p(n)$ is of order $n^{-1/2}$ (see the discussion below Theorem 4.7), but a proof of this is likely to require significantly more involved and sophisticated arguments than the present paper.

A simple lower bound for the emergence of the $\mathcal{CFS}$ property can be obtained from the fact that if $\Gamma \in \mathcal{CFS}$, then Γ must contain at least $n-3$ squares; if $p(n) \ll n^{-\frac{3}{4}}$, then by Markov's inequality a.a.s. a graph in $\mathcal{G}(n,p)$ contains fewer than $o(n)$ squares and thus cannot be in $\mathcal{CFS}$. Below, in Theorem 4.7, we prove a better lower bound, showing that the order of the upper bound we proved in Theorem 4.1 is not off by a factor of more than $(\log n)^{3/2}$.

Lemma 4.5. *Let Γ be a graph and let C be the subgraph of Γ supported on a given connected component of $\square(\Gamma)$. Then there exists an ordering $v_1 < v_2 < \cdots < v_{|C|}$ of the vertices of C such that for all $i \geq 3$, v_i is adjacent in Γ to at least two vertices preceding it in the order.*

Proof. As C is a component of $\square(\Gamma)$, it contains at least one induced 4–cycle. Let v_1, v_2 be a pair of non-adjacent vertices from such an induced 4–cycle. Then the two other vertices $\{v_3, v_4\}$ of the 4–cycle are both adjacent in Γ to both of v_1 and v_2. If this is all of C, then we are done. Otherwise, we know that each 4–cycle in C is "connected" to the cycle $F = \{v_1, v_2, v_3, v_4\}$ via a sequence of induced 4–cycles pairwise intersecting in pairwise non-adjacent vertices. In particular, there is some such 4–cycle whose intersection with F is either a pair of non-adjacent vertices in F or three vertices of F; either way, we may add the new vertex next in the order.

Continuing in this way and using the fact that the number of vertices not yet reached is a monotonically decreasing set of positive integers, the lemma follows. □

Proposition 4.6. *Let $\delta > 0$. Suppose $p \leq \frac{1}{\sqrt{n}\log n}$. Then a.a.s. for $\Gamma \in \mathcal{G}(n,p)$, no component of $\square(\Gamma)$ has support containing more than $4\log n$ vertices of Γ.*

Proof. Let $\delta > 0$. Let $m = \left\lceil \min\left(4\log n, 4\log\left(\frac{1}{p}\right)\right)\right\rceil$, with $p \leq 1/(\sqrt{n}\log n)$. We shall show that a.a.s. there is no ordered m–tuple of vertices $v_1 < v_2 < \cdots < v_m$ from Γ such that for every $i \geq 2$ each vertex v_i is adjacent to at least two vertices from $\{v_j : 1 \leq j < i\}$. By Lemma 4.5, this is enough to establish our claim.

Let $v_1 < v_2 < \cdots < v_m$ be an arbitrary ordered m–tuple of vertices from $V(\Gamma)$. For $i \geq 2$, let A_i be the event that v_i is adjacent to at least two vertices in the set $\{v_j : 1 \leq j < i\}$. We have:

$$\mathbb{P}(A_i) = \sum_{j=2}^{i-1} \binom{i-1}{j} p^j (1-p)^{i-j-1}. \tag{2}$$

As in the proof of Theorem 3.5 we consider the quotients of successive terms in the sum to show that its order is given by the term $j = 2$. To see this, observe:

$$\frac{\binom{i-1}{j+1} p^{j+1}(1-p)^{i-j-2}}{\binom{i-1}{j} p^j (1-p)^{i-j-1}} \leq \frac{i-j-1}{j+1} \cdot \frac{p}{1-p} < mp$$

where the final inequality holds for n sufficiently large and $p = p(n)$ satisfying our assumption. Since $m = O(\log n)$ and $p = o(n^{-1/2})$ we have, again for n large enough, that $mp = o(1)$, and we may bound the sum in equation (2) by a geometric series to obtain the bound:

$$\begin{aligned}\mathbb{P}(A_i) &= \binom{i-1}{2} p^2 (1-p)^{i-3}(1 + O(mp)) \\ &\leq \frac{(i-1)^2}{2} p^2 (1 + O(mp)).\end{aligned}$$

Now let $A = \bigcap_{i=1}^m A_i$. Note that the events A_i are mutually independent, since they are determined by disjoint edge-sets. Thus we have:

$$\begin{aligned}\mathbb{P}(A) = \prod_{i=3}^m \mathbb{P}(A_i) &\leq \prod_{i=3}^m \left(\frac{(i-1)^2}{2} p^2 (1 + O(mp))\right) \\ &= \frac{((m-1)!)^2 p^{2m-4}}{2^{m-2}} (1 + O(m^2 p)),\end{aligned}$$

where in the last line we used the equality $(1 + O(mp))^{m-2} = 1 + O(m^2 p)$ to bound the error term. Thus we have that the expected number X of ordered m–tuples of vertices from Γ for which A holds is at most:

$$\begin{aligned}\mathbb{E}(X) = \frac{n!}{(n-m)!}\mathbb{P}(A) &\le n^m 4e^2 \left(\frac{m^2 p^{2-\frac{4}{m}}}{e^2 2}\right)^m (1+O(m^2p)) \\ &= 4e^2 \left(\frac{nm^2p^{2-4/m}}{2e^2}\right)^m (1+O(m^2p)),\end{aligned}$$

where in the first line we used the inequality $(m-1)! \le e(m/e)^m$. We now consider the quantity

$$f(n,m,p) = \frac{nm^2p^{2-4/m}}{2e^2}$$

which is raised to the m^{th} power in the inequality above. We claim that $f(n,m,p) \le e^{-1+\log 2+o(1)}$. We have two cases to consider:

Case 1: $m = \lceil 4\log n \rceil$**.** Since $4\log n \le 4\log(1/p)$, we deduce that $p \le n^{-1}$. Then

$$f(n,m,p) = \frac{n(4\log n)^2 p^{2-o(1)}}{2e^2} \le n^{-1+o(1)} \le e^{-1+\log 2+o(1)}.$$

Case 2: $m = \lceil 4\log(1/p) \rceil$**.** First, note that $p^{-4/m} = \exp\left(\frac{4\log(1/p)}{\lceil 4\log 1/p \rceil}\right) \le e$. Also, for p in the range $[0, n^{-1/2}(\log n)^{-1}]$ and n large enough, $p^2(\log(1/p))^2$ is an increasing function of p and is thus at most:

$$\frac{1}{n(\log n)^2}\left(\frac{1}{2}\log n\right)^2\left(1+O\left(\frac{\log\log n}{\log n}\right)\right) = \frac{1}{4}n^{-1}(1+o(1)).$$

Plugging this into the expression for $f(n,m,p)$, we obtain:

$$\begin{aligned}f(n,m,p) &= (1+o(1))\frac{16n(\log(1/p))^2p^{2-4/m}}{2e^2} \\ &\le (1+o(1))\frac{2}{e} = e^{-1+\log 2+o(1)}.\end{aligned}$$

Thus, in both cases (1) and (2) we have $f(n,m,p) \le e^{-1+\log 2+o(1)}$, as claimed, whence

$$\mathbb{E}(X) \le 4e^2(f(n,m,p))^m(1+O(m^2p)) \le 4e^2 e^{-(1-\log 2)m+o(m)}(1+O(m^2p)) = o(1).$$

It follows from Markov's inequality that the non-negative, integer-valued random variable X is a.a.s. equal to 0. In other words, a.a.s. there is no ordered m–tuple of vertices in Γ for which the event A holds and, hence by Lemma 4.5, no component in $\square(\Gamma)$ covering more than $m \le 4\log n$ vertices of Γ. ☐

Theorem 4.7. *Suppose* $p \le \frac{1}{\sqrt{n}\log n}$. *Then a.a.s.* $\Gamma \in \mathcal{G}(n,p)$ *is not in* $\mathcal{CFS}$*.*

Proof. To show that $\Gamma \notin \mathcal{CFS}$, we first show that, for $p \le \frac{1}{\sqrt{n}\log n}$, a.a.s. there is no non-empty clique K such that $\Gamma = \Gamma' \star K$. Indeed the standard Chernoff bound guarantees that we have a.a.s. no vertex in Γ with degree greater than $\sqrt{n}$. Thus to prove the theorem, it is enough to show that a.a.s. there is no connected component C in $\square(\Gamma)$ containing all the vertices in Γ. Theorem 4.6 does this by establishing the stronger bound that a.a.s. there is no connected component C covering more than $4\log n$ vertices. ☐

While Theorem 4.7 improves on the trivial lower bound of $n^{-3/4}$, it is still off from the upper bound for the emergence of the $\mathcal{CFS}$ property established in Theorem 4.1. It is a natural question to ask where the correct threshold is located.

Remark 4.8. We strongly believe that there is a sharp threshold for the $\mathcal{CFS}$ property analogous to the one we established for the $\mathcal{AS}$ property. What is more, we believe this threshold should essentially coincide with the threshold for the emergence of a giant component in the auxiliary square graph $\square(\Gamma)$. Indeed, our arguments in Proposition 4.6 and Theorems 4.1 both focus on bounding the growth of a component in $\square(\Gamma)$. Heuristically, we would expect a giant component to emerge in $\square(\Gamma)$ to emerge for $p(n) = cn^{-1/2}$, for some constant $c > 0$, when the expected number of common neighbors of a pair of non-adjacent vertices in Γ is c^2, and thus the expected number of distinct vertices in 4–cycles which meet a fixed 4–cycle in a non-edge is $2c^2$. What the precise value of c should be is not entirely clear (a branching process heuristics suggests $\sqrt{\frac{\sqrt{17}-3}{2}}$ as a possible value, see Remark 5.2), however, and the dependencies in the square graph make its determination a delicate matter.

5. Experiments

Theorem 3.4 and Theorem 3.5 show that $(\log n/n)^{\frac{1}{3}}$ is a sharp threshold for the family $\mathcal{AS}$ and Theorem 4.1 shows that $n^{-\frac{1}{2}}$ is the right order of magnitude of the threshold for $\mathcal{CFS}$. Below we provide some empirical results on the behaviour of random graphs near the threshold for $\mathcal{AS}$ and the conjectured threshold for $\mathcal{CFS}$. We also compare our experimental data with analogous data at the connectivity threshold. Our experiments are based on various algorithms that we implemented in `C++`; the source code is available from the authors[1].

We begin with the observation that computer simulations of $\mathcal{AS}$ and $\mathcal{CFS}$ are tractable. Indeed, it is easily seen that there are polynomial-time algorithms for deciding whether a given graph is in $\mathcal{AS}$ and/or $\mathcal{CFS}$. Testing for $\mathcal{AS}$ by examining each block and determining whether it witnesses $\mathcal{AS}$ takes $O(n^5)$ steps, where n is the number of vertices. The $\mathcal{CFS}$ property is harder to detect, but essentially reduces to determining the component structure of the square graph. The square graph can be produced in polynomial time and, in polynomial time, one can find its connected components and check the support of these components in the original graph.

Using our software, we tested random graphs in $\mathcal{G}(n,p)$ for membership in $\mathcal{AS}$ for $n \in \{300 + 100k \mid 0 \le k \le 12\}$ and $\{p(n) = \alpha\,(\log n/n)^{\frac{1}{3}} \mid \alpha = 0.80 + 0.1k,\ 0 \le k \le 9\}$. For each pair (n,p) of this type, we generated 400 random graphs and tested each for membership in $\mathcal{AS}$. (This number of tests ensures that, with probability approximately 95%, the measured proportion of $\mathcal{AS}$ graphs is within at most 0.05 of the actual proportion.) The results are summarized in Figure 2. The data suggests that, fixing n, the probability that a random graph is in $\mathcal{AS}$ increases monotonically in the range of p we are considering, rising sharply from almost zero to almost one.

In Figure 3, we display the results of testing random graphs for membership in $\mathcal{CFS}$ for $n \in \{100 + 100k \mid 0 \le k \le 15\}$ and $\{p(n) = \alpha n^{-\frac{1}{2}} \mid \alpha = 0.700 + 0.025k,\ 0 \le k \le 8\}$. For each pair (n,p) of this type, we generated 400 random graphs and tested each for membership in $\mathcal{CFS}$. (This number of tests ensures that, with probability approximately 95%, the measured proportion of $\mathcal{AS}$ graphs is within at most 0.05 of the actual proportion.) The data suggests that, fixing n, the probability that a random graph is in $\mathcal{CFS}$ increases monotonically in considered range of p: rising sharply from almost zero to almost one inside a narrow window.

Remark 5.1 (Block and core sizes)**.** For each graph Γ tested, the $\mathcal{AS}$ software also keeps track of how many nonadjacent pairs $\{x,y\}$ — i.e., how many blocks — were tested before finding one sufficient to verify membership in $\mathcal{AS}$; if no such block is found, then all non-adjacent pairs have been tested and the graph is not in $\mathcal{AS}$.[2] At densities near the threshold, this number of

[1] All source code and data at `www.wescac.net/research.html` or `math.columbia.edu/~jason` .

[2] A set of such data comes with the source code, and more is available upon request.

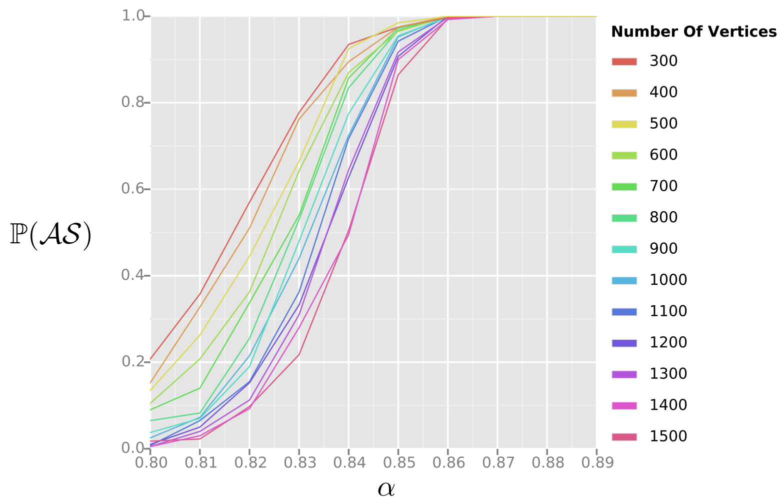

FIGURE 2. Experimental prevalence of $\mathcal{AS}$ at density $\alpha\left(\frac{\log n}{n}\right)^{\frac{1}{3}}$.

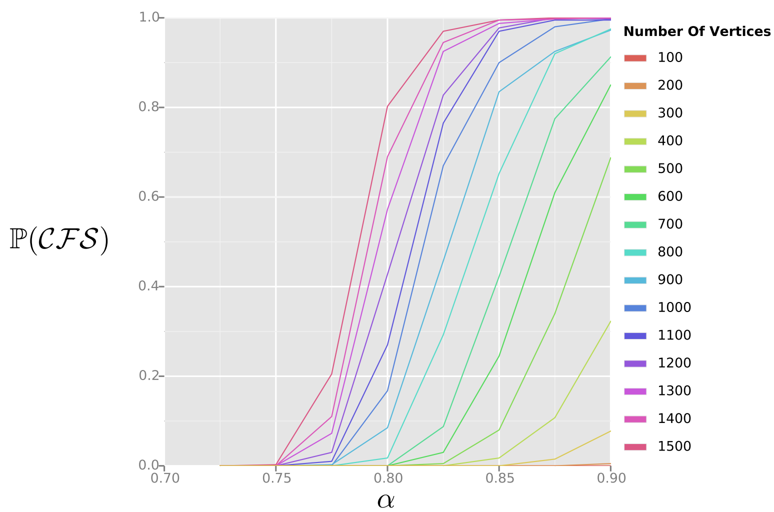

FIGURE 3. Experimental prevalence of $\mathcal{CFS}$ membership at density $\alpha n^{-\frac{1}{2}}$.

blocks is generally very large compared to the number of blocks tested at densities above the threshold. For example, in one instance with $(n, \alpha) = (1000, 0.89)$, verifying that the graph was in $\mathcal{AS}$ was accomplished after testing just 86 blocks, while at $(1000, 0.80)$, a typical test checked all 422961 blocks (expected number: 423397) before concluding that the graph is not $\mathcal{AS}$. At the same (n, α), another test found that the graph was in $\mathcal{AS}$, but only after 281332 tests. This data is consonant with the spirit of our proofs of Theorem 3.5 and Theorem 3.4: in the former case, we showed that no "good" block exists, while in the latter we show that every block is good. We believe that right at the threshold we should have some intermediate behavior, with the expected number of "good" blocks increasing continuously from 0 to $(1-p)\binom{n}{2}(1-o(1))$.

What is more, we expect that the more precise threshold for the $\mathcal{AS}$ property, coinciding with the appearance of a single "good" block, should be located "closer" to our lower bound than to our upper one, i.e., at $p(n) = ((1-\epsilon)\log n/n)^{1/3}$, where $\epsilon(n)$ is a sequence of strictly positive real numbers tending to 0 as $n \to \infty$ (most likely decaying at a rate just faster than $(\log n)^{-2}$, see below). Our experimental data, which exhibit a steep rise in $\mathbb{P}(\mathcal{AS})$ strictly before the value α hits one, gives some support to this guess.

Finally, our observations on the number of blocks suggests a natural way to understand the influence of higher-order terms on the emergence of the $\mathcal{AS}$ property: at exactly the threshold for $\mathcal{AS}$, the event $E(v,w)$ that a pair of non-adjacent vertices $\{v,w\}$ gives rise to a "good" block is rare and, despite some mild dependencies, the number N of pairs $\{v,w\}$ for which $E(v,w)$ occurs is very likely to be distributed approximatively like a Poisson random variable. The probability $\mathbb{P}(N \geq 1)$ would then be a very good approximation for $\mathbb{P}(\mathcal{AS})$. "Good" blocks would thus play a role for the emergence of the $\mathcal{AS}$ property in random graphs analogous to that of isolated vertices for connectivity in random graphs.

When $p = ((1-\epsilon)\log n/n)^{\frac{1}{3}}$, the expectation of N is roughly $ne^{-n^{\epsilon}(1-\epsilon)\log n}$. This expectation is $o(1)$ when $0 < \epsilon(n) = \Omega(1/n)$ and is $1/2$ when $\epsilon(n) = (1+o(1))\log 2/(\log n)^2$. This suggest that the emergence of $\mathcal{AS}$ should occur when $\epsilon(n)$ decays just a little faster than $(\log n)^2$.

Remark 5.2. Our data suggests that the prevalence of $\mathcal{CFS}$ is closely related to the emergence of a giant component in the square graph. Indeed, below the experimentally observed threshold for $\mathcal{CFS}$, not only is the support of the largest component in the square graph not all of $\Gamma \in \mathcal{G}(n,p)$, but in fact the size of the support of the largest component is an extemely small proportion of the vertices (see Figure 4).

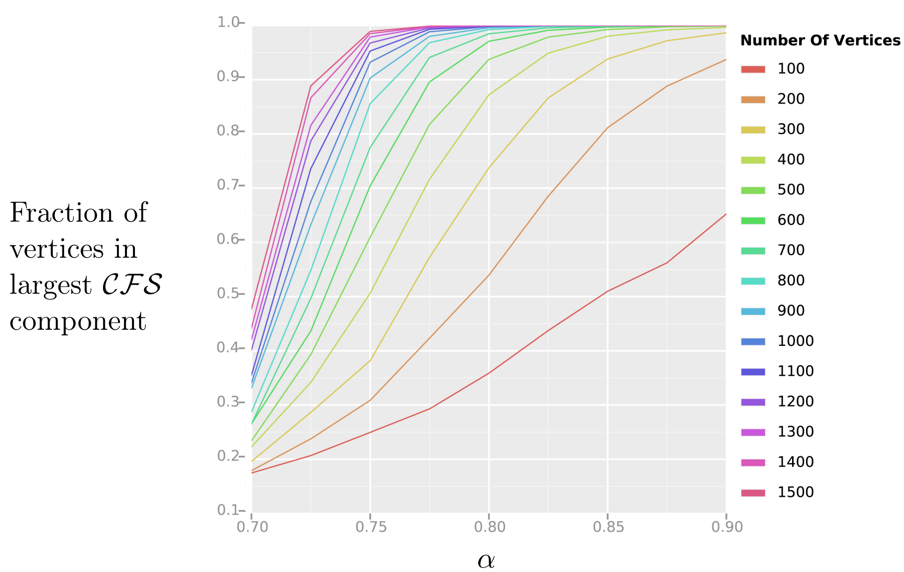

FIGURE 4. Fraction of vertices supporting the largest $\mathcal{CFS}$–subgraph at density $\alpha n^{-\frac{1}{2}}$.

In the Erdős–Rényi random graph, a giant component emerges when p is around $1/n$, i.e., when vertices begin to expect at least one neighbor; this corresponds to a paradigmatic condition of expecting at least one child for survival of a Galton–Watson process (see [9] for a modern treatment of the topic). The heuristic observation that when $p = cn^{-1/2}$ the vertices of a diagonal e in a fixed 4-cycle F expect to be adjacent to cn^2 vertices outside the 4-cycle, giving rise to an expected $\binom{cn^2+2}{2} - 1$ new 4-cycles connected to F through e in $\square(\Gamma)$ suggests that $c = \sqrt{\frac{\sqrt{17}-3}{2}} \approx 0.7494$ could be a reasonable guess for the location of the threshold for the $\mathcal{CFS}$ property. Our data, although not definitive, appears somewhat supportive of this guess: see Figure 4 which is based on the same underlying data set as Figure 3.

We note that unlike an Erdős–Rényi random graph, the square graph $\square(\Gamma)$ exhibits some strong local dependencies, which may make the determination of the exact location of its phase transition a much more delicate affair.

Remark 5.3. For comparison with the threshold data for $\mathcal{AS}$ and $\mathcal{CFS}$, we include below a similar figure of experimental data for connectivity for α from 0.8 to 1.4, where $p = \frac{\alpha \log n}{n}$. Given what we know about the thresholds for connectivity and the $\mathcal{AS}$ property, this last set of data together with Figure 2 should serve as a warning not to draw overly strong conclusions: the graphs we tested are sufficiently large for the broader picture to emerge, but probably not large enough to allow us to pinpoint the exact location of the threshold for $\mathcal{CFS}$.

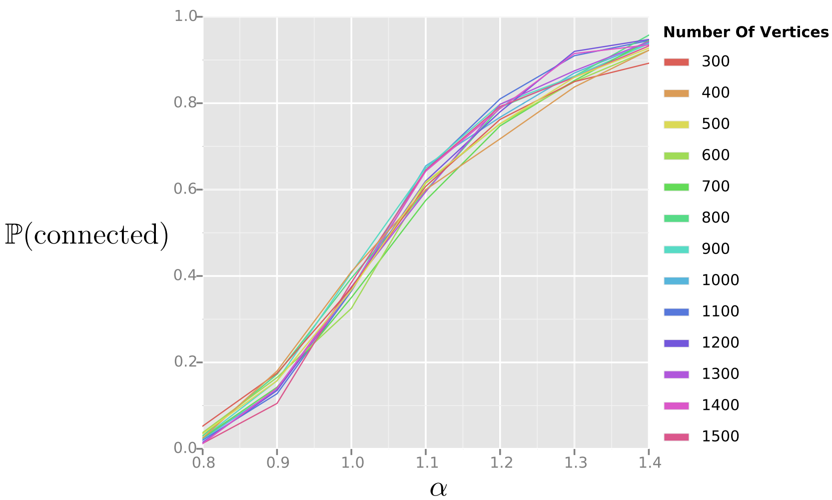

FIGURE 5. Experimental prevalence of connectedness at density $\alpha \frac{\log n}{n}$.

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