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Median structures on asymptotic cones and homomorphisms into mapping class groups

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Abstract

The main goal of this paper is a detailed study of asymptotic cones of the mapping class groups. In particular, we prove that every asymptotic cone of a mapping class group has a bi-Lipschitz equivariant embedding into a product of real trees, sending limits of hierarchy paths onto geodesics, and with image a median subspace. One of the applications is that a group with Kazhdan's property (T) can have only finitely many pairwise non-conjugate homomorphisms into a mapping class group. We also give a new proof of the rank conjecture of Brock and Farb (previously proved by Behrstock and Minsky, and independently by Hamenstaedt).

Contents

1. Introduction . . . 1
2. Background . . . 5
3. Tree-graded metric spaces . . . 12
4. Asymptotic cones of mapping class groups . . . 16
5. Actions on asymptotic cones of mapping class groups and splitting . . . 30
6. Subgroups with property (T) . . . 36
References . . . 50

1. *Introduction*

Mapping class groups of surfaces (denoted in this paper by $\mathcal{MCG}(S)$, where S always denotes a compact connected orientable surface) are very interesting geometric objects, whose geometry is not yet completely understood. Aspects of their geometry are especially striking when compared with lattices in semi-simple Lie groups. Mapping class groups are known to share some properties with lattices in rank 1 and some others with lattices in higher rank semi-simple Lie groups. For instance, the intersection pattern of quasi-flats in $\mathcal{MCG}(S)$ is reminiscent of intersection patterns of quasi-flats in uniform lattices of higher rank semi-simple groups [**7**]. On the other hand, the pseudo-Anosov elements in $\mathcal{MCG}(S)$ are 'rank 1 elements', that is, the cyclic subgroups generated by them are quasi-geodesics satisfying the Morse property [**4**, **26**, **33**].

Non-uniform lattices in rank 1 semi-simple groups are (strongly) relatively hyperbolic [**32**]. As mapping class groups act by isometries on their complex of curves, and the latter are hyperbolic [**43**], mapping class groups are weakly relatively hyperbolic with respect to finitely many stabilizers of multicurves. On the other hand, mapping class groups are not strongly relatively hyperbolic with respect to any collection of subgroups [**2**, **6**]; they are not even metrically relatively hyperbolic with respect to any collection of subsets [**6**]. Still, $\mathcal{MCG}(S)$

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share further properties with relatively hyperbolic groups. A subgroup of $\mathcal{MCG}(S)$ either contains a pseudo-Anosov element or it is parabolic, that is, it stabilizes a (multi-)curve in S (see [**40**]). A similar property is one of the main 'rank 1' properties of relatively hyperbolic groups [**27**].

Another form of rank 1 phenomenon, which is also a weaker version of relative hyperbolicity, is the existence of cut-points in the asymptotic cones. In an asymptotic cone of a relatively hyperbolic group every point is a cut-point. In general, a set of cut-points $\mathcal{C}$ in a geodesic metric space determines a *tree-graded structure* on that space [**27**, Lemma 2.30], that is, it determines a collection of proper geodesic subspaces, called *pieces*, such that every two pieces intersect in at most one point (contained in $\mathcal{C}$) and every simple loop is contained in one piece. One can consider as pieces maximal path-connected subsets with no cut-points in $\mathcal{C}$, and singletons. When taking the quotient of a tree-graded space $\mathbb{F}$ with respect to the closure of the equivalence relation 'two points are in the same piece' (this corresponds, roughly, to shrinking all pieces to points), one obtains a real tree $T_{\mathbb{F}}$ (see [**25**, Section 2.3]).

It was proved in [**5**] that in the asymptotic cones of mapping class groups every point is a cut-point; consequently such asymptotic cones are tree-graded. The pieces of the corresponding tree-graded structure, that is, the maximal path-connected subsets with no cut-points, are described in [**7**] (see Theorem 4.2 in this paper). Further information about the pieces is contained in Proposition 5.5. The canonical projection of an asymptotic cone $\mathcal{AM}(S)$ of a mapping class group onto the corresponding asymptotic cone $\mathcal{AC}(S)$ of the complex of curves (which is a real tree, since the complex of curves is hyperbolic) is a composition of the projection of $\mathcal{AM}(S)$ seen as a tree-graded space onto the quotient tree described above, which we denote by T_S, and a projection of T_S onto $\mathcal{AC}(S)$. The second projection has large pre-images of singletons, and is therefore very far from being injective (see Remark 4.3).

Asymptotic cones were used to prove quasi-isometric rigidity of lattices in higher rank semi-simple Lie groups (see [**23**, **42**]; also see [**30**, **31**, **55**] for proofs without the use of asymptotic cones) and of relatively hyperbolic groups [**24**, **27**]. Unsurprisingly therefore asymptotic cones of mapping class groups play a central part in the proof of the quasi-isometric rigidity of mapping class groups [**7**, **38**], as well as in the proof of the Brock–Farb rank conjecture that the rank of every quasi-flat in $\mathcal{MCG}(S)$ does not exceed $\xi(S) = 3g + p - 3$, where g is the genus of the surface S and p is the number of punctures [**8**, **38**]. Many useful results about the structure of asymptotic cones of mapping class groups can be found in [**7**, **8**, **38**].

In this paper, we continue the study of asymptotic cones of mapping class groups, and show that the natural metric on every asymptotic cone of $\mathcal{MCG}(S)$ can be deformed in an equivariant and bi-Lipschitz way, so that the new metric space is inside an ℓ_1-product of $\mathbb{R}$-trees, it is a median space and the limits of hierarchy paths become geodesics. To this end, the projection of the mapping class group onto mapping class groups of subsurfaces (see Paragraph 2.5.1) is used to define the projection of an asymptotic cone $\mathcal{AM}(S)$ onto limits $\mathcal{M}(\mathbf{U})$ of sequences of mapping class groups of subsurfaces $\mathbf{U} = (U_n)$. A limit $\mathcal{M}(\mathbf{U})$ is isometric to an asymptotic cone of $\mathcal{MCG}(Y)$ with Y a fixed subsurface; the latter is a tree-graded space, hence $\mathcal{M}(\mathbf{U})$ has a projection onto a real tree $T_{\mathbf{U}}$ obtained by shrinking pieces to points as described previously. These projections allow us to construct an embedding of $\mathcal{AM}(S)$ into an ℓ_1-product of $\mathbb{R}$-trees.

Theorem 1.1 (Theorems 4.16 and 4.25). *The map $\psi : \mathcal{AM}(S) \to \prod_{\mathbf{U}} T_{\mathbf{U}}$ whose components are the canonical projections of $\mathcal{AM}(S)$ onto $T_{\mathbf{U}}$ is a bi-Lipschitz map, when $\prod_{\mathbf{U}} T_{\mathbf{U}}$ is endowed with the ℓ^1-metric. Its image $\psi(\mathcal{AM}(S))$ is a median space. Moreover, ψ maps limits of hierarchy paths onto geodesics in $\prod_{\mathbf{U}} T_{\mathbf{U}}$.*

Remark. The subspaces of a median space that are strongly convex (i.e. for every two points in the subspace all geodesics connecting these points are in the subspace) are automatically median. Convex subspaces (i.e. containing one geodesic for every pair of points) are not necessarily median. The image of the asymptotic cone in Theorem 1.1 is convex but not strongly convex. Indeed in the asymptotic cone every point is a cut-point [Beh06], and the cone itself is not a tree. On the other hand, a strongly convex subspace Y with a (global) cut-point in a product of trees (possibly infinite uncountable) $\prod_{i\in I} T_i$ must be a tree factor $T_j \times \prod_{i\neq j}\{a_i\}$. Indeed, let x be a cut-point of Y and y, z two points in two distinct connected components of $Y \setminus \{x\}$. If there existed at least two indices $i, j \in I$ such that $y_i \neq z_i$ and $y_j \neq z_j$ then y, z could be connected by a geodesic in $Y \setminus \{x\}$, a contradiction. It follows that there exists a unique $i \in I$ such that $\{y_i, x_i, z_i\}$ are pairwise distinct. By repeating the argument for every two points in every two components of $Y \setminus \{x\}$ we obtain the result.

The bi-Lipschitz equivalence between the limit metric on $\mathcal{AM}(S)$ and the pullback of the ℓ^1-metric on $\prod_{\mathbf{U}} T_{\mathbf{U}}$ yields a distance formula in the asymptotic cone $\mathcal{AM}(S)$, similar to the Masur–Minsky distance formula for the marking complex [**44**].

The embedding ψ allows us to give in Subsection 4.2 an alternative proof of the Brock–Farb conjecture, which essentially follows the ideas outlined in [**4**]. We prove that the covering dimension of any locally compact subset of any asymptotic cone $\mathcal{AM}(S)$ does not exceed $\xi(S)$ (Theorem 4.24). This is done by showing that, for every compact subset K of $\mathcal{AM}(S)$ and every $\epsilon > 0$, there exists an *ϵ-map* $f : K \to X$ (that is, a continuous map with diameter of $f^{-1}(x)$ at most ϵ for every $x \in X$) from K to a product of finitely many $\mathbb{R}$-trees X such that $f(K)$ is of dimension at most $\xi(S)$. This, by a standard statement from dimension theory, implies that the dimension of $\mathcal{AM}(S)$ is at most $\xi(S)$.

One of the typical 'rank 1' properties of groups is the following result essentially due to Bestvina [**10**] and Paulin [**53**]: if a group A has infinitely many injective homomorphisms $\phi_1, \phi_2, \ldots$ into a hyperbolic group G which are pairwise non-conjugate in G, then A splits over a virtually abelian subgroup. The reason for this is that A acts without a global fixed point on an asymptotic cone of G (which is an $\mathbb{R}$-tree) by the natural action

$$a \cdot (x_i) = (\phi_i(a)x_i). \tag{1}$$

Similar statements hold for relatively hyperbolic groups (see [**9**, **22**, **25**, **34**–**36**, **52**]).

It is easy to see that this statement does not hold for mapping class groups. Indeed, consider the right-angled Artin group B corresponding to a finite graph Γ (B is generated by the set X of vertices of Γ subject to commutativity relations: two generators commute if and only if the corresponding vertices are adjacent in Γ). There clearly exist a surface S and a collection of curves X_S in one-to-one correspondence with X such that two curves α and β from X_S are disjoint if and only if the corresponding vertices in X are adjacent (see [**20**]). Let d_α, with $\alpha \in X$, be the Dehn twist corresponding to the curve α. Then every map $X \to \mathcal{MCG}(S)$, such that $\alpha \mapsto d_\alpha^{k_\alpha}$ for some integer k_α, extends to a homomorphism $B \to \mathcal{MCG}(S)$. For different choices of the k_α one obtains homomorphisms that are pairwise non-conjugate in $\mathcal{MCG}(S)$ (many of these homomorphisms are injective by [**20**]). This set of homomorphisms can be further increased by changing the collection of curves X_S (while preserving the intersection patterns), or by considering more complex subsurfaces of S than just simple closed curves (equivalently, annuli around those curves), and replacing Dehn twists with pseudo-Anosovs on those subsurfaces. Thus B has many pairwise non-conjugate homomorphisms into $\mathcal{MCG}(S)$. But the group B does not necessarily split over any 'nice' (for example, abelian, small, etc.) subgroup.

Nevertheless, if a group A has infinitely many pairwise non-conjugate homomorphisms into a mapping class group $\mathcal{MCG}(S)$, then it acts as in (1) without a global fixed point on an asymptotic cone $\mathcal{AM}(S)$ of $\mathcal{MCG}(S)$. Since $\mathcal{AM}(S)$ is a tree-graded space, we apply the theory of actions of groups on tree-graded spaces from [**25**]. We prove (Corollary 5.16) that in this case either A is virtually abelian, or it splits over a virtually abelian subgroup or the action (1) fixes a piece of $\mathcal{AM}(S)$ set-wise.

We also prove, in Section 6, that the action (1) of A has unbounded orbits, via an inductive argument on the complexity of the surface S. The main ingredient is a careful analysis of the sets of fixed points of pure elements in A, comprising a proof of the fact that a pure element in A with bounded orbits has fixed points (Lemmas 6.15 and 6.17), a complete description of the sets of fixed points of pure elements in A (Lemmas 6.16 and 6.18), and an argument showing that distinct pure elements in A with no common fixed point generate a group with infinite orbits (Lemma 6.23 and Proposition 6.8). The last argument follows the same outline as the similar result holding for isometries of an $\mathbb{R}$-tree, although it is much more complex.

The above and the fact that a group with property (T) cannot act on a median space with unbounded orbits (see Section 6 for references) allow us to apply our results in the case when A has property (T).

THEOREM 1.2 (Corollary 6.3). *A finitely generated group with property (T) has at most finitely many pairwise non-conjugate homomorphisms into a mapping class group.*

Groves announced the same result (personal communication).

A result similar to Theorem 1.2 is that, given a one-ended group A, there are finitely many pairwise non-conjugate injective homomorphisms $A \to \mathcal{MCG}(S)$ such that every non-trivial element in A has a pseudo-Anosov image [**15**, **17**, **21**]. Both this result and Theorem 1.2 should be seen as evidence that there are few subgroups (if any) with these properties in the mapping class group of a surface. We recall that the mapping class group itself does not have property (T) (see [**1**]).

Via Theorem 1.2, an affirmative answer to the following natural question would yield a new proof of Andersen's result that $\mathcal{MCG}(S)$ does not have property (T).

QUESTION 1.3. Given a surface S, do there exist a surface S' and an infinite set of pairwise non-conjugate homomorphisms from $\mathcal{MCG}(S)$ into $\mathcal{MCG}(S')$?

See [**3**] for some interesting constructions of non-trivial homomorphisms from the mapping class group of one surface into the mapping class group of another, but note that their constructions only yield finitely many conjugacy classes of homomorphisms for fixed S and S'.

Organization of the paper. In Section 2 we recall results on asymptotic cones, complexes of curves and mapping class groups, while in Section 3 we recall properties of tree-graded metric spaces and prove new results on groups of isometries of such spaces. In Section 4 we prove Theorem 1.1, and we give a new proof that the dimension of any locally compact subset of any asymptotic cone of $\mathcal{MCG}(S)$ is at most $\xi(S)$. This provides a new proof of the Brock–Farb conjecture. In Section 5 we describe further the asymptotic cones of $\mathcal{MCG}(S)$ and deduce that for groups not virtually abelian nor splitting over a virtually abelian subgroup sequences of pairwise non-conjugate homomorphism into $\mathcal{MCG}(S)$ induce an action on the asymptotic cone fixing a piece (Corollary 5.16). In Section 6 we study in more detail actions on asymptotic cones of $\mathcal{MCG}(S)$ induced by sequences of pairwise non-conjugate homomorphisms, focusing on the relationship between bounded orbits and existence either of fixed points or of fixed multicurves. We prove that if a group G is not virtually cyclic and has infinitely many pairwise non-conjugate homomorphisms into $\mathcal{MCG}(S)$, then G acts on the asymptotic cone of $\mathcal{MCG}(S)$ viewed as a median space with unbounded orbits. This allows us to prove Theorem 1.2.

2. *Background*

2.1. *Asymptotic cones*

A *non-principal ultrafilter* ω over a countable set I is a finitely additive measure on the class $\mathcal{P}(I)$ of subsets of I, such that each subset has measure either 0 or 1 and all finite sets have measure 0. Since we only use non-principal ultrafilters, the word non-principal will be omitted in what follows.

If a statement $S(i)$ is satisfied for all i in a set J with $\omega(J) = 1$, then we say that $S(i)$ holds ω-almost surely (a.s.).

Given a sequence of sets $(X_n)_{n\in I}$ and an ultrafilter ω, the *ultraproduct corresponding to* ω, $\Pi X_n/\omega$, consists of equivalence classes of sequences $(x_n)_{n\in I}$, with $x_n \in X_n$, where two sequences (x_n) and (y_n) are identified if $x_n = y_n$ ω-a.s. The equivalence class of a sequence $x = (x_n)$ in $\Pi X_n/\omega$ is denoted either by x^ω or by $(x_n)^\omega$. In particular, if all X_n are equal to the same X, then the ultraproduct is called the *ultrapower* of X and it is denoted by $\Pi X/\omega$.

If G_n, with $n \geqslant 1$, are groups, then $\Pi G_n/\omega$ is again a group with the multiplication law $(x_n)^\omega (y_n)^\omega = (x_n y_n)^\omega$.

If $\Re$ is a relation on X, then one can define a relation $\Re_\omega$ on $\Pi X/\omega$ by setting $(x_n)^\omega \Re_\omega (y_n)^\omega$ if and only if $x_n \Re y_n$ ω-a.s.

Lemma 2.1 [**18**, Lemma 6.5]. *Let ω be an ultrafilter over I and let $(X_n)_{n\in I}$ be a sequence of sets which ω-a.s. have cardinality at most N. Then the ultraproduct $\Pi X_i/\omega$ has cardinality at most N.*

For every sequence of points $(x_n)_{n\in I}$ in a topological space X, its *ω-limit*, denoted by $\lim_\omega x_n$, is a point x in X such that every neighbourhood U of x contains x_n for ω-almost every n. If a metric space X is Hausdorff and (x_n) is a sequence in X, then when the ω-limit of (x_n) exists, it is unique. In a compact metric space every sequence has an ω-limit [**14**, § I.9.1].

Definition 2.2 (Ultraproduct of metric spaces). Let (X_n, dist_n), with $n \in I$, be a sequence of metric spaces and let ω be an ultrafilter over I. Consider the ultraproduct $\Pi X_n/\omega$. For every two points $x^\omega = (x_n)^\omega, y^\omega = (y_n)^\omega$ in $\Pi X_n/\omega$ let

$$D(x^\omega, y^\omega) = \lim_\omega \mathrm{dist}_n(x_n, y_n).$$

The function $D : \Pi X_n/\omega \times \Pi X_n/\omega \to [0, \infty]$ is a pseudo-metric, that is, it satisfies all the properties of a metric except that it can take infinite values and need not satisfy $D(x^\omega, y^\omega) = 0 \Rightarrow x^\omega = y^\omega$. We may make the function D take only finite values by restricting it to a subset of the ultrapower in the following way.

Consider an *observation point* $e = (e_n)^\omega$ in $\Pi X_n/\omega$ and define $\Pi_e X_n/\omega$ to be the subset of $\Pi X_n/\omega$ consisting of elements that are finite distance from e with respect to D. Note that transitivity of D implies that the distance is finite between any pair of points in $\Pi_e X_n/\omega$.

Note that if X is a group G endowed with a word metric, then $\Pi_1 G/\omega$ is a subgroup of the ultrapower of G.

Definition 2.3 (Ultralimit of metric spaces). The *ω-limit of the metric spaces* (X_n, dist_n) *relative to the observation point* e is the metric space obtained from $\Pi_e X_n/\omega$ by identifying all pairs of points x^ω, y^ω with $D(x^\omega, y^\omega) = 0$; this space is denoted by $\lim_\omega(X_n, e)$. When there is no need to specify the ultrafilter ω, then we also call $\lim_\omega(X_n, e)$ *ultralimit of* (X_n, dist_n) *relative to* e.

The equivalence class of a sequence $x = (x_n)$ in $\lim_\omega(X_n, e)$ is denoted either by $\lim_\omega x_n$ or by $\boldsymbol{x}$.

Note that if $e, e' \in \Pi X_n/\omega$ and $D(e, e') < \infty$, then $\lim_\omega(X_n, e) = \lim_\omega(X_n, e')$.

Definition 2.4. For a sequence $A = (A_n)_{n\in I}$ of subsets $A_n \subset X_n$, we write $\boldsymbol{A}$ or $\lim_\omega A_n$ to denote the subset of $\lim_\omega(X_n, e)$ consisting of all the elements $\lim_\omega x_n$ such that $\lim_\omega x_n \in A_n$.

Note that if $\lim_\omega \mathrm{dist}_n(e_n, A_n) = \infty$, then the set $\lim_\omega(A_n)$ is empty.

Any ultralimit of metric spaces is a complete metric space [**56**]. The same proof gives that $\boldsymbol{A} = \lim_\omega(A_n)$ is always a closed subset of the ultralimit $\lim_\omega(X_n, e)$.

Definition 2.5 (Asymptotic cone). Let (X, dist) be a metric space, ω be an ultrafilter over a countable set I and $e = (e_n)^\omega$ be an observation point. Consider a sequence of numbers $d = (d_n)_{n\in I}$ called *scaling constants* satisfying $\lim_\omega d_n = \infty$.

The space $\lim_\omega(X, (1/d_n)\mathrm{dist}, e)$ is called an *asymptotic cone of X*. It is denoted by $\mathrm{Con}^\omega(X; e, d)$.

Remark 2.6. Let G be a finitely generated group endowed with a word metric.

(1) The group $\Pi_1 G/\omega$ acts on $\mathrm{Con}^\omega(G; 1, d)$ transitively by isometries:

$$(g_n)^\omega \lim_\omega(x_n) = \lim_\omega(g_n x_n).$$

(2) Given an arbitrary sequence of observation points e, the group $e^\omega(\Pi_1 G/\omega)(e^\omega)^{-1}$ acts transitively by isometries on the asymptotic cone $\mathrm{Con}^\omega(G; e, d)$. In particular, every asymptotic cone of G is homogeneous.

Convention 2.7. By the above remark, when we consider an asymptotic cone of a finitely generated group, it is no loss of generality to assume that the observation point is $(1)^\omega$. We shall do this unless explicitly stated otherwise.

2.2. *The complex of curves*

Throughout $S = S_{g,p}$ will denote a compact connected orientable surface of genus g and with p boundary components. Subsurfaces $Y \subset S$ will always be considered to be essential (that is, with non-trivial fundamental group which injects into the fundamental group of S), also they will not be assumed to be proper unless explicitly stated. We will often measure the *complexity of a surface* by $\xi(S_{g,p}) = 3g + p - 3$; this complexity is additive under disjoint union. Surfaces and curves are always considered up to homotopy unless explicitly stated otherwise; we refer to a pair of curves (surfaces, etc.) intersecting if they have non-trivial intersection independent of the choice of representatives.

The (1-skeleton of the) *complex of curves* of a surface S, denoted by $\mathcal{C}(S)$, is defined as follows. The set of vertices of $\mathcal{C}(S)$, denoted by $\mathcal{C}_0(S)$, is the set of homotopy classes of essential non-peripheral simple closed curves on S. When $\xi(S) > 1$, a collection of $n+1$ vertices span an n-simplex if the corresponding curves can be realized (by representatives of the homotopy classes) disjointly on S. A simplicial complex is quasi-isometric to its 1-skeleton, so when it is convenient, we abuse notation and use the term complex of curves to refer to its 1-skeleton.

A *multicurve* on S are the homotopy classes of curves associated to a simplex in $\mathcal{C}(S)$.

If $\xi(S) = 1$, then two vertices are connected by an edge if they can be realized, so that they intersect in the minimal possible number of points on the surface S (that is, (1) if $S = S_{1,1}$ and (2) if $S = S_{0,4}$). If $\xi(S) = 0$, then $S = S_{1,0}$ or $S = S_{0,3}$. In the first case we do the same as for $\xi(S) = 1$ and in the second case the curve complex is empty since this surface does not support any essential simple closed curve. The complex is also empty if $\xi(S) \leqslant -2$.

Finally if $\xi(S) = -1$, then S is an annulus. We only consider the case when the annulus is a subsurface of a surface S'. In this case we define $\mathcal{C}(S)$ by looking in the annular cover $\tilde{S}'$ in which S lifts homeomorphically. We use the compactification of the hyperbolic plane as the closed unit disk to obtain a closed annulus $\hat{S}'$.

We define the vertices of $\mathcal{C}(S)$ to be the homotopy classes of arcs connecting the two boundary components of $\hat{S}'$, where the homotopies are required to fix the endpoints. We define a pair of vertices to be connected by an edge if they have representatives which can be realized with disjoint interior. Metrizing edges to be isometric to the unit interval, this space is quasi-isometric to $\mathbb{Z}$.

A fundamental result on the curve complex is the following theorem.

Theorem 2.8 (Masur and Minsky [**43**]). *For any surface S, the complex of curves of S is an infinite-diameter δ-hyperbolic space* (*as long as it is non-empty*), *with δ depending only on $\xi(S)$.*

There is a particular family of geodesics in a complex of curves $\mathcal{C}(S)$ called *tight geodesics*. We use Bowditch's notion of tight geodesic, as defined in [**16**, § 1]. Although $\mathcal{C}(S)$ is a locally infinite complex, some finiteness phenomena appear when restricting to the family of tight geodesics [**16**, **44**].

2.3. *Projection on the complex of curves of a subsurface*

We shall need the natural projection of a curve (multicurve) γ onto an essential subsurface $Y \subseteq S$ defined in [**44**]. By definition, the projection $\pi_{\mathcal{C}(Y)}(\gamma)$ is a possibly empty subset of $\mathcal{C}(Y)$.

The definition of this projection is from [**44**, § 2] and is given below. Roughly speaking, the projection consists of all closed curves of the intersections of γ with Y together with all the arcs of $\gamma \cap Y$ combined with parts of the boundary of Y (to form essential non-peripheral closed curves).

Definition 2.9. Fix an essential subsurface $Y \subset S$. Given an element $\gamma \in \mathcal{C}(S)$, we define the projection $\pi_{\mathcal{C}(Y)}(\gamma) \in 2^{\mathcal{C}(Y)}$ as follows.

(1) If either $\gamma \cap Y = \emptyset$, or Y is an annulus and γ is its core curve (that is, γ is the unique homotopy class of essential simple closed curve in Y), then we define $\pi_{\mathcal{C}(Y)}(\gamma) = \emptyset$.

(2) If Y is an annulus that transversally intersects γ, then we define $\pi_{\mathcal{C}(Y)}(\gamma)$ to be the union of the elements of $\mathcal{C}(Y)$ defined by every lifting of γ to an annular cover as in the definition of the curve complex of an annulus.

(3) In all remaining cases, consider the arcs and simple closed curves obtained by intersecting Y and γ. We define $\pi_{\mathcal{C}(Y)}(\gamma)$ to be the set of vertices in $\mathcal{C}(Y)$ consisting of:

- (i) the essential non-peripheral simple closed curves in the collection above;
- (ii) the essential non-peripheral simple closed curves obtained by taking arcs from the above collection union a subarc of ∂Y (that is, those curves that can be obtained by resolving the arcs in $\gamma \cap Y$ to curves).

One of the basic properties is the following result of Masur–Minsky (see [**44**, Lemma 2.3] for the original proof, in [**46**] the bound was corrected from 2 to 3).

Lemma 2.10 [**44**]. *If Δ is a multicurve in S and Y is a subsurface of S intersecting non-trivially every homotopy class of curves composing Δ, then*

$$\mathrm{diam}_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\Delta)) \leqslant 3.$$

Notation 2.11. Given Δ, Δ' a pair of multicurves in $\mathcal{C}(S)$, for brevity we often write $\mathrm{dist}_{\mathcal{C}(Y)}(\Delta, \Delta')$ instead of $\mathrm{dist}_{\mathcal{C}(Y)}(\pi_Y(\Delta), \pi_Y(\Delta'))$.

2.4. *Mapping class groups*

The *mapping class group* $\mathcal{MCG}(S)$ of a surface S of finite type is the quotient of the group of homeomorphisms of S by the subgroup of homeomorphisms isotopic to the identity. Since the mapping class group of a surface of finite type is finitely generated [**12**], we may consider a word metric on the group; this metric is unique up to bi-Lipschitz equivalence. Note that $\mathcal{MCG}(S)$ acts on $\mathcal{C}(S)$ by simplicial automorphisms (in particular by isometries) and with finite quotient, and that the family of tight geodesics is invariant with respect to this action.

Recall that, according to the Nielsen–Thurston classification, any element $g \in \mathcal{MCG}(S)$ satisfies one of the following three properties, where the first two are not mutually exclusive:

(1) g has finite order;
(2) there exists a multicurve Δ in $\mathcal{C}(S)$ invariant by g (in this case g is called *reducible*);
(3) g is pseudo-Anosov.

We call an element $g \in \mathcal{MCG}(S)$ *pure* if there exists a multicurve Δ (possibly empty) component-wise invariant by g and such that g does not permute the connected components of $S \setminus \Delta$, and it induces on each component of $S \setminus \Delta$ and on each annulus with core curve in Δ either a pseudo-Anosov or the identity map (we use the convention that a Dehn twist on an annulus is considered to be pseudo-Anosov). In particular every pseudo-Anosov is pure.

Theorem 2.12 [**40**, Corollary 1.8; **41**, Theorem 7.1.E]. *Consider the homomorphism from $\mathcal{MCG}(S)$ to the finite group $\mathrm{Aut}(H_1(S, \mathbb{Z}/k\mathbb{Z}))$ defined by the action of diffeomorphisms on homology.*

If $k \geqslant 3$, then the kernel $\mathcal{MCG}_k(S)$ of the homomorphism is composed only of pure elements; in particular it is torsion free.

We now show that versions of some of the previous results hold for the ultrapower $\Pi\mathcal{MCG}(S)/\omega$ of a mapping class group. The elements in $\mathcal{MCG}(S)^\omega$ can also be classified into finite order, reducible and pseudo-Anosov elements, according to whether their components satisfy that property ω-a.s.

Similarly, one may define pure elements in $\mathcal{MCG}(S)^\omega$. Note that non-trivial pure elements both in $\mathcal{MCG}(S)$ and in its ultrapower are of infinite order.

Theorem 2.12 implies the following statements.

Lemma 2.13. (1) *The ultrapower $\mathcal{MCG}(S)^\omega$ contains a finite index normal subgroup $\mathcal{MCG}(S)^\omega_p$ which consists only of pure elements.*

(2) *The orders of finite subgroups in the ultrapower $\mathcal{MCG}(S)^\omega$ are bounded by a constant $N = N(S)$.*

Proof. (1) The homomorphism in Theorem 2.12, for $k \geqslant 3$, induces a homomorphism from $\mathcal{MCG}(S)^\omega$ to a finite group whose kernel consists only of pure elements.

(2) Since any finite subgroup of $\mathcal{MCG}(S)^\omega$ has trivial intersection with the group of pure elements $\mathcal{MCG}(S)^\omega_p$, it follows that it injects into the quotient group, hence its cardinality is at most the index of $\mathcal{MCG}(S)^\omega_p$. □

2.5. *The marking complex*

For most of what follows, we do not work with the mapping class group directly, but rather with a particular quasi-isometric model which is a graph called the *marking complex* $\mathcal{M}(S)$ which is defined as follows.

The vertices of the marking graph are called *markings*. Each marking $\mu \in \mathcal{M}(S)$ consists of the following pair of data.

(i) *Base curves*: a multicurve consisting of $3g + p - 3$ components, that is, a maximal simplex in $\mathcal{C}(S)$. This collection is denoted by $\operatorname{base}(\mu)$.

(ii) *Transversal curves*: to each curve $\gamma \in \operatorname{base}(\mu)$ is associated an essential curve in the complex of curves of the annulus with core curve γ with a certain compatibility condition. More precisely, letting T denote the complexity 1 component of $S \setminus \bigcup_{\alpha\in \operatorname{base}\mu, \alpha\neq\gamma} \alpha$, the transversal curve to γ is any curve $t(\gamma) \in \mathcal{C}(T)$ with $\operatorname{dist}_{\mathcal{C}(T)}(\gamma, t(\gamma)) = 1$; since $t(\gamma) \cap \gamma \neq \emptyset$, the curve $t(\gamma)$ is a representative of a point in the curve complex of the annulus about γ, that is, $t(\gamma) \in \mathcal{C}(\gamma)$.

We define two vertices $\mu, \nu \in \mathcal{M}(S)$ to be connected by an edge if either of the following two conditions holds.

(1) *Twists*: μ and ν differ by a Dehn twist along one of the base curves; that is, $\operatorname{base}(\mu) = \operatorname{base}(\nu)$ and all their transversal curves agree except for about one element $\gamma \in \operatorname{base}(\mu) = \operatorname{base}(\nu)$, where $t_\mu(\gamma)$ is obtained from $t_\nu(\gamma)$ by twisting once about the curve γ.

(2) *Flips*: the base curves and transversal curves of μ and ν agree except for one pair $(\gamma, t(\gamma)) \in \mu$ for which the corresponding pair consists of the same pair but with the roles of base and transversal reversed. Note that the second condition to be a marking requires that each transversal curve intersect exactly one base curve, but the flip move may violate this condition. It is shown in [**44**, Lemma 2.4] that there is a finite set of natural ways to resolve this issue, yielding a finite (in fact uniformly bounded) number of flip moves which can be obtained by flipping the pair $(\gamma, t(\gamma)) \in \mu$; an edge connects each of these possible flips to μ.

The following result is due to Masur and Minsky [**44**].

THEOREM 2.14. *The graph $\mathcal{M}(S)$ is locally finite and the mapping class group acts cocompactly and properly discontinuously on it. In particular the mapping class group of S endowed with a word metric is quasi-isometric to $\mathcal{M}(S)$ endowed with the simplicial distance, denoted by* $\operatorname{dist}_{\mathcal{M}(S)}$.

NOTATION 2.15. In what follows, we sometimes denote $\operatorname{dist}_{\mathcal{M}(S)}$ by $\operatorname{dist}_{\mathcal{M}}$, when there is no possibility of confusion.

The subsurface projections introduced in Subsection 2.2 allow one to consider the projection of a marking on S to the curve complex of a subsurface $Y \subseteq S$. Given a marking $\mu \in \mathcal{M}(S)$, we define $\pi_{\mathcal{C}(S)}(\mu)$ to be $\operatorname{base}\mu$. More generally, given a subsurface $Y \subset S$, we define $\pi_{\mathcal{C}(Y)}(\mu) = \pi_{\mathcal{C}(Y)}(\operatorname{base}(\mu))$, if Y is not an annulus about an element of $\operatorname{base}(\mu)$; if Y is an annulus about an element $\gamma \in \operatorname{base}(\mu)$, then we define $\pi_{\mathcal{C}(Y)}(\mu) = t(\gamma)$, the transversal curve to γ.

NOTATION 2.16. For two markings $\mu, \nu \in \mathcal{M}(S)$ we often use the following standard simplification of notation:

$$\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu) = \mathrm{dist}_{\mathcal{C}(Y)}(\pi_{\mathcal{C}(Y)}(\mu), \pi_{\mathcal{C}(Y)}(\nu)).$$

REMARK 2.17. By Lemma 2.10, for every marking μ and every subsurface $Y \subseteq S$, the diameter of the projection of μ into $\mathcal{C}(Y)$ is at most 3. This implies that the difference between $\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu)$ as defined above and the Hausdorff distance between $\pi_{\mathcal{C}(Y)}(\mu)$ and $\pi_{\mathcal{C}(Y)}(\nu)$ in $\mathcal{C}(Y)$ is at most 6.

Hierarchies. In the marking complex, there is an important family of quasi-geodesics called *hierarchy paths* which have several useful geometric properties. The concept of hierarchy was first developed by Masur and Minsky [**44**], which the reader may consult for further details. For a survey, see [**45**]. We recall below the properties of hierarchies that we shall use in what follows.

Given two subsets $A, B \subset \mathbb{R}$, a map $f : A \to B$ is said to be *coarsely increasing* if there exists a constant D such that, for each a and b in A satisfying $a + D < b$, we have that $f(a) \leqslant f(b)$. Similarly, we define *coarsely decreasing* and *coarsely monotonic* maps. We say a map between quasi-geodesics is coarsely monotonic if it defines a coarsely monotonic map between suitable nets in their domain.

We say a quasi-geodesic $\mathfrak{g}$ in $\mathcal{M}(S)$ is $\mathcal{C}(U)$*-monotonic* for some subsurface $U \subset S$ if one can associate a geodesic $\mathfrak{t}_U$ in $\mathcal{C}(U)$ which *shadows* $\mathfrak{g}$ in the sense that $\mathfrak{t}_U$ is a path from a vertex of $\pi_U(\mathrm{base}(\mu))$ to a vertex of $\pi_U(\mathrm{base}(\nu))$, and there is a coarsely monotonic map $v : \mathfrak{g} \to \mathfrak{t}_U$ such that $v(\rho)$ is a vertex in $\pi_U(\mathrm{base}(\rho))$ for every vertex $\rho \in \mathfrak{g}$.

Any pair of points $\mu, \nu \in \mathcal{M}(S)$ are connected by at least one hierarchy path. Hierarchy paths are quasi-geodesics with uniform constants depending only on the surface S. One of the important properties of hierarchy paths is that they are $\mathcal{C}(U)$-monotonic for every $U \subseteq S$ and, moreover, the geodesic onto which they project is a tight geodesic.

The following is an immediate consequence of [**44**, Lemma 6.2].

LEMMA 2.18. *There exists a constant $M = M(S)$ such that, if Y is an essential proper subsurface of S and μ and ν are two markings in $\mathcal{M}(S)$ satisfying $\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu) > M$, then any hierarchy path $\mathfrak{g}$ connecting μ to ν contains a marking ρ such that the multicurve $\mathrm{base}(\rho)$ includes the multicurve ∂Y. Furthermore, there exists a vertex v in the geodesic $\mathfrak{t}_{\mathfrak{g}}$ shadowed by $\mathfrak{g}$ for which $v \in \mathrm{base}(\rho)$, and hence satisfying $Y \subseteq S \setminus v$.*

DEFINITION 2.19. Given a constant $K \geqslant M(S)$, where $M(S)$ is the constant from Lemma 2.18, and a pair of markings μ, ν, the subsurfaces $Y \subseteq S$, for which $\mathrm{dist}_{\mathcal{C}(Y)}(\mu, \nu) > K$, are called the *$K$-large domains* for the pair (μ, ν). We say that a hierarchy path *contains a domain* $Y \subseteq S$ if Y is an $M(S)$-large domain between some pair of points on the hierarchy path. Note that, for every such domain, the hierarchy contains a marking whose base contains ∂Y.

The following useful lemma is one of the basic ingredients in the structure of hierarchy paths; it is an immediate consequence of [**44**, Theorem 4.7] and the fact that a chain of nested subsurfaces has length at most $\xi(S)$.

LEMMA 2.20. *Let $\mu, \nu \in \mathcal{M}(S)$ and let ξ be the complexity of S. Then, for every $M(S)$-large domain Y in a hierarchy path $[\mu, \nu]$, there exist at most 2ξ domains that are $M(S)$-large and that contain Y in that path.*

A pair of subsurfaces Y,Z are said to *overlap* if $Y \cap Z \neq \emptyset$ and neither of the two subsurfaces is a subsurface of the other; when Y and Z overlap we write $Y \pitchfork Z$, when they do not overlap we write $Y \not\pitchfork Z$. The following useful theorem was proved by Behrstock [5].

THEOREM 2.21 (Projection estimates [5]). *There exists a constant D depending only on the topological type of a surface S such that, for any two overlapping subsurfaces Y and Z in S, with $\xi(Y) \neq 0 \neq \xi(Z)$, and for any $\mu \in \mathcal{M}(S)$*

$$\min\{\mathrm{dist}_{\mathcal{C}(Y)}(\partial Z, \mu), \mathrm{dist}_{\mathcal{C}(Z)}(\partial Y, \mu)\} \leqslant D.$$

CONVENTION 2.22. In what follows, we assume that the constant $M = M(S)$ from Lemma 2.18 is larger than the constant D from Theorem 2.21.

NOTATION 2.23. Let $a > 1$, b, x, y be positive real numbers. We write $x \leqslant_{a,b} y$ if

$$x \leqslant ay + b.$$

We write $x \approx_{a,b} y$ if and only if $x \leqslant_{a,b} y$ and $y \leqslant_{a,b} x$.

NOTATION 2.24. Let $K, N > 0$ be real numbers. We define $\{\!\{N\}\!\}_K$ to be N if $N \geqslant K$, and 0 otherwise.

The following result is fundamental in studying the metric geometry of the marking complex. It provides a way to compute distance in the marking complex from the distances in the curve complexes of the large domains.

THEOREM 2.25 (Masur and Minsky [44]). *If $\mu, \nu \in \mathcal{M}(S)$, then there exists a constant $K(S)$, depending only on S, such that, for each $K > K(S)$, there exists $a \geqslant 1$ and $b \geqslant 0$ for which*:

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu, \nu) \approx_{a,b} \sum_{Y \subseteq S} \{\!\{\mathrm{dist}_{\mathcal{C}(Y)}(\pi_Y(\mu), \pi_Y(\nu))\}\!\}_K . \quad (2)$$

We now define an important collection of subsets of the marking complex. In what follows by *topological type of a multicurve* Γ we mean the topological type of the pair (S, Γ).

NOTATION 2.26. Let Δ be a simplex in $\mathcal{C}(S)$. We define $\mathcal{Q}(\Delta)$ to be the set of elements of $\mathcal{M}(S)$ whose bases contain Δ.

REMARK 2.27. As noted in [8] a consequence of the distance formula is that the space $\mathcal{Q}(\Delta)$ is quasi-isometric to a coset of a stabilizer in $\mathcal{MCG}(S)$ of a multicurve with the same topological type as Δ. To see this, fix a collection $\Gamma_1, \ldots, \Gamma_n$ of multicurves, where each topological type of multicurve is represented exactly once in this list. Given any multicurve Δ, fix an element $f \in \mathcal{MCG}$ for which $f(\Gamma_i) = \Delta$, for the appropriate $1 \leqslant i \leqslant n$. Now, up to a bounded Hausdorff distance, we have an identification of $\mathcal{Q}(\Delta)$ with $f\mathrm{stab}(\Gamma_i)$ given by the natural quasi-isometry between $\mathcal{M}(S)$ and $\mathcal{MCG}(S)$.

2.6. *Marking projections*

2.6.1. *Projection on the marking complex of a subsurface.* Given any subsurface $Z \subset S$, we define a projection $\pi_{\mathcal{M}(Z)} : \mathcal{M}(S) \to 2^{\mathcal{M}(Z)}$, which sends elements of $\mathcal{M}(S)$ to subsets of $\mathcal{M}(Z)$. Given any $\mu \in \mathcal{M}(S)$, we build a marking on Z in the following way. Choose an element $\gamma_1 \in \pi_Z(\mu)$, and then recursively choose γ_n from $\pi_{Z \setminus \bigcup_{i<n} \gamma_i}(\mu)$, for each $n \leqslant \xi(Z)$. Now take these γ_i to be the base curves of a marking on Z. For each γ_i we define its transversal $t(\gamma_i)$ to be an element of $\pi_{\gamma_i}(\mu)$. This process yields a marking; see [**5**] for details.

Arbitrary choices were made in this construction, but it is proved in [**5**] that there is a uniform constant depending only on $\xi(S)$, so that, given any $Z \subset S$ and any μ, any two choices in building $\pi_{\mathcal{M}(Z)}(\mu)$ lead to elements of $\mathcal{M}(Z)$ whose distance is bounded by this uniform constant. Thus, in what follows the choices made in the construction will be irrelevant.

REMARK 2.28. Given two nested subsurfaces $Y \subset Z \subset S$, the projection of an arbitrary marking μ onto $C(Y)$ is at uniformly bounded distance from the projection of $\pi_{\mathcal{M}(Z)}(\mu)$ onto $C(Y)$. This follows from the fact that in the choice of $\pi_{\mathcal{M}(Z)}(\mu)$ one can start with a base curve in S which intersects Y and hence also determines up to diameter 3 the projection of μ to $C(Y)$.

A similar argument implies that $\pi_{\mathcal{M}(Y)}(\mu)$ is at uniformly bounded distance from $\pi_{\mathcal{M}(Y)}(\pi_{\mathcal{M}(Z)}(\mu))$.

An easy consequence of the distance formula in Theorem 2.25 is the following.

COROLLARY 2.29. *There exist $A \geqslant 1$ and $B \geqslant 0$ depending only on S such that, for any subsurface $Z \subset S$ and any two markings $\mu, \nu \in \mathcal{M}(S)$, the following holds:*

$$\mathrm{dist}_{\mathcal{M}(Z)}(\pi_{\mathcal{M}(Z)}(\mu), \pi_{\mathcal{M}(Z)}(\nu)) \leqslant_{A,B} \mathrm{dist}_{\mathcal{M}(S)}(\mu, \nu).$$

2.6.2. *Projection on a set $\mathcal{Q}(\Delta)$.* Given a marking μ and a multicurve Δ, the projection $\pi_{\mathcal{M}(S\setminus\Delta)}(\mu)$ can be defined as in Paragraph 2.5.1. This allows one to construct a point $\mu' \in \mathcal{Q}(\Delta)$ which, up to a uniformly bounded error, is closest to μ. See [**8**] for details. The marking μ' is obtained by taking the union of the (possibly partial collection of) base curves Δ with transversal curves given by $\pi_\Delta(\mu)$ together with the base curves and transversals given by $\pi_{\mathcal{M}(S\setminus\Delta)}(\mu)$. Note that the construction of μ' requires, for each subsurface W determined by the multicurve Δ, the construction of a projection $\pi_{\mathcal{M}(W)}(\mu)$. As explained in Paragraph 2.5.1 each $\pi_{\mathcal{M}(W)}(\mu)$ is determined up to uniformly bounded distance in $\mathcal{M}(W)$; thus μ' is well defined up to a uniformly bounded ambiguity depending only on the topological type of S.

3. *Tree-graded metric spaces*

3.1. *Preliminaries*

A subset A in a geodesic metric space X is called *geodesic* if every two points in A can be joined by a geodesic contained in A.

DEFINITION 3.1 (Druţu and Sapir [**27**]). Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic proper subsets, called *pieces*, covering $\mathbb{F}$. We say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$ if the following two properties are satisfied.

(T_1) Every two different pieces have at most one point in common.

(T_2) Every simple non-trivial geodesic triangle in $\mathbb{F}$ is contained in one piece.

When there is no risk of confusion as to the set $\mathcal{P}$, we simply say that $\mathbb{F}$ is *tree-graded*.

Note that one can drop the requirement that pieces cover X because one can always add to the collection of pieces $\mathcal{P}$ all the 1-element subsets of X. In some important cases, as for asymptotic cones of metrically relatively hyperbolic spaces [**24**], the pieces of the natural tree-graded structure do not cover X.

We discuss in what follows some of the properties of tree-graded spaces that we shall need further on.

PROPOSITION 3.2 [**27**, Proposition 2.17]. *Property (T_2) can be replaced by the following property.*

(T_2') *For every topological arc $\mathfrak{c} : [0, d] \to \mathbb{F}$ and $t \in [0, d]$, let $\mathfrak{c}[t - a, t + b]$ be a maximal subarc of $\mathfrak{c}$ containing $\mathfrak{c}(t)$ and contained in one piece. Then every other topological arc with the same endpoints as $\mathfrak{c}$ must contain the points $\mathfrak{c}(t - a)$ and $\mathfrak{c}(t + b)$.*

CONVENTION 3.3. In what follows, when speaking about cut-points we always mean global cut-points.

Any complete geodesic metric space with a global cut-point provides an example of a tree-graded metric space, as the following result points out.

LEMMA 3.4 [**27**, Lemma 2.30]. *Let X be a complete geodesic metric space containing at least two points and let $\mathcal{C}$ be a non-empty set of cut-points in X.*

The set $\mathcal{P}$ of all maximal path-connected subsets that are either singletons or such that none of their cut-points belongs to $\mathcal{C}$ is a set of pieces for a tree-graded structure on X.

Moreover, the intersection of any two distinct pieces from $\mathcal{P}$ is either empty or a point from $\mathcal{C}$.

LEMMA 3.5 [**27**, Section 2.1]. *Let x be an arbitrary point in $\mathbb{F}$ and let T_x be the set of points $y \in \mathbb{F}$ which can be joined to x by a topological arc intersecting every piece in at most one point.*

The subset T_x is a real tree and a closed subset of $\mathbb{F}$, and every topological arc joining two points in T_x is contained in T_x. Moreover, for every $y \in T_x$, $T_y = T_x$.

DEFINITION 3.6. A subset T_x as in Lemma 3.5 is called a *transversal tree* in $\mathbb{F}$.

A geodesic segment, ray or line contained in a transversal tree is called a *transversal geodesic*.

Throughout the rest of the section, $(\mathbb{F}, \mathcal{P})$ is a tree-graded space.

The following statement is an immediate consequence of [**27**, Corollary 2.11].

LEMMA 3.7. *Let A and B be two pieces in $\mathcal{P}$. There exists a unique pair of points $a \in A$ and $b \in B$ such that any topological arc joining A and B contains a and b. In particular* $\mathrm{dist}(A, B) = \mathrm{dist}(a, b)$.

For every tree-graded space there can be defined a canonical $\mathbb{R}$-tree quotient [**25**].

NOTATION. Let x and y be two arbitrary points in $\mathbb{F}$. We define $\widetilde{\mathrm{dist}}(x,y)$ to be $\mathrm{dist}(x,y)$ minus the sum of lengths of non-trivial subarcs which appear as intersections of one (any) geodesic $[x,y]$ with pieces.

The function $\widetilde{\mathrm{dist}}(x,y)$ is well defined (independent of the choice of a geodesic $[x,y]$), symmetric, and it satisfies the triangle inequality [**25**].

The relation $\approx$ defined by

$$x \approx y \text{ if and only if } \widetilde{\mathrm{dist}}(x,y) = 0 \tag{3}$$

is a closed equivalence relation.

LEMMA 3.8 [**25**]. (1) *The quotient $T = \mathbb{F}/\approx$ is an $\mathbb{R}$-tree with respect to the metric induced by* $\widetilde{\mathrm{dist}}$.

(2) *Every geodesic in $\mathbb{F}$ projects onto a geodesic in T. Conversely, for every non-trivial geodesic $\mathfrak{g}$ in T there exists a non-trivial geodesic $\mathfrak{p}$ in $\mathbb{F}$ such that its projection on T is $\mathfrak{g}$.*

(3) *If $x \neq y$ are in the same transversal tree of $\mathbb{F}$, then $\mathrm{dist}(x,y) = \widetilde{\mathrm{dist}}(x,y)$. In particular, $x \not\approx y$. Thus every transversal tree projects into T isometrically.*

Following [**25**, Definitions 2.6 and 2.9], given a topological arc $\mathfrak{g}$ in $\mathbb{F}$, we define the *set of cut-points on* $\mathfrak{g}$, which we denote by $\mathrm{Cutp}\,(\mathfrak{g})$, as the subset of $\mathfrak{g}$ which is complementary to the union of all the interiors of subarcs appearing as intersections of $\mathfrak{g}$ with pieces. Given two points x and y in $\mathbb{F}$, we define the *set of cut-points separating x and y*, which we denote by $\mathrm{Cutp}\,\{x,y\}$, as the set of cut-points of some (any) topological arc joining x and y. Note that if g is an isometry of $\mathbb{F}$ that permutes pieces in a given tree-grading, then $g(\mathrm{Cutp}\,\{x,y\}) = \mathrm{Cutp}\,\{gx,gy\}$

3.2. *Isometries of tree-graded spaces*

For all the results on tree-graded metric spaces that we use in what follows, we refer to [**27**], mainly to Section 2 in that paper.

LEMMA 3.9. *Let x and y be two distinct points, and assume that $\mathrm{Cutp}\,\{x,y\}$ does not contain a point at equal distance from x and y. Let a be the farthest from x point in $\mathrm{Cutp}\,\{x,y\}$ with $\mathrm{dist}(x,a) \leqslant \mathrm{dist}(x,y)/2$, and let b be the farthest from y point in $\mathrm{Cutp}\,\{x,y\}$ with $\mathrm{dist}(y,b) \leqslant \mathrm{dist}(x,y)/2$. Then there exists a unique piece P containing $\{a,b\}$, and P contains all points at equal distance from x and y.*

Proof. Since $\mathrm{Cutp}\,\{x,y\}$ does not contain a point at equal distance from x and y it follows that $a \neq b$. The choice of a,b implies that $\mathrm{Cutp}\,\{a,b\} = \{a,b\}$, whence $\{a,b\}$ is contained in a piece P, and P is the unique piece with this property, by property (T_1) of a tree-graded space.

Let $m \in \mathbb{F}$ be such that $\mathrm{dist}(x,m) = \mathrm{dist}(y,m) = \mathrm{dist}(x,y)/2$. Then any union of geodesics $[x,m] \cup [m,y]$ is a geodesic. By property (T_2') of a tree-graded space, $a \in [x,m], b \in [m,y]$. The subgeodesic $[a,m] \cup [m,b]$ has endpoints in the piece P and therefore is entirely contained in P, since pieces are convex, that is, each geodesic with endpoints in a piece is entirely contained in that piece [**27**, Lemma 2.6]. $\square$

DEFINITION 3.10. Let x and y be two distinct points. If $\mathrm{Cutp}\,\{x,y\}$ contains a point at equal distance from x and y, then we call that point the *middle cut-point* of x,y.

If $\mathrm{Cutp}\,\{x,y\}$ does not contain such a point, then we call the piece defined in Lemma 3.9 the *middle cut-piece* of x,y.

If $x = y$, then we say that x, y have the middle cut-point x.

Let P and Q be two distinct pieces, and let $x \in P$ and $y \in Q$ be the unique pair of points minimizing the distance by Lemma 3.7. The *middle cut-point* (or *middle cut-piece*) of P, Q is the middle cut-point (respectively, the middle cut-piece) of x, y.

If $P = Q$, then we say that P, Q have the middle cut-piece P.

Lemma 3.11. *Let g be an isometry permuting pieces of a tree-graded space $\mathbb{F}$, such that the cyclic group $\langle g\rangle$ has bounded orbits.*

(1) *If x is a point such that $gx \neq x$, then g fixes the middle cut-point or the middle cut-piece of x, gx.*

(2) *If P is a piece such that $gP \neq P$, then g fixes the middle cut-point or the middle cut-piece of P, gP.*

Proof. (1) Let e be the farthest from gx point in $\operatorname{Cutp}\{x, gx\} \cap \operatorname{Cutp}\{gx, g^2x\}$ and let $d = \operatorname{dist}(x, gx) > 0$.

(a) Assume that x, gx have a middle cut-point $m \in \operatorname{Cutp}\{x, gx\}$. If the intersection $\operatorname{Cutp}\{x, gx\} \cap \operatorname{Cutp}\{gx, g^2x\}$ contains m, then $gm = m$ because g takes m is the (unique) point from $\operatorname{Cutp}\{gx, g^2x\}$ at distance $\operatorname{dist}(x, gx)/2$ from gx. We argue by contradiction and assume that $gm \neq m$, whence $\operatorname{Cutp}\{x, gx\} \cap \operatorname{Cutp}\{gx, g^2x\}$ does not contain m. Then $\operatorname{dist}(gx, e) = d/2 - \epsilon$ for some $\epsilon > 0$.

Assume that $\mathfrak{p} = [x, e] \sqcup [e, g^2x]$ is a topological arc. If $e \in \operatorname{Cutp}\mathfrak{p}$, then $e \in \operatorname{Cutp}\{x, g^2x\}$. It follows that $\operatorname{dist}(x, g^2x) = \operatorname{dist}(x, e) + \operatorname{dist}(e, g^2x) = 2(d/2 + \epsilon) = d + 2\epsilon$. An induction argument will then give that $\operatorname{Cutp}\{x, g^nx\}$ contains $e, ge, \ldots, g^{n-2}e$, hence that $\operatorname{dist}(x, g^nx) = d + 2\epsilon(n-1)$. This contradicts the hypothesis that the orbits of $\langle g\rangle$ are bounded.

If $e \notin \operatorname{Cutp}\mathfrak{p}$, then there exist $a \in \operatorname{Cutp}\{x, e\}$ and $b \in \operatorname{Cutp}\{e, g^2x\}$ such that a, e are the endpoints of a non-trivial intersection of any geodesic $[x, gx]$ with a piece P, and e, b are the endpoints of a non-trivial intersection of any geodesic $[gx, g^2x]$ with the same piece P. Note that $a \neq b$, otherwise the choice of e would be contradicted. Also, since $m \in \operatorname{Cutp}\{x, e\}$ and $m \neq e$, it follows that $m \in \operatorname{Cutp}\{x, a\}$. Similarly, $gm \in \operatorname{Cutp}\{g^2x, b\}$. It follows that $\operatorname{dist}(x, a)$ and $\operatorname{dist}(g^2x, b)$ are at least $d/2$. Since $[x, a] \sqcup [a, b] \sqcup [b, g^2x]$ is a geodesic (by [27, Lemma 2.28]), it follows that $\operatorname{dist}(x, g^2x) \geqslant \operatorname{dist}(x, a) + \operatorname{dist}(a, b) + \operatorname{dist}(b, g^2x) \geqslant d + \operatorname{dist}(a, b)$. An induction argument gives that $[x, a] \sqcup \bigsqcup_{k=0}^{n-2}([g^ka, g^kb] \sqcup [g^kb, g^{k+1}a]) \sqcup [g^{n-1}a, g^nx]$ is a geodesic. This implies that $\operatorname{dist}(x, g^nx) \geqslant d + (n-1)\operatorname{dist}(a, b)$, contradicting the hypothesis that $\langle g\rangle$ has bounded orbits. Note that the argument in this paragraph applies as soon as we find the points x, e, p as above.

Assume that $\mathfrak{p} = [x, e] \sqcup [e, g^2x]$ is not a topological arc. Then $[x, e] \cap [e, g^2x]$ contains a point $y \neq e$. According to the choice of e, y is either not in $\operatorname{Cutp}\{x, e\}$ or not in $\operatorname{Cutp}\{e, g^2x\}$, and $\operatorname{Cutp}\{e, y\}$ must be $\{e, y\}$. It follows that y, e are in the same piece P. If we consider the endpoints of the (non-trivial) intersections of any geodesics $[x, e]$ and $[e, g^2x]$ with the piece P, points a, e and e, b, respectively, then we are in the setup described in the previous case and we can apply the same argument.

(b) Assume that x, gx have a middle cut-piece Q. Then there exist two points i, o in $\operatorname{Cutp}\{x, gx\}$, the entrance and the exit points of any geodesic $[x, gx]$ in the piece Q, respectively, such that the midpoint of $[x, gx]$ is in the interior of the subgeodesic $[i, o]$ of $[x, gx]$ (for any choice of the geodesic $[x, gx]$).

If $e \in \operatorname{Cutp}\{x, i\}$, then g stabilizes $\operatorname{Cutp}\{e, ge\}$ and, since g is an isometry, g fixes the middle cut-piece Q of e, ge. Assume on the contrary that $gQ \neq Q$. Then $e \in \operatorname{Cutp}\{o, gx\}$. If $\mathfrak{p} = [x, e] \sqcup [e, g^2x]$ is a topological arc and $e \in \operatorname{Cutp}\mathfrak{p}$, then as in (a) we may conclude that $\langle g\rangle$ has an unbounded orbit. If either $e \notin \operatorname{Cutp}\mathfrak{p}$ or $\mathfrak{p} = [x, e] \sqcup [e, g^2x]$ is not a topological arc, then as in (a) we may conclude that there exist $a \in \mathrm{Cutp}\{x, e\}$ and $b \in \mathrm{Cutp}\{e, g^2x\}$ such that a, b, e are pairwise distinct and contained in the same piece P.

Assume that $e = o$. Then $a = i$ and $P = Q$. By hypothesis $P \neq gQ$. It follows that any geodesic $[b, g^2x]$ intersects gQ, whence $ga, ge \in \mathrm{Cutp}\{b, g^2x\}$. Since $[x, a] \sqcup [a, b] \sqcup [b, g^2x]$ is a geodesic (by [27, Lemma 2.28]), we have that $\mathrm{dist}(x, g^2x) = \mathrm{dist}(x, a) + \mathrm{dist}(a, b) + \mathrm{dist}(b, g^2x) \geqslant \mathrm{dist}(x, a) + \mathrm{dist}(a, b) + \mathrm{dist}(ga, g^2x) = d + \mathrm{dist}(a, b)$. An inductive argument gives that for any $n \geqslant 1$, the union of geodesics $[x, a] \sqcup \bigsqcup_{k=0}^{n-2}([g^k a, g^k b] \sqcup [g^k b, g^{k+1}a]) \sqcup [g^{n-1}a, g^n x]$ is a geodesic, hence $\mathrm{dist}(x, g^n x) \geqslant d + (n-1)\mathrm{dist}(a, b)$, contradicting the hypothesis that $\langle g \rangle$ has bounded orbits.

Assume that $e \neq o$. If $e = gi$, hence $b = go$ and $P = gQ$, then an argument as before gives that, for every $n \geqslant 1$, $\mathrm{dist}(x, g^n x) \geqslant d + (n-1)\mathrm{dist}(a, b)$, which is a contradiction.

If $e \notin \{gi, o\}$, then $i, o \in \mathrm{Cutp}\{x, a\}$ and $gi, go \in \mathrm{Cutp}\{b, g^2x\}$, whence both $\mathrm{dist}(x, a)$ and $\mathrm{dist}(b, g^2x)$ are larger than $d/2$. It follows that the union $[x, a] \sqcup \bigsqcup_{k=0}^{n-2}([g^k a, g^k b] \sqcup [g^k b, g^{k+1}a]) \sqcup [g^{n-1}a, g^n x]$ is a geodesic, therefore $\mathrm{dist}(x, g^n x)$ is at least $d + (n-1)\mathrm{dist}(a, b)$, which is a contradiction.

(2) Let $x \in P$ and $y \in gP$ be the pair of points realizing the distance which exist by Lemma 3.7. Assume that $gx \neq y$. Then for every geodesics $[x, y]$ and $[y, gx]$ it holds that $[x, y] \sqcup [y, gx]$ is a geodesic. Similarly, $[x, y] \sqcup [y, gx] \sqcup [gx, gy] \sqcup [gy, g^2x]$ is a geodesic. An easy induction argument gives that $\bigsqcup_{k=0}^{n-1}[g^k x, g^k y]$ is a geodesic. In particular $\mathrm{dist}(x, g^n x) \geqslant n\mathrm{dist}(y, gx)$, contradicting the hypothesis that $\langle g \rangle$ has bounded orbits.

It follows that $y = gx$. If $gx = x$, then we are done. If $gx \neq x$, then we apply (1). □

LEMMA 3.12. *Let $g_1, \dots, g_n$ be isometries of a tree-graded space permuting pieces and generating a group with bounded orbits. Then $g_1, \dots, g_n$ have a common fixed point or (set-wise) fixed piece.*

Proof. We argue by induction on n. For $n = 1$ it follows from Lemma 3.11. Assume that the conclusion holds for n and take $g_1, \dots, g_{n+1}$ isometries generating a group with bounded orbits and permuting pieces. By the induction hypothesis $g_1, \dots, g_n$ fix either a point x or a piece P.

Assume that they fix a point x, and that $g_{n+1}x \neq x$. Assume that $x, g_{n+1}x$ have a middle cut-point m. Then $g_{n+1}m = m$ by Lemma 3.11(1). For every $i \in \{1, \dots, n\}$, $g_{n+1}g_i x = g_{n+1}x$ therefore $g_{n+1}g_i m = m$, hence $g_i m = m$. If $x, g_{n+1}x$ have a middle cut-piece Q, then it is shown similarly that $g_1, \dots, g_{n+1}$ fix Q set-wise. In the case when $g_1, \dots, g_n$ fix a piece P, Lemma 3.11 also allows to prove that $g_1, \dots, g_{n+1}$ fix the middle cut-point or middle cut-piece of $P, g_{n+1}P$. □

4. *Asymptotic cones of mapping class groups*

4.1. *Distance formula in asymptotic cones of mapping class groups*

Fix an arbitrary asymptotic cone $\mathcal{AM}(S) = \mathrm{Con}^\omega(\mathcal{M}(S); (x_n), (d_n))$ of $\mathcal{M}(S)$.

We fix a point ν_0 in $\mathcal{M}(S)$ and define the map $\mathcal{MCG}(S) \to \mathcal{M}(S)$, $g \mapsto g\nu_0$, which according to Theorem 2.14 is a quasi-isometry. There exists a sequence $g_0 = (g_n^0)$ in $\mathcal{MCG}(S)$ such that $x_n = g_n^0 \nu_0$, which we shall later use to discuss the ultrapower of the mapping class group.

NOTATION. By Remark 2.6, the group $g_0^\omega(\Pi_1 \mathcal{MCG}(S)/\omega)(g_0^\omega)^{-1}$ acts transitively by isometries on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(S); (x_n), (d_n))$. We denote this group by $\mathcal{GM}$.

Definition 4.1. A path in $\mathcal{AM}$, obtained by taking an ultralimit of hierarchy paths, is, by a slight abuse of notation, also called a *hierarchy path*.

It was proved in [5] that the asymptotic cone $\mathcal{AM}$ has cut-points and is thus a tree-graded space. Since $\mathcal{AM}$ is tree-graded, one can define the collection of pieces in the tree-graded structure of $\mathcal{AM}$ as the collection of maximal subsets in $\mathcal{AM}$ without cut-points. That set of pieces can be described as follows [7, § 7], where the equivalence to the third item below is an implicit consequence of the proof in [7] of the equivalence of the first two items.

Theorem 4.2 (Behrstock, Kleiner, Minsky and Mosher [7]). *Fix a pair of points $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{AM}(S)$. If $\xi(S) \geqslant 2$, then the following are equivalent.*

(1) *No point of $\mathcal{AM}(S)$ separates $\boldsymbol{\mu}$ from $\boldsymbol{\nu}$.*

(2) *There exist points $\boldsymbol{\mu}'$ and $\boldsymbol{\nu}'$ arbitrarily close to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively, for which there exist representative sequences (μ'_n) and (ν'_n) satisfying*

$$\lim_\omega d_{\mathcal{C}(S)}(\mu'_n, \nu'_n) < \infty.$$

(3) *For every hierarchy path $\boldsymbol{H} = \lim_\omega h_n$ connecting $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ there exist points $\boldsymbol{\mu}'$ and $\boldsymbol{\nu}'$ on $\boldsymbol{H}$ which are arbitrarily close to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively, and for which there exist representative sequences (μ'_n) and (ν'_n) with μ'_n and ν'_n on h_n satisfying*

$$\lim_\omega d_{\mathcal{C}(S)}(\mu'_n, \nu'_n) < \infty.$$

Thus for every two points μ and ν in the same piece of $\mathcal{AM}$ there exists a sequence of pairs $\boldsymbol{\mu}^{(k)}, \boldsymbol{\nu}^{(k)}$ such that:

(i) $\epsilon_k = \max(\mathrm{dist}_{\mathrm{Con}^\omega(\mathcal{M}(S);(x_n),(d_n))}(\boldsymbol{\mu}, \boldsymbol{\mu}^{(k)}), \mathrm{dist}_{\mathrm{Con}^\omega(\mathcal{M}(S);(x_n),(d_n))}(\boldsymbol{\nu}, \boldsymbol{\nu}^{(k)}))$ goes to zero;

(ii) $D^{(k)} = \lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\mu_n^{(k)}, \nu_n^{(k)})$ is finite for every $k \in \mathbb{N}$.

Remark 4.3. The projection of the marking complex $\mathcal{M}(S)$ onto the complex of curves $\mathcal{C}(S)$ induces a Lipschitz map from $\mathcal{AM}(S)$ onto the asymptotic cone $\mathcal{AC}(S) = \mathrm{Con}^\omega(\mathcal{C}(S); (\pi_{\mathcal{C}(S)} x_n), (d_n))$. By Theorem 4.2, pieces in $\mathcal{AM}(S)$ project onto singletons in $\mathcal{AC}(S)$. Therefore two points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}(S)$ for which $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 0$ (that is, satisfying the relation $\boldsymbol{\mu} \approx \boldsymbol{\nu}$; see Lemma 3.8) project onto the same point in $\mathcal{AC}(S)$. Thus the projection $\mathcal{AM}(S) \to \mathcal{AC}(S)$ induces a projection of $T_S = \mathcal{AM}(S)/\approx$ onto $\mathcal{AC}(S)$. The latter projection is not a bijection. This can be seen by taking, for instance, a sequence (γ_n) of geodesics in $\mathcal{C}(S)$ with one endpoint in $\pi_{\mathcal{C}(S)}(x_n)$ and of length $\sqrt{d_n}$, and considering elements g_n in $\mathcal{MCG}(S)$ obtained by performing $\lfloor \sqrt{d_n} \rfloor$. Dehn twists around each curve in γ_n consecutively. The projections of the limit points $\lim_\omega(x_n)$ and $\lim_\omega(g_n x_n)$ onto $\mathcal{AC}(S)$ coincide, while their projections onto T_S are distinct. Indeed the limit of the sequence of paths h_n joining x_n and $g_n x_n$ obtained by consecutive applications to x_n of the Dehn twists is a transversal path. Otherwise, if this path had a non-trivial intersection with a piece, then according to Theorem 4.2(3), there would exist points μ_n and ν_n on h_n such that $\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n) \geqslant \epsilon d_n$ for some $\epsilon > 0$ and such that $\mathrm{dist}_{\mathcal{C}(S)}(\mu_n, \nu_n)$ is bounded by a constant $D > 0$ uniform in n. But the latter inequality would imply that $\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n) \leqslant D\sqrt{d_n}$, contradicting the former inequality.

Let $\mathcal{U}$ be the set of all subsurfaces of S and let $\Pi\mathcal{U}/\omega$ be its ultrapower. For simplicity we denote by $\mathbf{S}$ the element in $\Pi\mathcal{U}/\omega$ given by the constant sequence (S). We define, for every $\mathbf{U} = (U_n)^\omega \in \Pi\mathcal{U}/\omega$, its complexity $\xi(\mathbf{U})$ to be $\lim_\omega \xi(U_n)$.

We say that an element $\mathbf{U} = (U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ is a *subsurface* of another element $\mathbf{Y} = (Y_n)^\omega$, and we denote it by $\mathbf{U} \subseteq \mathbf{Y}$ if ω-a.s. $U_n \subseteq Y_n$. An element $\mathbf{U} = (U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ is said to be a *strict subsurface* of another element $\mathbf{Y} = (Y_n)^\omega$, denoted by $\mathbf{U} \subsetneq \mathbf{Y}$, if ω-a.s. $U_n \subsetneq Y_n$; equivalently $\mathbf{U} \subsetneq \mathbf{Y}$ if and only if $\mathbf{U} \subseteq \mathbf{Y}$ and $\xi(\mathbf{Y}) - \xi(\mathbf{U}) \geqslant 1$.

For every $\mathbf{U} = (U_n)^\omega$ in $\Pi\mathcal{U}/\omega$ consider the ultralimit of the marking complexes of U_n with their own metric $\mathcal{M}^\omega\mathbf{U} = \lim_\omega(\mathcal{M}(U_n), (1), (d_n))$. Since there exists a surface U' such that ω-a.s. U_n is homeomorphic to U', the ultralimit $\mathcal{M}^\omega\mathbf{U}$ is isometric to the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(U'), (d_n))$. Consequently $\mathcal{M}^\omega\mathbf{U}$ is a tree-graded metric space.

Notation 4.4. We denote by $T_\mathbf{U}$ the quotient tree $\mathcal{M}^\omega\mathbf{U}/\approx$, as constructed in Lemma 3.8. We denote by $\mathrm{dist}_\mathbf{U}$ the metric on $\mathcal{M}^\omega\mathbf{U}$. We abuse notation slightly, by writing $\widetilde{\mathrm{dist}}_\mathbf{U}$ to denote both the pseudo-metric on $\mathcal{M}^\omega\mathbf{U}$ defined at the end of Section 3, and the metric this induces on $T_\mathbf{U}$.

Notation 4.5. We denote by $\mathcal{Q}(\partial\mathbf{U})$ the ultralimit $\lim_\omega(\mathcal{Q}(\partial U_n))$ in the asymptotic cone $\mathcal{AM}$, taken with respect to the basepoint obtained by projecting the basepoints we use for $\mathcal{AM}$ projected to $\mathcal{Q}(\partial\mathbf{U})$.

There exists a natural projection map $\pi_{\mathcal{M}^\omega\mathbf{U}}$ from $\mathcal{AM}$ to $\mathcal{M}^\omega\mathbf{U}$ sending any element $\boldsymbol{\mu} = \lim_\omega(\mu_n)$ to the element of $\mathcal{M}^\omega\mathbf{U}$ defined by the sequence of projections of μ_n onto $\mathcal{M}(U_n)$. This projection induces a well-defined projection between asymptotic cones with the same rescaling constants by Corollary 2.29.

Notation 4.6. For simplicity, we write $\mathrm{dist}_\mathbf{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ to denote the distance and the pseudo-distance in $\mathcal{M}^\omega\mathbf{U}$ between the images under the projection maps, $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\nu})$.

We denote by $\mathrm{dist}_{C(\mathbf{U})}(\boldsymbol{\mu}, \boldsymbol{\nu})$ the ultralimit $\lim_\omega(1/d_n)\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n)$.

The following is from [**5**, Theorem 6.5, Remark 6.3].

Lemma 4.7 [**5**]. *Given a point $\boldsymbol{\mu}$ in $\mathcal{AM}$, the transversal tree $T_{\boldsymbol{\mu}}$ as defined in Definition* 3.6 *contains the set*

$$\{\boldsymbol{\nu} \mid \mathrm{dist}_\mathbf{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 0 \ \forall \mathbf{U} \subsetneq \mathbf{S}\}.$$

Corollary 4.8. *For any two distinct points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}$ there exists at least one subsurface $\mathbf{U}$ in $\Pi\mathcal{U}/\omega$ such that $\mathrm{dist}_\mathbf{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ and, for every strict subsurface $\mathbf{Y} \subsetneq \mathbf{U}$, $\mathrm{dist}_{\mathcal{M}^\omega(\mathbf{Y})}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 0$. In particular, $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\nu})$ are in the same transversal tree.*

Proof. Indeed, every chain of nested subsurfaces of S contains at most ξ (the complexity of S) elements. It implies that the same is true for chains of subsurfaces $\mathbf{U}$ in $\Pi\mathcal{U}/\omega$. In particular, $\Pi\mathcal{U}/\omega$ with the inclusion order satisfies the descending chain condition. It remains to apply Lemma 4.7. □

Lemma 4.9. *There exists a constant t depending only on $\xi(S)$ such that for every $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}$ and $\mathbf{U} \in \Pi\mathcal{U}/\omega$ the following inequality holds*:

$$\mathrm{dist}_{C(\mathbf{U})}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant t\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu}, \boldsymbol{\nu}).$$

Proof. The inequality involves only the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ onto $\mathcal{M}^\omega(\mathbf{U})$. Also ω-a.s. U_n is homeomorphic to a fixed surface U, hence $\mathcal{M}^\omega(\mathbf{U})$ is isometric to some asymptotic cone of $\mathcal{M}(U)$, and it suffices to prove the inequality for $\mathbf{U}$ the constant sequence $\mathbf{S}$.

Let $([\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k])_{k\in K}$ be the set of non-trivial intersections of a geodesic $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ with pieces in $\mathcal{AM}$. Then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathrm{dist}(\boldsymbol{\mu}, \boldsymbol{\nu}) - \sum_{k\in K} \mathrm{dist}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$. For any $\epsilon > 0$ there exists a finite subset J in K such that $\sum_{k\in K\setminus J} \mathrm{dist}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) \leqslant \epsilon$. According to Theorem 4.2, for every $k \in K$ there exist $\boldsymbol{\alpha}'_k = \lim_\omega(\alpha'_{k,n})$ and $\boldsymbol{\beta}'_k = \lim_\omega(\beta'_{k,n})$ for which $[\boldsymbol{\alpha}'_k, \boldsymbol{\beta}'_k] \subset [\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k]$ and such that:

(1) $\lim_\omega \mathrm{dist}_{C(S)}(\alpha'_{k,n}, \beta'_{k,n}) < \infty$;

(2) $\sum_{k\in K} \mathrm{dist}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) - 2\epsilon \leqslant \sum_{k\in J} \mathrm{dist}(\boldsymbol{\alpha}'_k, \boldsymbol{\beta}'_k)$.

The second item above follows since Theorem 4.2 yields that $\boldsymbol{\alpha}'_k, \boldsymbol{\beta}'_k$ can be chosen so that $\sum_{k\in J} \mathrm{dist}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$ is arbitrarily close to $\sum_{k\in J} \mathrm{dist}(\boldsymbol{\alpha}'_k, \boldsymbol{\beta}'_k)$, and since the contributions from those entries indexed by $K - J$ are less than ϵ.

Assume that $J = \{1, 2, \ldots, m\}$ and that the points $\boldsymbol{\alpha}'_1, \boldsymbol{\beta}'_1, \boldsymbol{\alpha}'_2, \boldsymbol{\beta}'_2, \ldots, \boldsymbol{\alpha}'_m, \boldsymbol{\beta}'_m$ appear on the geodesic $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ in that order.

By the triangle inequality

$$\begin{aligned}\mathrm{dist}_{C(S)}(\mu_n, \nu_n) &\leqslant \mathrm{dist}_{C(S)}(\mu_n, \alpha'_{1,n}) + \mathrm{dist}_{C(S)}(\beta'_{m,n}, \nu_n) \\ &\quad + \sum_{j=1}^{m} \mathrm{dist}_{C(S)}(\alpha'_{j,n}, \beta'_{j,n}) + \sum_{j=1}^{m-1} \mathrm{dist}_{C(S)}(\beta'_{j,n}, \alpha'_{j+1,n}). \end{aligned} \tag{4}$$

Above we noted $\lim_\omega \sum_{j=1}^{m} \mathrm{dist}_{C(S)}(\alpha'_{j,n}, \beta'_{j,n}) < \infty$, thus if we rescale the above inequality by $1/d_n$ and take the ultralimit, we obtain

$$\mathrm{dist}_{C(\mathbf{S})}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant \mathrm{dist}_{C(\mathbf{S})}(\boldsymbol{\mu}, \boldsymbol{\alpha}'_1) + \mathrm{dist}_{C(\mathbf{S})}(\boldsymbol{\beta}'_m, \boldsymbol{\nu}) + \sum_{j=1}^{m-1} \mathrm{dist}_{C(\mathbf{S})}(\boldsymbol{\beta}'_j, \boldsymbol{\alpha}'_{j+1}). \tag{5}$$

The distance formula implies that up to some multiplicative constant t, the right-hand side of equation (5) is at most $\mathrm{dist}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\alpha}'_1) + \sum_{j=1}^{m-1} \mathrm{dist}_{\mathbf{S}}(\boldsymbol{\beta}'_j, \boldsymbol{\alpha}'_{j+1}) + \mathrm{dist}_{\mathbf{S}}(\boldsymbol{\beta}'_m, \boldsymbol{\nu})$, which is equal to $\mathrm{dist}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) - \sum_{j=1}^{m} \mathrm{dist}_{\mathbf{S}}(\boldsymbol{\alpha}'_j, \boldsymbol{\beta}'_j)$. Since we have noted above that $\sum_{k\in K} \mathrm{dist}(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k) - 2\epsilon \leqslant \sum_{j\in J} \mathrm{dist}(\boldsymbol{\alpha}'_k, \boldsymbol{\beta}'_k)$, it follows that

$$\mathrm{dist}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) - \sum_{j=1}^{m} \mathrm{dist}_{\mathbf{S}}(\boldsymbol{\alpha}'_j, \boldsymbol{\beta}'_j) \leqslant \widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) + 2\epsilon.$$

Thus, we have shown that, for every $\epsilon > 0$, we have $\mathrm{dist}_{C(\mathbf{S})}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant t\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) + 2t\epsilon$. This completes the proof. □

LEMMA 4.10. *Let μ and ν be two markings in $\mathcal{M}(S)$ at $\mathcal{C}(S)$-distance s and let $\alpha_1, \ldots, \alpha_{s+1}$ be the $s+1$ consecutive vertices (curves) of a tight geodesic in $\mathcal{C}(S)$ shadowed by a hierarchy path $\mathfrak{p}$ joining μ and ν. Then the $s+1$ proper subsurfaces $S_1, \ldots, S_{s+1}$ of S defined by $S_i = S \setminus \alpha_i$ satisfy the inequality*

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu, \nu) \leqslant C \sum_i \mathrm{dist}_{\mathcal{M}(S_i)}(\mu, \nu) + Cs + D \tag{6}$$

for some constants C and D depending only on S.

Proof. By Lemma 2.18, if $Y \subsetneq S$ is a proper subsurface that yields a term in the distance formula (see Theorem 2.25) for $\mathrm{dist}_{\mathcal{M}(S)}(\mu, \nu)$, then there exists at least one (and at most 3) $i \in \{1, \ldots, s+1\}$ for which $Y \cap \alpha_i = \emptyset$. Hence, any such Y occurs in the distance formula for $\mathrm{dist}_{\mathcal{M}(S_i)}(\mu, \nu)$, where $S_i = S \setminus \alpha_i$. Every term which occurs in the distance formula for $\mathrm{dist}_{\mathcal{M}(S)}(\mu, \nu)$, except for the $\mathrm{dist}_{\mathcal{C}(S)}(\mu, \nu)$ term, has a corresponding term (up to bounded multiplicative and additive errors) in the distance formula for at least one of the $\mathrm{dist}_{\mathcal{M}(S_i)}(\mu,\nu)$. Since $\mathrm{dist}_{\mathcal{C}(S)}(\mu,\nu)=s$, up to the additive and multiplicative bounds occurring in the distance formula this term in the distance formula for $\mathrm{dist}_{\mathcal{M}(S)}(\mu,\nu)$ is bounded above by s up to a bounded multiplicative and additive error. This implies inequality (6). □

NOTATION 4.11. For any subset $F\subset \Pi\mathcal{U}/\omega$ we define the map $\psi_F:\mathcal{AM}\to\prod_{\mathbf{U}\in F}T_{\mathbf{U}}$, where, for each $\mathbf{U}\in F$, the map $\psi_{\mathbf{U}}:\mathcal{AM}\to T_{\mathbf{U}}$ is the canonical projection of $\mathcal{AM}$ onto $T_{\mathbf{U}}$. In the particular case when F is finite equal to $\{\mathbf{U}_1,\dots,\mathbf{U}_k\}$ we also use the notation $\psi_{\mathbf{U}_1,\dots,\mathbf{U}_k}$.

LEMMA 4.12. *Let $\mathfrak{h}\subset\mathcal{AM}$ denote the ultralimit of a sequence of uniform quasi-geodesics in $\mathcal{M}$. Moreover, assume that the quasi-geodesics in the sequence are $\mathcal{C}(U)$-monotonic for every $U\subseteq S$, with constants that are uniform over the sequence.*

Any path $\mathfrak{h}:\ [0,a]\to\mathcal{AM}$, as above, projects onto a geodesic $\mathfrak{g}:\ [0,b]\to T_{\mathbf{U}}$ such that $\mathfrak{h}(0)$ projects onto $\mathfrak{g}(0)$ and, assuming that both $\mathfrak{h}$ and $\mathfrak{g}$ are parametrized by arc length, the map $[0,a]\to[0,b]$ defined by the projection is non-decreasing.

Proof. Fix a path $\mathfrak{h}$ satisfying the hypothesis of the lemma. It suffices to prove that, for every $\boldsymbol{x},\boldsymbol{y}$ on $\mathfrak{h}$ and every $\boldsymbol{\mu}$ on $\mathfrak{h}$ between $\boldsymbol{x}$ and $\boldsymbol{y}$, $\psi_{\mathbf{U}}(\boldsymbol{\mu})$ is on the geodesic joining $\psi_{\mathbf{U}}(\boldsymbol{x})$ to $\psi_{\mathbf{U}}(\boldsymbol{y})$ in $T_{\mathbf{U}}$. If the contrary were to hold then there would exist $\boldsymbol{\nu},\boldsymbol{\rho}$ on $\mathfrak{h}$ with $\boldsymbol{\mu}$ between them satisfying $\psi_{\mathbf{U}}(\boldsymbol{\nu})=\psi_{\mathbf{U}}(\boldsymbol{\rho})\neq\psi_{\mathbf{U}}(\boldsymbol{\mu})$. Without loss of generality we may assume that $\boldsymbol{\nu},\boldsymbol{\rho}$ are the endpoints of $\mathfrak{h}$. We denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the subarcs of $\mathfrak{h}$ of endpoints $\boldsymbol{\nu},\boldsymbol{\mu}$ and, $\boldsymbol{\mu},\boldsymbol{\rho}$, respectively.

The projection $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h})$ is by Corollary 2.29 a continuous path joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$ and containing $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$.

According to [**25**, Lemma 2.19] a geodesic $\bar{\mathfrak{g}}_1$ joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$ projects onto the geodesic $[\psi_{\mathbf{U}}(\boldsymbol{\nu}),\psi_{\mathbf{U}}(\boldsymbol{\mu})]$ in $T_{\mathbf{U}}$. Moreover, the set $\mathrm{Cutp}\,\bar{\mathfrak{g}}_1$ of cut-points of $\bar{\mathfrak{g}}_1$ in the tree-graded space $\mathcal{M}(\mathbf{U})$ projects onto $[\psi_{\mathbf{U}}(\boldsymbol{\nu}),\psi_{\mathbf{U}}(\boldsymbol{\mu})]$. By properties of tree-graded spaces [**27**] the continuous path $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h}_1)$ contains $\mathrm{Cutp}\,\bar{\mathfrak{g}}_1$.

Likewise if $\bar{\mathfrak{g}}_2$ is a geodesic joining $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$, then the set $\mathrm{Cutp}\,\bar{\mathfrak{g}}_2$ projects onto $[\psi_{\mathbf{U}}(\boldsymbol{\mu}),\psi_{\mathbf{U}}(\boldsymbol{\rho})]$, which is the same as the geodesic $[\psi_{\mathbf{U}}(\boldsymbol{\mu}),\psi_{\mathbf{U}}(\boldsymbol{\rho})]$ reversed, and the path $\pi_{\mathcal{M}(\mathbf{U})}(\mathfrak{h}_2)$ contains $\mathrm{Cutp}\,\bar{\mathfrak{g}}_2$. This implies that $\mathrm{Cutp}\,\bar{\mathfrak{g}}_1=\mathrm{Cutp}\,\bar{\mathfrak{g}}_2$ and that there exist $\boldsymbol{\nu}'$ on $\mathfrak{h}_1$ and $\boldsymbol{\rho}'$ on $\mathfrak{h}_2$ such that $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu}')=\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho}')$ and $\psi_{\mathbf{U}}(\boldsymbol{\nu}')=\psi_{\mathbf{U}}(\boldsymbol{\rho}')\neq\psi_{\mathbf{U}}(\boldsymbol{\mu})$. Without loss of generality we assume that $\boldsymbol{\nu}'=\boldsymbol{\nu}$ and $\boldsymbol{\rho}'=\boldsymbol{\rho}$.

Since $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu},\boldsymbol{\mu})>0$, it follows that $\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu},\boldsymbol{\mu})>0$. Since by construction μ_n is on a path joining ν_n and ρ_n, satisfying the hypotheses of the lemma, we know that up to a uniformly bounded additive error we have $\mathrm{dist}_{\mathcal{C}(Y)}(\nu_n,\mu_n)\leqslant\mathrm{dist}_{\mathcal{C}(Y)}(\nu_n,\rho_n)$ for every $Y\subset U_n$. It then follows from the distance formula that, for some positive constant C,

$$\frac{1}{C}\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu},\boldsymbol{\mu})\leqslant\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu},\boldsymbol{\rho}).$$

In particular, this implies that $\mathrm{dist}_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu},\boldsymbol{\rho})>0$, contradicting the fact that $\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\nu})=\pi_{\mathcal{M}(\mathbf{U})}(\boldsymbol{\rho})$. □

The following is an immediate consequence of Lemma 4.12 since by construction hierarchy paths satisfy the hypothesis of the lemma.

COROLLARY 4.13. *Every hierarchy path in $\mathcal{AM}$ projects onto a geodesic in $T_{\mathbf{U}}$ for every subsurface $\mathbf{U}$ as in Lemma 4.12.*

NOTATION 4.14. Let F and G be two finite subsets in the asymptotic cone $\mathcal{AM}$, and let K be a fixed constant larger than the constant $M(S)$ from Lemma 2.18.

We denote by $\mathcal{Y}(F,G)$ the set of elements $\mathbf{U}=(U_n)^\omega$ in the ultrapower $\Pi\mathcal{U}/\omega$ such that, for any two points $\boldsymbol{\mu}=\lim_\omega(\mu_n)\in F$ and $\boldsymbol{\nu}=\lim_\omega(\nu_n)\in G$, the subsurfaces U_n are ω-a.s. K-large domains for the pair (μ_n,ν_n), in the sense of Definition 2.19.

If $F=\{\boldsymbol{\mu}\}$ and $G=\{\boldsymbol{\nu}\}$, then we simplify the notation to $\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})$.

LEMMA 4.15. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ be two points in $\mathcal{AM}$ and let $\mathbf{U}=(U_n)^\omega$ be an element in $\Pi\mathcal{U}/\omega$. If $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})>0$, then $\lim_\omega(\mathrm{dist}_{C(U_n)}(\mu_n,\nu_n))=\infty$ (and thus $\mathbf{U}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})$). In particular the following holds:*

$$\sum_{\mathbf{U}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})}\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})=\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega}\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})$$

Proof. We establish the result by proving the contrapositive; thus we assume that $\lim_\omega(\mathrm{dist}_{C(U_n)}(\mu_n,\nu_n))<\infty$. Theorem 4.2 then implies that $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\mu})$ and $\pi_{\mathcal{M}^\omega\mathbf{U}}(\boldsymbol{\nu})$ are in the same piece, hence $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})=0$. □

We are now ready to prove a distance formula in the asymptotic cones.

THEOREM 4.16 (Distance formula for asymptotic cones). *There is a constant E, depending only on the constant K used to define $\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})$, and on the complexity $\xi(S)$, such that, for every $\boldsymbol{\mu},\boldsymbol{\nu}$ in $\mathcal{AM}$,*

$$\frac{1}{E}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu})\leqslant\sum_{\mathbf{U}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})}\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})\leqslant\sum_{\mathbf{U}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})}\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})\leqslant E\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu}). \quad (7)$$

Proof. Let us prove by induction on the complexity of S that

$$\sum_{\mathbf{U}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})}\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\boldsymbol{\nu})>\frac{1}{E}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu}) \quad (8)$$

for some $E>1$. Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ be two distinct elements in $\mathcal{AM}$. If $\mathcal{M}(S)$ is hyperbolic, then $\mathcal{AM}$ is a tree; hence there are no non-trivial subsets without cut-points and thus in this case we have $\widetilde{\mathrm{dist}}_{\mathbf{S}}=\mathrm{dist}_{\mathbf{S}}$. This gives the base for the induction.

We may assume that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})<\frac{1}{3}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu})$. Otherwise, we would have that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})>0$, which implies by Lemma 4.15 that $\mathbf{S}\in\mathcal{Y}(\boldsymbol{\mu},\boldsymbol{\nu})$, and we would be done by choosing $E=3$.

Since $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})$ is obtained from $\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu})$ by removing $\sum_{i\in I}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i)$, where $[\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i], i\in I$, are all the non-trivial intersections of a geodesic $[\boldsymbol{\mu},\boldsymbol{\nu}]$ with pieces, it follows that there exists $F\subset I$ finite such that $\sum_{i\in F}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i)\geqslant\frac{1}{2}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu})$. For simplicity assume that $F=\{1,2,\ldots,m\}$ and that the intersections $[\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i]$ appear on $[\boldsymbol{\mu},\boldsymbol{\nu}]$ in the increasing order of their index. According to property (T_2') of Proposition 3.2, the points $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ also appear on any path joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Therefore, without loss of generality, for the rest of the proof we will assume that $[\boldsymbol{\mu},\boldsymbol{\nu}]$ is a hierarchy path, and $[\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i]$ are subpaths of it (this is a slight abuse of notation since hierarchy paths are not geodesics). By Theorem 4.2, for every $i\in F$ there exists $[\boldsymbol{\alpha}_i',\boldsymbol{\beta}_i']\subset[\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i]$ with the following properties:

(i) there exists a number s such that $\forall i=1,\ldots,m$

$$\mathrm{dist}_{C(S)}(\alpha_{i,n}',\beta_{i,n}')<s \quad \omega\text{-a.s.};$$

(ii) $\sum_{i=1}^m\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\alpha}_i',\boldsymbol{\beta}_i')>\frac{1}{3}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu})$.

Let $l=m(s+1)$. By Lemma 4.10 there exists a sequence of proper subsurfaces $Y_1(n),\ldots,Y_l(n)$ of the form $Y_j(n)=S\setminus v_j(n)$ with $v_j(n)$ a vertex (curve) on the tight geodesic in $\mathcal{C}(S)$ shadowed by the hierarchy path $[\mu_n, \nu_n]$, such that ω-a.s.:

$$\sum_{i=1}^{m} \mathrm{dist}_{\mathcal{M}(S)}(\alpha'_{i,n}, \beta'_{i,n}) \leqslant C \sum_{i=1}^{m} \sum_{j=1}^{l} \mathrm{dist}_{\mathcal{M}(Y_j(n))}(\alpha'_{i,n}, \beta'_{i,n}) + Csm + Dm. \tag{9}$$

Let $\boldsymbol{Y}_j$ be the element in $\Pi\mathcal{U}/\omega$ given by the sequence of subsurfaces $(Y_j(n))$.

Rescaling (9) by $1/d_n$, passing to the ω-limit and applying Lemma 2.1, we deduce that

$$\frac{1}{3}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant C \sum_{i=1}^{m} \sum_{j=1}^{l} \mathrm{dist}_{\boldsymbol{Y}_j}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i). \tag{10}$$

As the complexity of $\boldsymbol{Y}_j$ is smaller than the complexity of S, according to the induction hypothesis the second term in (10) is at most

$$CE \sum_{i=1}^{m} \sum_{j=1}^{l} \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i), \mathbf{U} \subseteq \boldsymbol{Y}_j} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}'_i, \boldsymbol{\beta}'_i).$$

Lemma 4.12 implies that the non-zero terms in the latter sum correspond to the subsurfaces $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$, and that the sum is at most

$$CE \sum_{j=1}^{l} \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu}), \mathbf{U} \subseteq \boldsymbol{Y}_j} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}).$$

According to Lemmas 4.15 and 2.18 for every $\mathbf{U} = (U_n)^\omega \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ there exist at least one and at most three vertices (curves) on the tight geodesic in $\mathcal{C}(S)$ shadowed by the hierarchy path $[\mu_n, \nu_n]$, which are disjoint from U_n ω-a.s. In particular, for every $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ there exist at most three $j \in \{1, 2, \ldots, l\}$ such that $\mathbf{U} \subseteq Y_j$.

Therefore the previous sum is at most $3CE \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. We have thus obtained that $\frac{1}{3}\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant 3CE \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

The inequality

$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant \sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$$

immediately follows from the definition of $\widetilde{\mathrm{dist}}$.

In remains to prove the inequality

$$\sum_{\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})} \mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < E\mathrm{dist}(\boldsymbol{\mu}, \boldsymbol{\nu}). \tag{11}$$

It suffices to prove (11) for every possible finite subsum of the left-hand side of (11). Note that this would imply also that the set of $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ is countable, since it implies that the set of $\mathbf{U} \in \mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ with $\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 1/k$ has cardinality at most $kE\mathrm{dist}(\boldsymbol{\mu}, \boldsymbol{\nu})$.

Let $\mathbf{U}_1, \ldots, \mathbf{U}_m$ be elements in $\mathcal{Y}(\boldsymbol{\mu}, \boldsymbol{\nu})$ represented by sequences $(U_{i,n})$ of large domains of hierarchy paths connecting μ_n and ν_n, with $i = 1, \ldots, m$.

By definition, the sum

$$\mathrm{dist}_{\mathbf{U}_1}(\boldsymbol{\mu}, \boldsymbol{\nu}) + \ldots + \mathrm{dist}_{\mathbf{U}_m}(\boldsymbol{\mu}, \boldsymbol{\nu}) \tag{12}$$

is equal to

$$\begin{aligned} &\lim_\omega \left(\frac{\mathrm{dist}_{\mathcal{M}(U_{1,n})}(\mu_n, \nu_n)}{d_n} \right) + \ldots + \lim_\omega \left(\frac{\mathrm{dist}_{\mathcal{M}(U_{m,n})}(\mu_n, \nu_n)}{d_n} \right) \\ &= \lim_\omega \frac{1}{d_n} [\mathrm{dist}_{\mathcal{M}(U_{1,n})}(\mu_n, \nu_n) + \ldots + \mathrm{dist}_{\mathcal{M}(U_{m,n})}(\mu_n, \nu_n)]. \end{aligned} \tag{13}$$

According to the distance formula (Theorem 2.25), there exist constants a and b depending only on $\xi(S)$ so that the following holds:

$$\begin{aligned}&\mathrm{dist}_{\mathcal{M}(U_{1,n})}(\mu_n,\nu_n)+\ldots+\mathrm{dist}_{\mathcal{M}(U_{m,n})}(\mu_n,\nu_n)\\&\quad\leqslant_{a,b}\sum_{V\subseteq U_{1,n}}\mathrm{dist}_{C(V)}(\mu_n,\nu_n)+\ldots+\sum_{V\subseteq U_{m,n}}\mathrm{dist}_{C(V)}(\mu_n,\nu_n).\end{aligned}\tag{14}$$

Since each $U_{i,n}$ is a large domain in the hierarchy connecting μ_n and ν_n, and since, for each fixed n, all of the $U_{i,n}$ are different ω-a.s., we can apply Lemma 2.20, and conclude that each summand occurs in the right-hand side of (14) at most 2ξ times (where ξ denotes $\xi(S)$). Hence we can bound the right-hand side of (14) from above by

$$2\xi\sum_{V\subseteq S}\mathrm{dist}_{C(V)}(\mu_n,\nu_n)\leqslant_{a,b}2\xi\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,\nu_n).$$

Therefore the right-hand side in (13) does not exceed

$$2\xi\lim_\omega\left(\frac{1}{d_n}(a\,\mathrm{dist}_{\mathcal{M}(S)}(\mu_n,\nu_n)+b)\right)=2a\xi\mathrm{dist}_{\mathcal{AM}}(\boldsymbol{\mu},\boldsymbol{\nu}),$$

proving (11). □

Notation. Let $\boldsymbol{\mu}^0$ be a fixed point in $\mathcal{AM}$ and, for every $\mathbf{U}\in\Pi\mathcal{U}/\omega$, let $\boldsymbol{\mu}^0_{\mathbf{U}}$ be the image of $\boldsymbol{\mu}^0$ by canonical projection on $T_{\mathbf{U}}$. In $\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}$ we consider the subset $\mathcal{T}_0'=\{(x_{\mathbf{U}})\in\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}\;;\;x_{\mathbf{U}}\neq\boldsymbol{\mu}^0_{\mathbf{U}}$ for countably many $\mathbf{U}\in\Pi\mathcal{U}/\omega\}$, and $\mathcal{T}_0=\{(x_{\mathbf{U}})\in\mathcal{T}_0'\;;\;\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega}\widetilde{\mathrm{dist}}_{\mathbf{U}}(x_{\mathbf{U}},\boldsymbol{\mu}^0_{\mathbf{U}})<\infty\}$. We always consider $\mathcal{T}_0$ endowed with the ℓ^1 metric.

The following is an immediate consequence of Theorem 4.16 and Lemma 4.15.

Corollary 4.17. *Consider the map $\psi:\mathcal{AM}\to\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega}T_{\mathbf{U}}$ whose components are the canonical projections of $\mathcal{AM}$ onto $T_{\mathbf{U}}$. This map is a bi-Lipschitz homeomorphism onto its image in $\mathcal{T}_0$.*

Proposition 4.18. *Let $\mathfrak{h}\subset\mathcal{AM}$ denote the ultralimit of a sequence of quasi-geodesics in $\mathcal{M}$ each of which is $\mathcal{C}(U)$-monotonic for every $U\subseteq S$ with the quasi-geodesics and monotonicity constants all uniform over the sequence. Then $\psi(\mathfrak{h})$ is a geodesic in $\mathcal{T}_0$.*

In particular, for any hierarchy path $\mathfrak{h}\subset\mathcal{AM}$, its image under ψ is a geodesic in $\mathcal{T}_0$.

The first statement of this proposition is a direct consequence of the following lemma, which is an easy exercise in elementary topology. The second statement is a consequence of the first.

Lemma 4.19. *Let $(X_i,\mathrm{dist}_i)_{i\in I}$ be a collection of metric spaces. Fix a point $x=(x_i)\in\prod_{i\in I}X_i$, and consider the subsets*

$$S_0'=\left\{(y_i)\in\prod_{i\in I}X_i : y_i\neq x_i\text{ for countably many }i\in I\right\}$$

and $S_0=\{(y_i)\in S_0':\sum_{i\in I}\mathrm{dist}_i(y_i,x_i)<\infty\}$ endowed with the ℓ^1-distance $\mathrm{dist}=\sum_{i\in I}\mathrm{dist}_i$.

Let $\mathfrak{h}:[0,a]\to S_0$ be a non-degenerate parametrizations of a topological arc. For each $i\in I$ assume that $\mathfrak{h}$ projects onto a geodesic $\mathfrak{h}_i:[0,a_i]\to X_i$ such that $\mathfrak{h}(0)$ projects onto $\mathfrak{h}_i(0)$ and *the function* $\varphi_i = d_i(\mathfrak{h}_i(0), \mathfrak{h}_i(t)) : [0, a] \to [0, a_i]$ *is a non-decreasing function. Then* $\mathfrak{h}[0, a]$ *is a geodesic in* (S_0, dist).

NOTATION. We write $\widetilde{\mathrm{dist}}$ to denote the ℓ^1 metric on $\mathcal{T}_0$. We abuse notation slightly by also using $\widetilde{\mathrm{dist}}$ to denote both its restriction to $\psi(\mathcal{AM})$ and for the metric on $\mathcal{AM}$ which is the pullback via ψ of $\widetilde{\mathrm{dist}}$. We have that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ for every $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{AM}$, and that $\widetilde{\mathrm{dist}}$ is bi-Lipschitz equivalent to $\mathrm{dist}_{\mathcal{AM}}$, according to Theorem 4.16.

Note that the canonical map $\mathcal{AM} \to \prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega} \mathcal{C}(\mathbf{U})$, whose components are the canonical projections of $\mathcal{AM}$ onto $\mathcal{C}(\mathbf{U})$, ultralimit of complexes of curves, factors through the above bi-Lipschitz embedding. These maps were studied in [**4**], where among other things it was shown that this canonical map is not a bi-Lipschitz embedding (see also Remark 4.3).

4.2. *Dimension of asymptotic cones of mapping class groups*

LEMMA 4.20. *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be two elements in* $\Pi\mathcal{U}/\omega$ *such that either* $\mathbf{U}$ *and* $\mathbf{V}$ *overlap or* $\mathbf{U} \subsetneq \mathbf{V}$. *Then* $\mathcal{Q}(\partial\mathbf{U})$ *projects onto* $T_{\mathbf{V}}$ *in a unique point.*

Proof. Indeed, $\mathcal{Q}(\partial\mathbf{U})$ projects into $\mathcal{Q}(\pi_{\mathcal{M}^\omega\mathbf{V}}(\partial\mathbf{U}))$, which is contained in one piece of $\mathcal{M}^\omega\mathbf{V}$, hence it projects onto one point in $T_{\mathbf{V}}$. □

The following gives an asymptotic analogue of [**5**, Theorem 4.4].

THEOREM 4.21. *Consider a pair* $\mathbf{U}, \mathbf{V}$ *in* $\Pi\mathcal{U}/\omega$.
(1) *If* $\mathbf{U} \cap \mathbf{V} = \emptyset$, *then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is* $T_{\mathbf{U}} \times T_{\mathbf{V}}$.
(2) *If* $\mathbf{U}$ *and* $\mathbf{V}$ *overlap, then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is*

$$(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}}),$$

where u *is the point in* $T_{\mathbf{V}}$ *onto which projects* $\mathcal{Q}(\partial\mathbf{U})$ *and* v *is the point in* $T_{\mathbf{U}}$ *onto which projects* $\mathcal{Q}(\partial\mathbf{V})$ *(see Lemma 4.20).*
(3) *If* $\mathbf{U} \subsetneq \mathbf{V}$, $u \in T_{\mathbf{V}}$ *is the point onto which projects* $\mathcal{Q}(\partial\mathbf{U})$ *and* $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i\in I} \mathcal{C}_i$ *is the decomposition into connected components, then the image of* $\psi_{\mathbf{U},\mathbf{V}}$ *is*

$$\left(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I} (\{t_i\} \times \mathcal{C}_i \right),$$

where t_i *are points in* $T_{\mathbf{U}}$.

Proof. Case (1) is obvious. We prove (2). Let $\boldsymbol{\mu}$ be a point in $\mathcal{AM}$ whose projection on $T_{\mathbf{U}}$ is different from v. Then $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \partial\mathbf{V}) > 0$, which implies that $\lim_\omega \mathrm{dist}_{C(U_n)}(\mu_n, \partial V_n) = +\infty$. Theorem 2.21 implies that ω-a.s. $\mathrm{dist}_{C(V_n)}(\mu_n, \partial U_n) \leqslant D$. Hence $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu})$ and $\mathcal{Q}(\pi_{\mathcal{M}^\omega\mathbf{V}}(\partial\mathbf{U}))$ are in the same piece of $\mathcal{M}^\omega\mathbf{V}$, so $\boldsymbol{\mu}$ projects on $T_{\mathbf{V}}$ in u. The set of $\boldsymbol{\mu}$ in $\mathcal{AM}$ projecting on $T_{\mathbf{U}}$ in v contains $\mathcal{Q}(\partial\mathbf{V})$, hence their projection on $T_{\mathbf{V}}$ is surjective.

We now prove (3). As before the set of $\boldsymbol{\mu}$ projecting on $T_{\mathbf{V}}$ in u contains $\mathcal{Q}(\partial\mathbf{U})$, hence it projects on $T_{\mathbf{U}}$ surjectively.

For every $i \in I$ we choose $\boldsymbol{\mu}_i \in \mathcal{AM}$ whose projection on $T_{\mathbf{V}}$ is in $\mathcal{C}_i$. Then every $\boldsymbol{\mu}$ with projection on $T_{\mathbf{V}}$ in $\mathcal{C}_i$ has the property that any topological arc joining $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu}_i)$ does not intersect the piece containing $\mathcal{Q}(\partial\mathbf{U})$. Otherwise, by property (T_2') of tree-graded spaces the geodesic joining $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu})$ to $\pi_{\mathcal{M}^\omega\mathbf{V}}(\boldsymbol{\mu}_i)$ in $\mathcal{M}^\omega\mathbf{V}$ would intersect the same piece, and since geodesics in $\mathcal{M}^\omega \mathbf{V}$ project onto geodesics in $T_{\mathbf{V}}$ (see [25]) the geodesic in $T_{\mathbf{V}}$ joining the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ would contain u. This would contradict the fact that both projections are in the same connected component $\mathcal{C}_i$.

Take (μ_n) representatives of $\boldsymbol{\mu}$ and (μ_n^i) representatives of $\boldsymbol{\mu}_i$. The above and Lemma 2.18 imply that ω-a.s. $\mathrm{dist}_{C(U_n)}(\mu_n, \mu_n^i) \leqslant M$, hence the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ onto $\mathcal{M}^\omega \mathbf{U}$ are in the same piece. Therefore the projections of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_i$ onto $T_{\mathbf{U}}$ coincide. Thus all elements in $\mathcal{C}_i$ project in $T_{\mathbf{U}}$ in the same point t_i, which is the projection of $\boldsymbol{\mu}_i$. □

Remark 4.22. Note that in cases (2) and (3) the image of $\psi_{\mathbf{U},\mathbf{V}}$ has dimension 1. In case (3) this is due to the Hurewicz–Morita–Nagami Theorem [47, Theorem III.6].

We shall need the following classical Dimension Theory result.

Theorem 4.23 [29]. *Let K be a compact metric space. If, for every $\epsilon > 0$, there exists an ϵ-map $f : K \to X$ (that is, a continuous map with diameter of $f^{-1}(x)$ at most ϵ for every $x \in X$) such that $f(K)$ is of dimension at most n, then K has dimension at most n.*

We give another proof of the following theorem.

Theorem 4.24 (Dimension Theorem [8]). *Every locally compact subset of every asymptotic cone of the mapping class group of a surface S has dimension at most $\xi(S)$.*

Proof. Since every subset of an asymptotic cone is itself a metric space, it is paracompact. This implies that every locally compact subset of the asymptotic cone is a free union of σ-compact subspaces [28, Theorem 7.3, p. 241]. Thus, it suffices to prove that every compact subset in $\mathcal{AM}$ has dimension at most $\xi(S)$. Let K be such a compact subset. For simplicity we see it as a subset of $\psi(\mathcal{AM}) \subset \Pi_{\mathbf{V} \in \Pi\mathcal{U}/\omega} T_{\mathbf{V}}$.

Fix $\epsilon > 0$. Let N be a finite $(\epsilon/4)$-net for $(K, \widetilde{\mathrm{dist}})$, that is, a finite subset such that $K = \bigcup_{a \in N} B_{\widetilde{\mathrm{dist}}}(a, \epsilon/4)$. There exists a finite subset $J_\epsilon \subset \Pi\mathcal{U}/\omega$ such that, for every $a, b \in N$, $\sum_{\mathbf{U} \notin J_\epsilon} \widetilde{\mathrm{dist}}_{\mathbf{U}}(a,b) < \epsilon/2$. Then, for every $x, y \in K$, $\sum_{\mathbf{U} \notin J_\epsilon} \widetilde{\mathrm{dist}}_{\mathbf{U}}(x,y) < \epsilon$. In particular this implies that the projection $\pi_{J_\epsilon} : \Pi_{\mathbf{V} \in \Pi\mathcal{U}/\omega} T_{\mathbf{V}} \to \Pi_{\mathbf{V} \in J_\epsilon} T_{\mathbf{V}}$ restricted to K is an ϵ-map.

We now prove that, for every finite subset $J \subset \Pi\mathcal{U}/\omega$, the projection $\pi_J(K)$ has dimension at most $\xi(S)$, by induction on the cardinality of J. This will complete the proof, by Theorem 4.23.

If the subsurfaces in J are pairwise disjoint, then the cardinality of J is at most $3g+p-3$ and thus the dimension bound follows. So, suppose we have a pair of subsurfaces $\mathbf{U}, \mathbf{V}$ in J which are not disjoint: then they are either nested or overlapping. We deal with the two cases separately.

Suppose $\mathbf{U}, \mathbf{V} \in J$ overlap. Then according to Theorem 4.21, $\psi_{\mathbf{U},\mathbf{V}}(\mathcal{AM})$ is $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$, hence we can write $K = K_{\mathbf{U}} \cup K_{\mathbf{V}}$, where $\pi_{\mathbf{U},\mathbf{V}}(K_{\mathbf{U}}) \subset T_{\mathbf{U}} \times \{u\}$ and $\pi_{\mathbf{U},\mathbf{V}}(K_{\mathbf{V}}) \subset \{v\} \times T_{\mathbf{V}}$. Now $\pi_J(K_{\mathbf{U}}) = \pi_{J \setminus \{\mathbf{U}\}}(K_{\mathbf{U}}) \times \{u\} \subset \pi_{J \setminus \{\mathbf{U}\}}(K) \times \{u\}$, which is of dimension at most $\xi(S)$ by induction hypothesis. Likewise $\pi_J(K_{\mathbf{V}}) = \pi_{J \setminus \{\mathbf{V}\}}(K_{\mathbf{V}}) \times \{v\} \subset \pi_{J \setminus \{\mathbf{V}\}}(K) \times \{v\}$ is of dimension at most $\xi(S)$. It follows that $\pi_J(K)$ is of dimension at most $\xi(S)$.

Assume that $\mathbf{U} \subsetneq \mathbf{V}$. Let u be the point in $T_{\mathbf{V}}$ onto which projects $\mathcal{Q}(\partial \mathbf{U})$ and $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i \in I} \mathcal{C}_i$ be the decomposition into connected components. By Theorem 4.21, $\psi_{\mathbf{U},\mathbf{V}}(\mathcal{AM})$ is $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i \in I} (\{t_i\} \times \mathcal{C}_i)$, where t_i are points in $T_{\mathbf{U}}$. We prove that $\pi_J(K)$ is of dimension at most $\xi(S)$ by means of Theorem 4.23. Let $\delta > 0$. We shall construct a 2δ-map on $\pi_J(K)$ with image of dimension at most $\xi(S)$. Let N be a finite δ-net of $(K, \widetilde{\mathrm{dist}})$. There exist $i_1, \dots, i_m$ in I such that $\pi_{\mathbf{U},\mathbf{V}}(N)$ is contained in $\mathfrak{T} = (T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{j=1}^m (\{t_{i_j}\} \times \mathcal{C}_{i_j})$. The set $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i \in I}(\{t_i\} \times \mathcal{C}_i)$ endowed with the ℓ^1-metric is a tree and $\mathfrak{T}$ is a subtree in it. We consider the nearest point retraction map

$$\mathrm{retr} : \left(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i \in I}(\{t_i\} \times \mathcal{C}_i \right) \longrightarrow \mathfrak{T}$$

which is, moreover, a contraction. This defines a contraction

$$\mathrm{retr}_J : \psi_J(\mathcal{AM}) \longrightarrow \psi_{J \setminus \{\mathbf{U},\mathbf{V}\}}(\mathcal{AM}) \times \mathfrak{T}, \ \mathrm{retr}_J = \mathrm{id} \times \mathrm{retr}.$$

The set $\pi_J(K)$ splits as $K_{\mathfrak{T}} \sqcup K'$, where $K_{\mathfrak{T}} = \pi_J(K) \cap \pi_{\mathbf{U},\mathbf{V}}^{-1}(\mathfrak{T})$ and K' is its complementary set. Every $x \in K'$ has $\pi_{\mathbf{U},\mathbf{V}}(x)$ in some $\{t_i\} \times \mathcal{C}_i$ with $i \in I \setminus \{i_1, \dots, i_m\}$. Since there exists $n \in N$ such that x is at distance smaller than δ from $\pi_J(n)$, it follows that $\pi_{\mathbf{U},\mathbf{V}}(x)$ is at distance smaller than δ from $\mathfrak{T}$, hence at distance smaller than δ from $(t_i, u) = \mathrm{retr}(\pi_{\mathbf{U},\mathbf{V}}(x))$. We conclude that $\mathrm{retr}(\pi_{\mathbf{U},\mathbf{V}}(K')) \subset \{t_i \mid i \in I\} \times \{u\} \cap \pi_{\mathbf{U}}(K) \times \{u\}$, hence $\mathrm{retr}_J(K') \subset \pi_{J \setminus \{\mathbf{V}\}}(K) \times \{u\}$, which is of dimension at most $\xi(S)$ by the induction hypothesis.

By definition $\mathrm{retr}_J(K_{\mathfrak{T}}) = K_{\mathfrak{T}}$. The set $K_{\mathfrak{T}}$ splits as $K_{\mathbf{U}} \sqcup \bigsqcup_{j=1}^m K_j$, where $K_{\mathbf{U}} = \pi_J(K) \cap \pi_{\mathbf{U},\mathbf{V}}^{-1}(T_{\mathbf{U}} \times \{u\})$ and $K_j = \pi_J(K) \cap \pi_{\mathbf{U},\mathbf{V}}^{-1}(\{t_{i_j}\} \times \mathcal{C}_{i_j})$. Now $K_{\mathbf{U}} \subset \pi_{J \setminus \{\mathbf{V}\}}(K) \times \{u\}$, while $K_j \subset \pi_{J \setminus \{\mathbf{U}\}}(K) \times \{t_{i_j}\}$ for $j = 1, \dots, m$, hence by the induction hypothesis they have dimension at most $\xi(S)$. Consequently $K_{\mathfrak{T}}$ has dimension at most $\xi(S)$.

We have obtained that the map retr_J restricted to $\pi_J(K)$ is a 2δ-map with image $K_{\mathfrak{T}} \cup \mathrm{retr}_J(K')$ of dimension at most $\xi(S)$. It follows that $\pi_J(K)$ is of dimension at most $\xi(S)$. □

4.3. *The median structure*

More can be said about the structure of $\mathcal{AM}$ endowed with $\widetilde{\mathrm{dist}}$. We recall that a *median space* is a metric space for which, given any triple of points, there exists a unique *median point*, which is a point that is simultaneously between any two points in that triple. A point x is said to be *between* two other points a and b in a metric space (X, dist) if $\mathrm{dist}(a, x) + \mathrm{dist}(x, b) = \mathrm{dist}(a, b)$. See [19] for details.

THEOREM 4.25. *The asymptotic cone $\mathcal{AM}$ endowed with the metric $\widetilde{\mathrm{dist}}$ is a median space. Moreover hierarchy paths (that is, ultralimits of hierarchy paths) are geodesics in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$.*

The second statement follows from Proposition 4.18. Note that the first statement is equivalent to that of $\psi(\mathcal{AM})$ being a median subspace of the median space $(\mathcal{T}_0, \widetilde{\mathrm{dist}})$. The proof is done in several steps.

LEMMA 4.26. *Let $\boldsymbol{\nu}$ be in $\mathcal{AM}$ and $\boldsymbol{\Delta} = (\Delta_n)^\omega$, where Δ_n is a multicurve. Let $\boldsymbol{\nu}'$ be the projection of $\boldsymbol{\nu}$ on $\mathcal{Q}(\boldsymbol{\Delta})$. Then, for every subsurface $\mathbf{U}$ such that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$ (that is, $\mathbf{U}$ does not overlap Δ), the distance $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}') = 0$.*

Proof. The projection of $\boldsymbol{\nu}$ on $\mathcal{Q}(\boldsymbol{\Delta})$ is defined as limit of projections described in Paragraph 2.5.2. Since $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$, the subsurface $\mathbf{U} = (U_n)^\omega$ is contained in a component, $\mathbf{V} = (V_n)^\omega$, of $S \setminus \boldsymbol{\Delta} = (S \setminus \Delta_n)^\omega$. The marking ν_n', by construction, does not differ from the intersection of ν_n with V_n, and since $U_n \subseteq V_n$ the same is true for U_n, hence $\mathrm{dist}_{C(U_n)}(\nu_n, \nu_n') = O(1)$. On the other hand, if $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}') > 0$, then by Lemma 4.15 $\lim_\omega \mathrm{dist}_{C(U_n)}(\boldsymbol{\nu}_n, \boldsymbol{\nu}_n') = +\infty$, whence a contradiction. □

LEMMA 4.27. *Let $\boldsymbol{\nu} = \lim_\omega(\nu_n)$ and $\boldsymbol{\rho} = \lim_\omega(\rho_n)$ be two points in $\mathcal{AM}$, let $\boldsymbol{\Delta} = (\Delta_n)^\omega$, where Δ_n is a multicurve, and let $\boldsymbol{\nu}', \boldsymbol{\rho}'$ be the respective projections of $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ on $\mathcal{Q}(\boldsymbol{\Delta})$. Assume there exist $\mathbf{U}_1 = (U_n^1)^\omega, \dots, \mathbf{U}_k = (U_n^k)^\omega$ subsurfaces such that $\Delta_n = \partial U_n^1 \cup \dots \cup \partial U_n^k$, and $\mathrm{dist}_{C(U_n^i)}(\nu_n, \rho_n) > M$ ω-a.s. for every $i = 1, \dots, k$, where M is the constant in Lemma 2.18.*

Then for every $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ hierarchy paths joining $\boldsymbol{\nu}, \boldsymbol{\nu}'$, $\boldsymbol{\nu}', \boldsymbol{\rho}'$ and $\boldsymbol{\rho}', \boldsymbol{\rho}$, respectively, the path, $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$.

Proof. Let $\mathbf{V} \in \Pi\mathcal{U}/\omega$ be an arbitrary subsurface. According to Lemma 4.12, $\psi_{\mathbf{V}}(\mathfrak{h}_i)$, $i = 1, 2, 3$, is a geodesic in $T_{\mathbf{V}}$. We shall prove that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$.

There are two cases: either $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}$ (that is, $\mathbf{V}$ does not overlap Δ) or $\mathbf{V} \pitchfork \boldsymbol{\Delta}$ (that is, $\mathbf{V}$ overlaps Δ). In the first case, by Lemma 4.26 the projections $\psi_{\mathbf{V}}(\mathfrak{h}_1)$ and $\psi_{\mathbf{V}}(\mathfrak{h}_3)$ are singletons, and there is nothing to prove.

Assume now that $\mathbf{V} \pitchfork \boldsymbol{\Delta}$. Then $\mathbf{V} \pitchfork \partial\mathbf{U}_i$ for some $i \in \{1, \dots, k\}$.

We have that ω-a.s.

$$\mathrm{dist}_{C(U_n)}(\nu_n', \rho_n') \leqslant \mathrm{dist}_{C(U_n)}(\nu_n', \Delta_n) + \mathrm{dist}_{C(U_n)}(\rho_n', \Delta_n) = O(1).$$

Lemma 4.15 then implies that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}', \boldsymbol{\rho}') = 0$. Hence, $\psi_{\mathbf{V}}(\mathfrak{h}_2)$ reduces to a singleton x which is the projection onto $T_{\mathbf{V}}$ of both $\mathcal{Q}(\boldsymbol{\Delta})$ and $\mathcal{Q}(\partial\mathbf{U}_i)$. It remains to prove that $\psi_{\mathbf{V}}(\mathfrak{h}_1)$ and $\psi_{\mathbf{V}}(\mathfrak{h}_3)$ have in common only x. Assume on the contrary that they are both geodesic with a common non-trivial subgeodesic containing x. Then the geodesic in $T_{\mathbf{V}}$ joining $\psi_{\mathbf{V}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\rho})$ does not contain x. On the other hand, by hypothesis and Lemma 2.18 any hierarchy path joining $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ contains a point in $\mathcal{Q}(\partial\mathbf{U}_i)$. Lemma 4.12 implies that the geodesic in $T_{\mathbf{V}}$ joining $\psi_{\mathbf{V}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\rho})$ contains x, yielding a contradiction.

Thus $\psi_{\mathbf{V}}(\mathfrak{h}_1) \cap \psi_{\mathbf{V}}(\mathfrak{h}_3) = \{x\}$ and $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$ also in this case.

We proved that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$ for every $\mathbf{V} \in \Pi\mathcal{U}/\omega$. This implies that $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{\boldsymbol{\nu}'\}$ and $\mathfrak{h}_2 \cap \mathfrak{h}_3 = \{\boldsymbol{\rho}'\}$, and that $\mathfrak{h}_1 \cap \mathfrak{h}_3 = \emptyset$ if $\mathfrak{h}_2$ is non-trivial, while $\mathfrak{h}_1 \cap \mathfrak{h}_3 = \{\boldsymbol{\nu}'\}$, if $\mathfrak{h}_2$ reduces to a singleton $\boldsymbol{\nu}'$. Indeed if, for instance, $\mathfrak{h}_1 \cap \mathfrak{h}_2$ contained a point $\boldsymbol{\mu} \neq \boldsymbol{\nu}'$ then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$ whence $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$ for some subsurface $\mathbf{V}$. It would follow that $\psi_{\mathbf{V}}(\mathfrak{h}_1), \psi_{\mathbf{V}}(\mathfrak{h}_2)$ have in common a non-trivial subgeodesic, contradicting the proven statement.

Also, if $\mathfrak{h}_1 \cap \mathfrak{h}_3$ contains a point $\boldsymbol{\mu} \neq \boldsymbol{\nu}'$, then, for some subsurface $\mathbf{V}$ such that $\mathbf{V} \cap \boldsymbol{\Delta} \neq \emptyset$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\nu}') > 0$. Since $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\nu}', \boldsymbol{\rho}') = 0$, it follows that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\rho}') > 0$ and that $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is not a geodesic in $T_{\mathbf{V}}$. A similar contradiction occurs if $\boldsymbol{\mu} \neq \boldsymbol{\rho}'$. Therefore if $\boldsymbol{\mu}$ is a point in $\mathfrak{h}_1 \cap \mathfrak{h}_3$, then we must have $\boldsymbol{\mu} = \boldsymbol{\nu}' = \boldsymbol{\rho}'$, in particular $\mathfrak{h}_2$ reduces to a point, which is the only point that $\mathfrak{h}_1$ and $\mathfrak{h}_3$ have in common.

Thus in all cases $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a topological arc. Since $\mathfrak{h}_i$, for $i = 1, 2, 3$, each satisfy the hypotheses of Lemma 4.18 and, for every $\mathbf{V} \in \Pi\mathcal{U}/\omega$, $\psi_{\mathbf{V}}(\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3)$ is a geodesic in $T_{\mathbf{V}}$, it follows that $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ also satisfies the hypotheses of Lemma 4.18. We may therefore conclude that $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$. □

DEFINITION 4.28. A point $\boldsymbol{\mu}$ in $\mathcal{AM}$ is *between* the points $\boldsymbol{\nu}, \boldsymbol{\rho}$ in $\mathcal{AM}$ if, for every $\mathbf{U} \in \Pi\mathcal{U}/\omega$, the projection $\psi_{\mathbf{U}}(\boldsymbol{\mu})$ is in the geodesic joining $\psi_{\mathbf{U}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{U}}(\boldsymbol{\rho})$ in $T_{\mathbf{U}}$ (possibly identical to one of its endpoints).

LEMMA 4.29. *For every triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$, every choice of a pair $\boldsymbol{\nu}, \boldsymbol{\rho}$ in the triple and every finite subset F in $\Pi\mathcal{U}/\omega$ of pairwise disjoint subsurfaces, there exists a point $\boldsymbol{\mu}$ between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu}), \psi_F(\boldsymbol{\rho}), \psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U} \in F} T_{\mathbf{U}}$.*

Proof. Let $F = \{\mathbf{U}_1, \dots, \mathbf{U}_k\}$, where $\mathbf{U}_i = (U_n^i)^\omega$. We argue by induction on k. If $k = 1$, then the statement follows immediately from Lemma 4.12. We assume that the statement is true for all $i < k$, where $k \geqslant 2$, and we prove it for k.

We consider the multicurve $\Delta_n = \partial U_n^1 \cup \ldots \cup \partial U_n^k$. We denote the set $\{1, 2, \dots, k\}$ by I. If, for some $i \in I$, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\rho}) = 0$, then the median point of $\psi_{\mathbf{U}_i}(\boldsymbol{\nu})$, $\psi_{\mathbf{U}_i}(\boldsymbol{\rho})$, $\psi_{\mathbf{U}_i}(\boldsymbol{\sigma})$ is $\psi_{\mathbf{U}_i}(\boldsymbol{\nu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\rho})$. By the induction hypothesis there exists $\boldsymbol{\mu}$ between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{i\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F\setminus\{i\}}(\boldsymbol{\nu}), \psi_{F\setminus\{i\}}(\boldsymbol{\rho}), \psi_{F\setminus\{i\}}(\boldsymbol{\sigma})$. Since $\psi_{\mathbf{U}_i}(\boldsymbol{\mu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\nu}) = \psi_{\mathbf{U}_i}(\boldsymbol{\rho})$, it follows that the desired statement holds not just for $F \setminus \{i\}$, but for all of F as well.

Assume now that, for all $i \in I$, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\rho}) > 0$. Lemma 4.15 implies that $\lim_\omega \mathrm{dist}_{C(U_n^i)}(\nu_n, \rho_n) = \infty$.

Let $\boldsymbol{\nu}', \boldsymbol{\rho}', \boldsymbol{\sigma}'$ be the respective projections of $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ onto $\mathcal{Q}(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta} = (\Delta_n)^\omega$. According to Lemma 4.26, $\widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\nu}, \boldsymbol{\nu}') = \widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\rho}, \boldsymbol{\rho}') = \widetilde{\mathrm{dist}}_{\mathbf{U}_i}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = 0$ for every $i \in I$, whence $\psi_F(\boldsymbol{\nu}) = \psi_F(\boldsymbol{\nu}'), \psi_F(\boldsymbol{\rho}) = \psi_F(\boldsymbol{\rho}'), \psi_F(\boldsymbol{\sigma}) = \psi_F(\boldsymbol{\sigma}')$. This and Lemma 4.27 imply that it suffices to prove the statement for $\boldsymbol{\nu}', \boldsymbol{\rho}', \boldsymbol{\sigma}'$. Thus, without loss of generality, we may assume that $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ are in $\mathcal{Q}(\boldsymbol{\Delta})$. Also without loss of generality we may assume that $\{U_n^1, U_n^2, \dots, U_n^k\}$ are all the connected components of $S \setminus \Delta_n$ and all the annuli with core curves in Δ_n. If not, we may add the missing subsurfaces.

For every $i \in I$ we consider the projections $\nu_n^i, \rho_n^i \in \mathcal{M}(U_n^i)$ of ν_n and, respectively, ρ_n on $\mathcal{M}(U_n^i)$. Let $\mathfrak{g}_n^i$ be a hierarchy path in $\mathcal{M}(U_n^i)$ joining ν_n^i, ρ_n^i and let $\mathfrak{g}^i = \lim_\omega(\mathfrak{g}_n^i)$ be the limit hierarchy path in $\mathcal{M}(\mathbf{U}_i)$. According to Lemma 4.12, for every $i \in I$ there exists μ_n^i on $\mathfrak{g}_n^i$ such that $\boldsymbol{\mu^i} = \lim_\omega(\mu_n^i)$ projects on $T_{\mathbf{U}_i}$ on the median point of the projections of $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$. Let $\mathfrak{p}_n^i$ and $\mathfrak{q}_n^i$ be the subpaths of $\mathfrak{g}_n^i$, respectively, preceding and succeeding μ_n^i on $\mathfrak{g}_n^i$, and let $\mathfrak{p}^i = \lim_\omega(\mathfrak{p}_n^i)$ and $\mathfrak{q}^i = \lim_\omega(\mathfrak{q}_n^i)$ be the limit hierarchy paths in $\mathcal{M}(\mathbf{U}_i)$.

Let $\widetilde{\mathfrak{p}}_n^1$ be a path in $\mathcal{M}(S)$ that starts at ν_n and then continues on a path obtained by markings whose restriction to U_1^n are given by $\mathfrak{p}_n^1$ and in the complement of U_1^n are given by the restriction of ν_n. Continue this path by concatenating a path $\widetilde{\mathfrak{p}}_n^2$, obtained by starting from the terminal point of $\widetilde{\mathfrak{p}}_n^1$ and then continuing by markings which are all the same in the complement of U_n^2, while their restriction to U_n^2 are given by $\mathfrak{p}_n^2$. Similarly, we obtain $\mathfrak{p}_n^j$ from $\mathfrak{p}_n^{j-1}$ for any $j \leqslant k$. Note that, for any $1 \leqslant j \leqslant k$ and any $i \neq j$, the path $\mathfrak{p}_n^j$ restricted to U_n^i is constant. Note that the starting point of $\widetilde{\mathfrak{p}}_n^1 \sqcup \ldots \sqcup \widetilde{\mathfrak{p}}_n^k$ is ν_n and the terminal point is the marking μ_n with the property that it projects on $\mathcal{M}(U_n^i)$ in μ_n^i for every $i \in I$. Now consider $\widetilde{\mathfrak{q}}_n^1$ the path with starting point μ_n obtained following $\mathfrak{q}_n^1$ (and keeping the projections onto U_n^i with $i \in I \setminus \{1\}$ unchanged), then $\widetilde{\mathfrak{q}}_n^2, \dots, \widetilde{\mathfrak{q}}_n^k$ constructed such that the starting point of $\widetilde{\mathfrak{q}}_n^j$ is the terminal point of $\widetilde{\mathfrak{q}}_n^{j-1}$, and $\widetilde{\mathfrak{q}}_n^j$ is obtained following $\mathfrak{q}_n^j$ (and keeping the projections onto U_n^i with $i \in I \setminus \{j\}$ unchanged).

Let $\widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{j}} = \lim_\omega(\widetilde{\mathfrak{p}}_n^j)$ and $\widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{j}} = \lim_\omega(\widetilde{\mathfrak{q}}_n^j)$. We prove that, for every subsurface $\mathbf{V} = (V_n)^\omega$, the path $\boldsymbol{\mathfrak{h}} = \widetilde{\boldsymbol{\mathfrak{p}}}^{\mathbf{1}} \sqcup \ldots \sqcup \widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{k}} \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\mathbf{1}} \sqcup \ldots \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{k}}$ projects onto a geodesic in $T_{\mathbf{V}}$. For any $i \neq j$, we have that $\widetilde{\mathfrak{p}}_n^i \cup \widetilde{\mathfrak{q}}_n^i$ and $\widetilde{\mathfrak{p}}_n^j \cup \widetilde{\mathfrak{q}}_n^j$ have disjoint support. Hence, for each $i \in I$ we have that the restriction to U_n^i of the entire path is the same as the restriction to U_n^i of $\widetilde{\mathfrak{p}}_n^i \cup \widetilde{\mathfrak{q}}_n^i$. Since the latter is by construction the hierarchy path $\mathfrak{g}_n^i$, if $\mathbf{V} \subset \mathbf{U}_i$ for some $i \in I$, then it follows from Lemma 4.18 that $\boldsymbol{\mathfrak{h}}$ projects to a geodesic in $T_{\mathbf{V}}$. If $\mathbf{V}$ is disjoint from $\mathbf{U}_i$, then all the markings composing $\mathfrak{h}_n'$ have the same intersection with V_n, whence the diameter of $\mathfrak{h}_n$ with respect to $\mathrm{dist}_{C(V_n)}$ must be uniformly bounded. This and Lemma 4.15 implies that $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is a singleton. Lastly, if $\mathbf{V}$ contains or overlaps $\mathbf{U}_i$, then since all the markings in $\mathfrak{h}_n'$ contain ∂U_n^i, the diameter of $\mathfrak{h}_n$ with respect to $\mathrm{dist}_{C(V_n)}$ must be uniformly bounded, leading again to the conclusion that $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is a singleton. Thus, the only case when $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}')$ is not a singleton is when $\mathbf{V} \subseteq \mathbf{U}_i$.

Now let $\mathbf{V}$ denote an arbitrary subsurface. If it is not contained in any $\mathbf{U}_i$, then $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}})$ is a singleton. The other situation is when $\mathbf{V}$ is contained in some $\mathbf{U}_i$, hence disjoint from all $\mathbf{U}_j$ with $j \neq i$. Then $\psi_{\mathbf{V}}(\boldsymbol{\mathfrak{h}}) = \psi_{\mathbf{V}}(\widetilde{\boldsymbol{\mathfrak{p}}}^{\boldsymbol{i}} \sqcup \widetilde{\boldsymbol{\mathfrak{q}}}^{\boldsymbol{i}}) = \psi_{\mathbf{V}}^i(\mathfrak{p}^i \sqcup \mathfrak{q}^i) = \psi_{\mathbf{V}}^i(\mathfrak{g}^i)$, which is a geodesic. Note that in the last two inequalities the map $\psi_{\mathbf{V}}^i$ is the natural projection of $\mathcal{M}(\mathbf{U}_i)$ onto $T_{\mathbf{V}}$ which exists when $\mathbf{V} \subseteq \mathbf{U}_i$.

Thus we have shown that $\mathfrak{h}$ projects onto a geodesic in $T_{\mathbf{V}}$ for every $\mathbf{V}$, whence $\boldsymbol{\mu}$ is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$. By construction, for every $i \in I$, $\psi_{\mathbf{U}_i}(\boldsymbol{\mu})$ is the median point of $\psi_{\mathbf{U}_i}(\boldsymbol{\nu}), \psi_{\mathbf{U}_i}(\boldsymbol{\rho}), \psi_{\mathbf{U}_i}(\boldsymbol{\sigma})$; equivalently $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu})$, $\psi_F(\boldsymbol{\rho})$, $\psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$. $\square$

We now generalize the last lemma by removing the hypothesis that the subsurfaces are disjoint.

LEMMA 4.30. *For every triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$, every choice of a pair $\boldsymbol{\nu}, \boldsymbol{\rho}$ in the triple and every finite subset F in $\Pi\mathcal{U}/\omega$ there exists a point $\boldsymbol{\mu}$ in $\mathcal{AM}$ between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu}), \psi_F(\boldsymbol{\rho}), \psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$.*

Proof. We prove the statement by induction on the cardinality of F. When $\operatorname{card} F = 1$ it follows from Lemma 4.12. Assume that it is true whenever $\operatorname{card} F < k$ and consider F of cardinality $k \geqslant 2$. If the subsurfaces in F are pairwise disjoint, then we can apply Lemma 4.29, hence we may assume that there exists a pair of subsurfaces $\mathbf{U}, \mathbf{V}$ in F which either overlap or are nested.

First, assume that $\mathbf{U}$ and $\mathbf{V}$ overlap. Then $\psi_{\mathbf{U},\mathbf{V}}$ is equal to $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$, by Theorem 4.21. We write $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$ to denote the image $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\nu})$ and let $\boldsymbol{\nu}_{\mathbf{U}}$ and $\boldsymbol{\nu}_{\mathbf{V}}$ denote its coordinates (that is, $\psi_{\mathbf{U}}(\boldsymbol{\nu})$ and $\psi_{\mathbf{V}}(\boldsymbol{\nu})$). We use similar notation for $\boldsymbol{\rho}, \boldsymbol{\sigma}$. If the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is not (v, u) then it is either some point (x, u) with $x \in T_{\mathbf{U}} \setminus \{v\}$, or (v, y) with $y \in T_{\mathbf{V}} \setminus \{u\}$. In the first case, by the induction hypothesis, there exists a point $\boldsymbol{\mu}_1$ between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\nu}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\rho}), \psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\sigma})$. In particular, $\psi_{\mathbf{U}}(\boldsymbol{\mu}) = x$, hence $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is a point in $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$ having the first coordinate x. Since there exists only one such point (x, u), it follows that $\psi_{\mathbf{V}}(\boldsymbol{\mu}) = u$. Thus, for every $\mathbf{Y} \in F$, the point $\psi_{\mathbf{Y}}(\boldsymbol{\mu})$ is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This is equivalent to the fact that $\psi_F(\boldsymbol{\mu})$ is the median point in $\prod_{\mathbf{Y}} T_{\mathbf{Y}}$ of $\psi_F(\boldsymbol{\nu})$, $\psi_F(\boldsymbol{\rho})$ and $\psi_F(\boldsymbol{\sigma})$. A similar argument works when the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}, \boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is a point (v, y) with $y \in T_{\mathbf{V}} \setminus \{u\}$.

Hence, we may assume that the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is (v, u). Let $\boldsymbol{\mu}_1$ be a point between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\nu})$, $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\rho})$ and $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\sigma})$, and let $\boldsymbol{\mu}_2$ be a point between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\nu})$, $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\rho})$ and $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\sigma})$. In particular $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_1) = (v, y)$ with $y \in T_{\mathbf{V}}$ and $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_2) = (x, u)$ with $x \in T_{\mathbf{U}}$. Any hierarchy path joining $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ is mapped by $\psi_{\mathbf{U},\mathbf{V}}$ onto a path joining (v, y) and (x, u) in $(T_{\mathbf{U}} \times \{u\}) \cup (\{v\} \times T_{\mathbf{V}})$. Therefore it contains a point $\boldsymbol{\mu}$ such that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is (v, u). According to Lemma 4.12 $\boldsymbol{\mu}$ is between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, hence it is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$; moreover, for every $\mathbf{Y} \in F \setminus \{\mathbf{U}, \mathbf{V}\}$, $\psi_{\mathbf{Y}}(\boldsymbol{\mu}) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_1) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_2)$, and it is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This and the fact that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}) = (v, u)$ is the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ complete the argument in this case.

We now consider the case where $\mathbf{U} \subsetneq \mathbf{V}$. Let u be the point in $T_{\mathbf{V}}$ which is the projection of $\mathcal{Q}(\partial\mathbf{U})$ and let $T_{\mathbf{V}} \setminus \{u\} = \bigsqcup_{i\in I} \mathcal{C}_i$ be the decomposition into connected components. By Theorem 4.21, the image of $\psi_{\mathbf{U},\mathbf{V}}$ is $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$ where t_i are points in $T_{\mathbf{U}}$. If the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is not in the set $\{(t_i, u) \mid i \in I\}$, then we are done as in the previous case using the induction hypothesis as well as the fact that for such points there are no other points having the same first coordinate or the same second coordinate.

Thus, we may assume that the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ is (t_i, u) for some $i \in I$. Let $\boldsymbol{\mu}_1$ be a point between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\mu}_1)$ is the median point of $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\nu})$, $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\rho})$ and $\psi_{F\setminus\{\mathbf{V}\}}(\boldsymbol{\sigma})$, and let $\boldsymbol{\mu}_2$ be a point between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\mu})$ is the median point of $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\nu})$, $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\rho})$ and $\psi_{F\setminus\{\mathbf{U}\}}(\boldsymbol{\sigma})$. In particular, $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_1) = (t_i, y)$ with $y \in C_i$ and $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}_2) = (x, u)$ with $x \in T_{\mathbf{U}}$. Any hierarchy path joining $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ is mapped by $\psi_{\mathbf{U},\mathbf{V}}$ onto a path joining (t_i, y) and (x, u) in $(T_{\mathbf{U}} \times \{u\}) \cup \bigsqcup_{i\in I}(\{t_i\} \times \mathcal{C}_i)$. It contains a point $\boldsymbol{\mu}$ such that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu})$ is (t_i, u). By Lemma 4.12, the point $\boldsymbol{\mu}$ is between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, hence in particular it is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$. Moreover, for every $\mathbf{Y} \in F \setminus \{\mathbf{U}, \mathbf{V}\}$, $\psi_{\mathbf{Y}}(\boldsymbol{\mu}) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_1) = \psi_{\mathbf{Y}}(\boldsymbol{\mu}_2)$ and hence $\boldsymbol{\mu}$ is the median point in $T_{\mathbf{Y}}$ of $\psi_{\mathbf{Y}}(\boldsymbol{\nu})$, $\psi_{\mathbf{Y}}(\boldsymbol{\rho})$ and $\psi_{\mathbf{Y}}(\boldsymbol{\sigma})$. This and the fact that $\psi_{\mathbf{U},\mathbf{V}}(\boldsymbol{\mu}) = (t_i, u)$ is the median point of $\boldsymbol{\nu}_{\mathbf{U},\mathbf{V}}$, $\boldsymbol{\rho}_{\mathbf{U},\mathbf{V}}$ and $\boldsymbol{\sigma}_{\mathbf{U},\mathbf{V}}$ complete the argument. □

Proof of Theorem 4.25. Consider an arbitrary triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}$ in $\mathcal{AM}$. For every $\epsilon > 0$ there exists a finite subset F in $\Pi\mathcal{U}/\omega$ such that $\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) < \epsilon$ for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$. By Lemma 4.30 there exists $\boldsymbol{\mu}$ in $\mathcal{AM}$ between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ such that $\psi_F(\boldsymbol{\mu})$ is the median point of $\psi_F(\boldsymbol{\nu})$, $\psi_F(\boldsymbol{\rho})$ and $\psi_F(\boldsymbol{\sigma})$ in $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$. The latter implies that, for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$,

$$\sum_{\mathbf{U}\in F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) = \sum_{\mathbf{U}\in F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{\mu}) + \sum_{\mathbf{U}\in F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{b}).$$

Also, since $\boldsymbol{\mu}$ is between $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ it follows that

$$\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\mu}) < \epsilon \quad \text{and} \quad \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\rho}) < \epsilon,$$

whence

$$\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega\setminus F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\sigma}) < 2\epsilon.$$

It follows that, for every $\boldsymbol{a}, \boldsymbol{b}$ in $\{\boldsymbol{\nu}, \boldsymbol{\rho}, \boldsymbol{\sigma}\}$,

$$\sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{\mu}) + \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{b}) \leqslant \sum_{\mathbf{U}\in\Pi\mathcal{U}/\omega} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{a}, \boldsymbol{b}) + 3\epsilon.$$

That is, $\widetilde{\mathrm{dist}}(\boldsymbol{a}, \boldsymbol{\mu}) + \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{b}) \leqslant \widetilde{\mathrm{dist}}(\boldsymbol{a}, \boldsymbol{b}) + 3\epsilon$. This and [19, Section 2.3] imply that $\psi(\boldsymbol{\mu})$ is at distance at most 5ϵ from the median point of $\psi(\boldsymbol{\nu})$, $\psi(\boldsymbol{\rho})$ and $\psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$.

We have thus proved that for every $\epsilon > 0$ there exists a point $\psi(\boldsymbol{\mu})$ in $\psi(\mathcal{AM})$ at distance at most 5ϵ from the median point of $\psi(\boldsymbol{\nu})$, $\psi(\boldsymbol{\rho})$ and $\psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$. Now the asymptotic cone $\mathcal{AM}$ is a complete metric space with the metric $\mathrm{dist}_{\mathcal{AM}}$, hence the bi-Lipschitz equivalent metric space $\psi(\mathcal{AM})$ with the metric $\widetilde{\mathrm{dist}}$ is also complete. Since it is a subspace in the complete metric space $\mathcal{T}_0$, it follows that $\psi(\mathcal{AM})$ is a closed subset in $\mathcal{T}_0$. We may then conclude that $\psi(\mathcal{AM})$ contains the unique median point of $\psi(\boldsymbol{\nu})$, $\psi(\boldsymbol{\rho})$ and $\psi(\boldsymbol{\sigma})$ in $\mathcal{T}_0$. □

5. *Actions on asymptotic cones of mapping class groups and splitting*

5.1. *Pieces of the asymptotic cone*

LEMMA 5.1. *Let* $\lim_\omega(\mu_n)$, $\lim_\omega(\mu'_n)$, $\lim_\omega(\nu_n)$ *and* $\lim_\omega(\nu'_n)$ *be sequences of points in* $\mathcal{M}(S)$ *for which* $\lim_\omega \mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n) = \infty$. *For every* $M > 2K(S)$, *where* $K(S)$ *is the constant in Theorem* 2.25, *there exists a positive constant* $C = C(M) < 1$ *so that if*

$$\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \mu'_n) + \mathrm{dist}_{\mathcal{M}(S)}(\nu_n, \nu'_n) \leqslant C\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n),$$

then there exists a sequence of subsurfaces $Y_n \subseteq S$ *such that for* ω-a.e. n *both* $\mathrm{dist}_{\mathcal{C}(Y_n)}(\mu_n, \nu_n) > M$ *and* $\mathrm{dist}_{\mathcal{C}(Y_n)}(\mu'_n, \nu'_n) > M$.

Proof. Assume that ω-a.s. the sets of subsurfaces

$$\mathcal{Y}_n = \{Y_n \mid \mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) > 2M\} \quad \text{and} \quad \mathcal{Z}_n = \{Z_n \mid \mathrm{dist}_{C(Z_n)}(\mu'_n, \nu'_n) > M\}$$

are disjoint. Then, for every $Y_n \in \mathcal{Y}_n$, $\mathrm{dist}_{C(Y_n)}(\mu_n', \nu_n') \leqslant M$, which by the triangle inequality implies that $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n') + \mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n') \geqslant \mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) - M \geqslant \frac{1}{2}\mathrm{dist}_{C(Y_n)}(\mu_n, \nu_n) > M$. Hence either $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n')$ or $\mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n')$ is larger than $M/2 > K(S)$. Let a and b be the constants appearing in (2) for $K = M/2$, and let A and B be the constants appearing in the same formula for $K' = 2M$. According to the above, we may then write

$$\begin{aligned} \mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \mu_n') + \mathrm{dist}_{\mathcal{M}(S)}(\nu_n, \nu_n') &\geqslant_{a,b} \sum_{Y \in \mathcal{Y}_n} \{\!\{\mathrm{dist}_{\mathcal{C}(Y)}(\mu_n, \mu_n')\}\!\}_K \\ &\quad + \sum_{Y \in \mathcal{Y}_n} \{\!\{\mathrm{dist}_{\mathcal{C}(Y)}(\nu_n, \nu_n')\}\!\}_K \\ &\geqslant \frac{1}{4} \sum_{Y \in \mathcal{Y}_n} \mathrm{dist}_{\mathcal{C}(Y)}(\mu_n, \nu_n) \geqslant_{A,B} \frac{1}{4}\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \nu_n). \end{aligned}$$

The coefficient $\frac{1}{4}$ is accounted for by the case when one of the two distances $\mathrm{dist}_{C(Y_n)}(\mu_n, \mu_n')$ and $\mathrm{dist}_{C(Y_n)}(\nu_n, \nu_n')$ is larger than $K = M/2$ while the other is not.

When C is small enough we thus obtain a contradiction of the hypothesis; hence ω-a.s. $\mathcal{Y}_n \cap \mathcal{Z}_n \neq \emptyset$. $\square$

Definition 5.2. For any $g = (g_n)^\omega \in \mathcal{M}(S)^\omega_b$ let us denote by $U(g)$ the set of points $\boldsymbol{h} \in \mathcal{AM}$ such that, for some representative $(h_n)^\omega \in \mathcal{M}(S)^\omega_b$ of $\boldsymbol{h}$,

$$\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(h_n, g_n) < \infty.$$

The set $U(g)$ is called the *g-interior*. This set is non-empty since $g \in U(g)$.

Lemma 5.3. *Let P be a piece in $\mathcal{AM} = \mathrm{Con}^\omega(\mathcal{M}(S); (d_n))$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be distinct points in P. Then there exists $g = (g_n)^\omega \in \mathcal{M}(S)^\omega_b$ such that $U(g) \subseteq P$; moreover, the intersection of any hierarchy path $[\boldsymbol{x}, \boldsymbol{y}]$ with $U(g)$ contains $[\boldsymbol{x}, \boldsymbol{y}] \setminus \{\boldsymbol{x}, \boldsymbol{y}\}$.*

Proof. Consider arbitrary representatives $(x_n)^\omega, (y_n)^\omega$ of $\boldsymbol{x}$ and respectively $\boldsymbol{y}$, and let $[\boldsymbol{x}, \boldsymbol{y}]$ be the limit of a sequence of hierarchy paths $[x_n, y_n]$. Since $\boldsymbol{x}, \boldsymbol{y} \in P$, there exist sequences of points $\boldsymbol{x}(k) = \lim_\omega(x_n(k)), \boldsymbol{y}(k) = \lim_\omega(y_n(k))$, $x_n(k), y_n(k) \in [x_n, y_n]$ and a sequence of numbers $C(k) > 0$ such that, for ω-almost every n. we have

$$\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), y_n(k)) < C(k)$$

and

$$\begin{aligned} \mathrm{dist}_{\mathcal{M}(S)}(x_n(k), x_n) &< \frac{d_n \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{x}, \boldsymbol{y})}{k}, \\ \mathrm{dist}_{\mathcal{M}(S)}(y_n(k), y_n) &< \frac{d_n \mathrm{dist}_{\mathcal{AM}}(\boldsymbol{x}, \boldsymbol{y})}{k}. \end{aligned} \tag{15}$$

Let $[x_n(k), y_n(k)]$ be the subpath of $[x_n, y_n]$ connecting $x_n(k)$ and $y_n(k)$. Let g_n be the midpoint of $[x_n(n), y_n(n)]$. Then $\lim_\omega(g_n) \in [\boldsymbol{x}, \boldsymbol{y}]$. Let $g = (g_n)^\omega \in \mathcal{M}(S)^\omega_b$. Let us prove that $U(g)$ is contained in P.

Since $\boldsymbol{x}, \boldsymbol{y} \in P$, it is enough to show that any point $\boldsymbol{z} = \lim_\omega(z_n)$ from $U(g)$ is in the same piece with $\boldsymbol{x}$ and in the same piece with $\boldsymbol{y}$ (because distinct pieces cannot have two points in common).

By the definition of $U(g)$, we can assume that ω-a.s. $\mathrm{dist}_{\mathcal{C}(S)}(z_n, g_n) \leqslant C_1$ for some constant C_1. For every $k > 0$, $\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), y_n(k)) \leqslant C(k)$, so

$$\mathrm{dist}_{\mathcal{C}(S)}(x_n(k), z_n), \mathrm{dist}_{\mathcal{C}(S)}(y_n(k), z_n) \leqslant C(k) + C_1$$

ω-a.s. By (15) and Theorem 4.2, $\boldsymbol{x} = \lim_\omega(x_n)$, $\boldsymbol{z} = \lim_\omega(z_n)$ and $\boldsymbol{y} = \lim_\omega(y_n)$ are in the same piece.

Note that $\lim_\omega(x_n(k))$ and $\lim_\omega(y_n(k))$ are in $U(g)$ for every k. Now let $(x'_n)^\omega$ and $(y'_n)^\omega$ be other representatives of $\boldsymbol{x}$ and $\boldsymbol{y}$, and let $x'_n(k)$ and $y'_n(k)$ be chosen as above on a sequence of hierarchy paths $[x'_n, y'_n]$. Let $g' = (g'_n)^\omega$, where g'_n is the point in the middle of the hierarchy path $[x'_n, y'_n]$. We show that $U(g') = U(g)$. Indeed, the sequence of quadruples $x_n(k), y_n(k), y'_n(k), x'_n(k)$ satisfies the conditions of Lemma 5.1 for large enough k. Therefore the subpaths $[x_n(k), y_n(k)]$ and $[x'_n(k), y'_n(k)]$ share a large domain ω-a.s. Since the entrance points of these subpaths in this domain are at a uniformly bounded $\mathcal{C}(S)$-distance, the same holds for g_n, g'_n. Hence $U(g') = U(g)$. This completes the proof of the lemma. □

Lemma 5.3 shows that, for every two points $\boldsymbol{x}, \boldsymbol{y}$ in a piece P of $\mathcal{AM}$, there exists an interior $U(g)$ depending only on these points and contained in P. We shall denote $U(g)$ by $U(\boldsymbol{x}, \boldsymbol{y})$.

Lemma 5.4. *Let $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ be three different points in a piece $P \subseteq \mathcal{AM}$. Then*

$$U(\boldsymbol{x}, \boldsymbol{y}) = U(\boldsymbol{y}, \boldsymbol{z}) = U(\boldsymbol{x}, \boldsymbol{z}).$$

Proof. Let $(x_n)^\omega$, $(y_n)^\omega$ and $(z_n)^\omega$ be representatives of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. Choose hierarchy paths $[x_n, y_n], [y_n, z_n], [x_n, z_n]$. By Theorem 2.25, the hierarchy path $[x_n, y_n]$ shares a large domain with either $[x_n, z_n]$ or $[y_n, z_n]$ for all n ω-a.s. Then, by Lemma 5.3, $U(\boldsymbol{x}, \boldsymbol{y})$ coincides either with $U(\boldsymbol{x}, \boldsymbol{z})$ or with $U(\boldsymbol{y}, \boldsymbol{z})$. Repeating the argument with $[\boldsymbol{x}, \boldsymbol{y}]$ replaced either by $[\boldsymbol{y}, \boldsymbol{z}]$ or by $[\boldsymbol{x}, \boldsymbol{z}]$, we conclude that all three interiors coincide. □

Proposition 5.5. *Every piece P of the asymptotic cone $\mathcal{AM}$ contains a unique interior $U(g)$, and P is the closure of $U(g)$.*

Proof. Let $U(g)$ and $U(g')$ be two interiors inside P. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two distinct points in $U(g)$, and $\boldsymbol{z}$ and $\boldsymbol{t}$ be two distinct points in $U(g')$. If $\boldsymbol{y} \neq \boldsymbol{z}$, then we apply Lemma 5.4 to the triples $(\ x, \boldsymbol{y}, \boldsymbol{z})$ and $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$, and conclude that $U(g) = U(g')$. If $\boldsymbol{y} = \boldsymbol{z}$, then we apply Lemma 5.4 to the triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$. The fact that the closure of $U(g)$ is P follows from Lemma 5.3. □

5.2. *Actions and splittings*

We recall a theorem proved by Guirardel [**37**], that we shall use in what follows.

Definition 5.6. The *height* of an arc in an $\mathbb{R}$-tree with respect to the action of a group G on it is the maximal length of a decreasing chain of subarcs with distinct stabilizers. If the height of an arc is zero, then it follows that all subarcs of it have the same stabilizer. In this case the arc is called *stable*.

The tree T is *of finite height* with respect to the action of some group G if any arc of it can be covered by finitely many arcs with finite height. If the action is minimal and G is finitely generated, then this condition is equivalent to the fact that there exists a finite collection of arcs $\mathcal{I}$ of finite height such that any arc is covered by finitely many translates of arcs in $\mathcal{I}$ (see [**37**]).

THEOREM 5.7 (Guirardel [37]). *Let Λ be a finitely generated group and let T be a real tree on which Λ acts minimally and with finite height. Suppose that the stabilizer of any non-stable arc in T is finitely generated.*

Then one of the following three situations occurs:

(1) *Λ splits over the stabilizer of a non-stable arc or over the stabilizer of a tripod*;
(2) *Λ splits over a virtually cyclic extension of the stabilizer of a stable arc*;
(3) *T is a line and Λ has a subgroup of index at most 2 that is the extension of the kernel of that action by a finitely generated free abelian group.*

In some cases stability and finite height follow from the algebraic structure of stabilizers of arcs, as the next lemma shows.

LEMMA 5.8 [25]. *Let G be a finitely generated group acting on an $\mathbb{R}$-tree T with finite of size at most D tripod stabilizers, and (finite of size at most D)-by-abelian arc stabilizers, for some constant D. Then:*

(1) *an arc with stabilizer of size $> (D+1)!$ is stable*;
(2) *every arc of T is of finite height (and so the action is of finite height and stable).*

We also recall the following two well-known results due to Bestvina [10, 11] and Paulin [53].

LEMMA 5.9. *Let Λ and G be two finitely generated groups, $A = A^{-1}$ be a finite set generating Λ and let* dist *be a word metric on G. Given $\phi_n : \Lambda \to G$ an infinite sequence of homomorphisms, one can associate to it a sequence of positive integers defined by*

$$d_n = \inf_{x\in G} \sup_{a\in A} \operatorname{dist}(\phi_n(a)x, x). \tag{16}$$

If (ϕ_n) are pairwise non-conjugate in Γ, then $\lim_{n\to\infty} d_n = \infty$.

REMARK 5.10. For every $n \in \mathbb{N}$, $d_n = \operatorname{dist}(\phi_n(a_n)x_n, x_n)$ for some $x_n \in \Gamma$ and $a_n \in A$.

Consider an arbitrary ultrafilter ω. According to Remark 5.10, there exist $a \in A$ and $x_n \in G$ such that $d_n = \operatorname{dist}(\phi_n(a)x_n, x_n)$ ω-a.s.

LEMMA 5.11. *Under the assumptions of Lemma 5.9, the group Λ acts on the asymptotic cone $\mathcal{K}_\omega = \operatorname{Con}^\omega(G; (x_n), (d_n))$ by isometries, without a global fixed point, as follows*:

$$g \cdot \lim_\omega(x_n) = \lim_\omega(\phi_n(g)x_n). \tag{17}$$

This defines a homomorphism ϕ_ω from Λ to the group $x^\omega(\Pi_1\Gamma/\omega)(x^\omega)^{-1}$ of isometries of $\mathcal{K}_\omega$.

Let S be a surface of complexity $\xi(S)$. When $\xi(S) \leqslant 1$ the mapping class group $\mathcal{MCG}(S)$ is hyperbolic and the well-known theory on homomorphisms into hyperbolic groups can be applied (see, for instance, [10, 53, 11] and references therein). Therefore we adopt the following convention for the rest of this section.

CONVENTION 5.12. In what follows, we assume that $\xi(S) \geqslant 2$.

Proposition 5.13. *Suppose that a finitely generated group $\Lambda = \langle A \rangle$ has infinitely many homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$. Let*

$$d_n = \inf_{\mu \in \mathcal{M}(S)} \sup_{a \in A} \operatorname{dist}(\phi_n(a)\mu, \mu), \tag{18}$$

and let μ_n be the point in $\mathcal{M}(S)$, where the above infimum is attained.

Then one of the following two situations occurs.

(1) *The sequence (ϕ_n) defines a non-trivial action of Λ on an asymptotic cone of the complex of curves $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (\ell_n))$.*

(2) *The action by isometries, without a global fixed point, of Λ on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined as in Lemma 5.11 fixes a piece set-wise.*

Proof. Let $\ell_n = \inf_{\gamma \in \mathcal{C}(S)} \sup_{a \in A} \operatorname{dist}_{\mathcal{C}(S)}(\phi_n(a)\gamma, \gamma)$. As before, there exists $b_0 \in A$ and $\gamma_n \in \mathcal{C}(S)$ such that $\ell_n = \operatorname{dist}_{\mathcal{C}(S)}(\phi_n(b_0)\gamma_n, \gamma_n)$ ω-a.s.

If $\lim_\omega \ell_n = +\infty$, then the sequence (ϕ_n) defines a non-trivial action of Λ on $\mathrm{Con}^\omega(\mathcal{C}(S); (\gamma_n), (\ell_n))$.

Assume now that there exists M such that, for every $b \in A$, $\operatorname{dist}_{\mathcal{C}(S)}(\phi_n(b)\gamma_n, \gamma_n) \leqslant M$ ω-a.s. . This implies that, for every $g \in \Lambda$, there exists M_g such that $\operatorname{dist}_{\mathcal{C}(S)}(\phi_n(g)\gamma_n, \gamma_n) \leqslant M_g$ ω-a.s.

Consider μ_n' the projection of μ_n onto $\mathcal{Q}(\gamma_n)$. A hierarchy path $[\mu_n, \mu_n']$ shadows a tight geodesic $\mathfrak{g}_n$ joining a curve in $\mathrm{base}(\mu_n)$ to a curve in $\mathrm{base}(\mu_n')$, the latter curve being at $C(S)$-distance 1 from γ_n. If the ω-limit of the $C(S)$-distance from μ_n to μ_n' is finite, then it follows that for every $b \in A$, $\operatorname{dist}_{\mathcal{C}(S)}(\phi_n(b)\mu_n, \mu_n) = O(1)$ ω-a.s. Then the action of Λ on $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined by the sequence (ϕ_n) preserves $U((\mu_n)^\omega)$, which is the interior of a piece, hence it fixes a piece set-wise. Therefore, in what follows we assume that the ω-limit of the $C(S)$-distance from μ_n to μ_n' is infinite.

Let b be an arbitrary element in the set of generators A. Consider a hierarchy path $[\mu_n, \phi_n(b)\mu_n]$. Consider the Gromov product

$$\tau_n(b) = \tfrac{1}{2}[\operatorname{dist}_{\mathcal{C}(S)}(\mu_n, \mu_n') + \operatorname{dist}_{\mathcal{C}(S)}(\mu_n, \phi_n(b)\mu_n) - \operatorname{dist}_{\mathcal{C}(S)}(\mu_n', \phi_n(b)\mu_n)],$$

and $\tau_n = \max_{b \in A} \tau_n(b)$.

The geometry of quadrangles in hyperbolic geodesic spaces combined with the fact that $\operatorname{dist}_{\mathcal{C}(S)}(\mu_n', \phi_n(b)\mu_n') \leqslant M$ implies that:

(i) every element ν_n on $[\mu_n, \mu_n']$ which is at $C(S)$-distance at least $\tau_n(b)$ from μ_n is at $C(S)$-distance $O(1)$ from an element ν_n' on $[\phi_n(b)\mu_n, \phi_n(b)\mu_n']$; it follows that $\operatorname{dist}_{\mathcal{C}(S)}(\nu_n', \phi_n(b)\mu_n') = \operatorname{dist}_{\mathcal{C}(S)}(\nu_n, \mu_n') + O(1)$, therefore $\operatorname{dist}_{\mathcal{C}(S)}(\nu_n', \phi_n(b)\nu_n) = O(1)$ and $\operatorname{dist}_{\mathcal{C}(S)}(\nu_n, \phi_n(b)\nu_n) = O(1)$;

(ii) the element $\rho_n(b)$ which is at $C(S)$-distance $\tau_n(b)$ from μ_n is at $C(S)$-distance $O(1)$ also from an element $\rho_n''(b)$ on $[\mu_n, \phi_n(b)\mu_n]$.

We have thus obtained that, for every element ν_n on $[\mu_n, \mu_n']$, which is at $C(S)$-distance at least τ_n from μ_n, $\operatorname{dist}_{\mathcal{C}(S)}(\nu_n, \phi_n(b)\nu_n) = O(1)$ for every $b \in B$ and ω-almost every n. In particular this holds for the point ρ_n on $[\mu_n, \mu_n']$ which is at $C(S)$-distance τ_n from μ_n. Let $a \in A$ be such that $\tau_n(a) = \tau_n$ and $\rho_n = \rho_n(a)$, and let $\rho_n'' = \rho_n''(a)$ be the point on $[\mu_n, \phi_n(a)\mu_n]$ at $C(S)$-distance $O(1)$ from $\rho_n(a)$. It follows that $\operatorname{dist}_{\mathcal{C}(S)}(\rho_n'', \phi_n(b)\rho_n'') = O(1)$ for every $b \in B$ and ω-almost every n. Moreover, since ρ'' is a point on $[\mu_n, \phi_n(a)\mu_n]$, its limit is a point in $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$. It follows that the action of Λ on $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (d_n))$ defined by the sequence (ϕ_n) preserves $U((\rho_n'')^\omega)$, which is the interior of a piece, hence it fixes a piece set-wise. $\square$

LEMMA 5.14. *Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ be two distinct points in an asymptotic cone of the complex of curves* $\mathrm{Con}^\omega(\mathcal{C}(S);(\gamma_n),(d_n))$. *The stabilizer* $\mathrm{stab}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$ *in the ultrapower* $\mathcal{MCG}(S)^\omega$ *is the extension of a finite subgroup of cardinality at most $N = N(S)$ by an abelian group.*

Proof. Let $\mathfrak{q}_n$ be a geodesic joining γ_n and γ_n', and let x_n and y_n be points at distance ϵd_n from γ_n and γ_n', respectively, where $\epsilon > 0$ is small enough.

Let $\boldsymbol{g} = (g_n)^\omega$ be an element in $\mathrm{stab}(\boldsymbol{\gamma},\boldsymbol{\gamma}')$. Then

$$\delta_n(\boldsymbol{g}) = \max(\mathrm{dist}_{\mathcal{C}(S)}(\gamma_n, g_n\gamma_n),\ \mathrm{dist}(\gamma_n', g_n\gamma_n'))$$

satisfies $\delta_n(\boldsymbol{g}) = o(d_n)$.

Since $\mathcal{C}(S)$ is a Gromov hyperbolic space, it follows that the subgeodesic of $\mathfrak{q}_n$, with endpoints x_n, y_n, is contained in a finite radius tubular neighbourhood of $g_n\mathfrak{q}_n$. Since x_n is ω-a.s. at distance $O(1)$ from a point x_n' on g_nq_n, define $\ell_x(g_n)$ as $(-1)^\epsilon \mathrm{dist}_{\mathcal{C}(S)}(x_n, g_nx_n)$, where $\epsilon = 0$ if x_n' is nearer to $g_n\mu_n$ than g_nx_n, and $\epsilon = 1$ otherwise.

Let $\ell_x : \mathrm{stab}(\mu,\nu) \to \Pi\mathbb{R}/\omega$ be defined by $\ell_x(\boldsymbol{g}) = (\ell_x(g_n))^\omega$. It is easy to see that ℓ_x is a quasi-morphism, that is,

$$|\ell_x(\boldsymbol{gh}) - \ell_x(\boldsymbol{g}) - \ell_x(\boldsymbol{h})| \leqslant_\omega O(1). \tag{19}$$

It follows that $|\ell_x([\boldsymbol{g},\boldsymbol{h}])| \leqslant_\omega O(1)$.

The above and a similar argument for y_n imply that, for every commutator, $\boldsymbol{c} = \lim_\omega(c_n)$, in the stabilizer of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, $\mathrm{dist}_{\mathcal{C}(S)}(x_n, c_nx_n)$ and $\mathrm{dist}_{\mathcal{C}(S)}(y_n, c_ny_n)$ are at most $O(1)$. Lemma 2.1, together with Bowditch's acylindricity result [**16**, Theorem 1.3], implies that the set of commutators of $\mathrm{stab}(\boldsymbol{\mu},\boldsymbol{\nu})$ has uniformly bounded cardinality, say, N. Then any finitely generated subgroup G of $\mathrm{stab}(\boldsymbol{\mu},\boldsymbol{\nu})$ has conjugacy classes of cardinality at most N, that is, G is an FC-group [**48**]. By [**48**] the set of all torsion elements of G is finite, and the derived subgroup of G is finite of cardinality at most $N(S)$ (by Lemma 2.13). □

LEMMA 5.15. *Let $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ be the vertices of a non-trivial tripod in an asymptotic cone of the complex of curves* $\mathrm{Con}^\omega(\mathcal{C}(S);(\gamma_n),(d_n))$. *The stabilizer* $\mathrm{stab}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ *in the ultrapower* $\mathcal{MCG}(S)^\omega$ *is a finite subgroup of cardinality at most $N = N(S)$.*

Proof. Since $\mathcal{C}(S)$ is δ-hyperbolic, for every $a > 0$ there exists $b > 0$ such that, for any triple of points $x, y, z \in \mathcal{C}(S)$, the intersection of the three a-tubular neighbourhoods of geodesics $[x, y]$, $[y, z]$ and $[z, a]$ is a set $C_a(x, y, z)$ of diameter at most b.

Let $\boldsymbol{g} = (g_n)^\omega$ be an element in $\mathrm{stab}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$. Then

$$\delta_n(\boldsymbol{g}) = \max(\mathrm{dist}_{\mathcal{C}(S)}(\alpha_n, g_n\alpha_n), \mathrm{dist}(\beta_n, g_n\beta_n), \mathrm{dist}(\gamma_n, g_n\gamma_n))$$

satisfies $\delta_n(\boldsymbol{g}) = o(d_n)$. While the distance between each pair of points among α_n, β_n and γ_n is at least λd_n for some $\lambda > 0$. It follows that if (x_n, y_n) is any of the pairs (α_n, β_n), (α_n, γ_n), (γ_n, β_n), then away from an $o(d_n)$-neighbourhood of the endpoints the two geodesics $[x_n, y_n]$ and $[g_nx_n, g_ny_n]$ are uniformly Hausdorff close. This in particular implies that away from an $o(d_n)$-neighbourhood of the endpoints, the a-tubular neighbourhood of $[g_nx_n, g_ny_n]$ is contained in the A-tubular neighbourhood of $[x_n, y_n]$ for some $A > a$. Since $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ are the vertices of a non-trivial tripod, for any $a > 0$, $C_a(\alpha_n, \beta_n, \gamma_n)$ is ω-a.s. disjoint of $o(d_n)$-neighbourhoods of $\alpha_n, \beta_n, \gamma_n$. The same holds for $\boldsymbol{g\alpha}, \boldsymbol{g\beta}, \boldsymbol{g\gamma}$. It follows that $C_a(g_n\alpha_n, g_n\beta_n, g_n\gamma_n)$ is contained in $C_A(\alpha_n, \beta_n, \gamma_n)$, hence it is at Hausdorff distance at most $B > 0$ from $C_a(\alpha_n, \beta_n, \gamma_n)$.

Thus we may find a point $\tau_n \in [\alpha_n, \beta_n]$ such that $\mathrm{dist}(\tau_n, g_n\tau_n) = O(1)$, while the distance from τ_n to $\{\alpha_n, \beta_n\}$ is at least $2\epsilon d_n$. Let $\eta_n \in [\tau_n, \alpha_n]$ be a point at distance ϵd_n from τ_n. Then $g_n\eta_n \in [g_n\tau_n, g_n\alpha_n]$ is a point at distance ϵd_n from $g\tau_n$. On the other hand, since η_n is at distance at least ϵd_n from α_n, it follows that there exists $\eta_n' \in [g_n\tau_n, g_n\alpha_n]$ at distance

$O(1)$ from η_n. It follows that η'_n is at distance $\epsilon d_n + O(1)$ from $g_n\tau_n$, hence η'_n is at distance $O(1)$ from $g_n\eta_n$. Thus we have obtained that $g_n\eta_n$ is at distance $O(1)$ from η_n. This, the fact that $g_n\tau_n$ is at distance $O(1)$ from τ_n as well, and the fact that $\mathrm{dist}(\tau_n, \eta_n) = \epsilon d_n$, together with Bowditch's acylindricity result [**16**, Theorem 1.3] and Lemma 2.1 imply that the stabilizer $\mathrm{stab}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ has uniformly bounded cardinality. □

COROLLARY 5.16. *Suppose that a finitely generated group $\Lambda = \langle A\rangle$ has infinitely many injective homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$.*

Then one of the following two situations occurs:

(1) *Λ is virtually abelian, or it splits over a virtually abelian subgroup;*

(2) *the action by isometries, without a global fixed point, of Λ on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{MCG}(S); (\mu_n), (d_n))$, defined as in Lemma* 5.11, *fixes a piece set-wise.*

Proof. It suffices to prove that case (1) from Proposition 5.13 implies (1) from Corollary 5.13. According to Proposition 5.13(1), the group Λ acts non-trivially on a real tree, by Lemma 5.14 we know that the stabilizers of non-trivial arcs are virtually abelian, and by Lemma 5.15 we know that the stabilizers of non-trivial tripods are finite.

On the other hand, since Λ injects into $\mathcal{MCG}(S)$, it follows immediately from the results of Birman, Lubotzky and McCarthy [**13**] that virtually abelian subgroups in Λ satisfy the ascending chain condition, and are always finitely generated. By Theorem 5.7 we thus have that Λ is either virtually abelian or it splits over a virtually solvable subgroup. One of the main theorems of [**13**] is that any virtual solvable subgroup of the mapping class group is virtually abelian, completing the argument. □

In fact the proof of Corollary 5.16 allows one to remove the injectivity assumption by replacing Λ by its natural image in $\mathcal{MCG}(S)^\omega_e$ as follows:

COROLLARY 5.17. *Suppose that a finitely generated group $\Lambda = \langle A\rangle$ has infinitely many homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$. Let N be the intersections of kernels of these homomorphisms*

Then one of the following two situations occurs:

(1) *Λ/N is virtually abelian, or it splits over a virtually abelian subgroup*;

(2) *the action by isometries, without a global fixed point, of Λ/N on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{MCG}(S); (\mu_n), (d_n))$, defined as in Lemma* 5.11, *fixes a piece set-wise.*

Corollary 5.17 immediately implies the following statement because every quotient of a group with property (T) has property (T).

COROLLARY 5.18. *Suppose that a finitely generated group $\Lambda = \langle A\rangle$ with property (T) has infinitely many injective homomorphisms ϕ_n into a mapping class group $\mathcal{MCG}(S)$, which are pairwise non-conjugate in $\mathcal{MCG}(S)$.*

Then the action by isometries, without a global fixed point, of Λ on the asymptotic cone $\mathrm{Con}^\omega(\mathcal{MCG}(S); (\mu_n), (d_n))$, defined as in Lemma 5.11, *fixes a piece set-wise.*

6. *Subgroups with property (T)*

Property (T) can be characterized in terms of isometric actions on median spaces.

THEOREM 6.1 [**19**, Theorem 1.2]. *A locally compact, second countable group has property (T) if and only if any continuous action by isometries of that group on any metric median space has bounded orbits.*

In our argument, the direct implication of Theorem 6.1 plays the most important part. This implication for discrete countable groups first appeared in [**51**] (see also [**49**, **50**] for implicit proofs, and [**54**] for median algebras). The same direct implication, for locally compact second countable groups, follows directly by combining [**39**, Theorem 20, Chapter 5] with [**57**, Theorem V.2.4].

Our main result in this section is the following.

THEOREM 6.2. *Let Λ be a finitely generated group and let S be a surface.*
If there exists an infinite collection Φ of homomorphisms $\phi\colon\Lambda\to\mathcal{MCG}(S)$ pairwise non-conjugate in $\mathcal{MCG}(S)$, then Λ acts on an asymptotic cone of $\mathcal{MCG}(S)$ with unbounded orbits.

COROLLARY 6.3. *If Λ is a finitely generated group with Kazhdan's property (T) and S is a surface, then the set of homomorphisms of Λ into $\mathcal{MCG}(S)$ up to conjugation in $\mathcal{MCG}(S)$ is finite.*

REMARKS 6.4. (1) Theorems 6.1 and 4.25 imply the following: for a finitely generated group Λ such that any action on a median space has bounded orbits, the space of homomorphisms from Λ to $\mathcal{MCG}(S)$ is finite modulo conjugation in $\mathcal{MCG}(S)$. By Theorem 6.1, the above hypothesis on Λ is equivalent to property (T).

(2) Corollary 6.3 suggests that $\mathcal{MCG}(S)$ contains few subgroups with property (T).

In order to prove Theorem 6.2, we need two easy lemmas and a proposition. Before formulating them, we introduce some notation and terminology.

Since the group $\mathcal{MCG}(S)$ acts co-compactly on $\mathcal{M}(S)$, there exists a compact subset K of $\mathcal{M}(S)$ containing the basepoint ν_0 fixed in Subsection 4.1 and such that $\mathcal{MCG}(S)K=\mathcal{M}(S)$.

Consider an asymptotic cone $\mathcal{AM}=\mathrm{Con}^\omega(\mathcal{M}(S);(\mu_n^0),(d_n))$. According to the above there exists $x_n\in\mathcal{MCG}(S)$ such that x_nK contains μ_n^0. We denote by x^ω the element $(x_n)^\omega$ in the ultrapower of $\mathcal{MCG}(S)$. According to the Remark 2.6(2), the subgroup $x^\omega(\Pi_1\mathcal{MCG}(S)/\omega)(x^\omega)^{-1}$ of the ultrapower of $\mathcal{MCG}(S)$ acts transitively by isometries on $\mathcal{AM}$. The action is isometric both with respect to the metric $\mathrm{dist}_{\mathcal{AM}}$ and with respect to the metric $\widetilde{\mathrm{dist}}$.

NOTATION 6.5. We denote for simplicity the subgroup $x^\omega(\Pi_1\mathcal{MCG}(S)/\omega)(x^\omega)^{-1}$ by $\mathcal{MCG}(S)^\omega_e$.

We say that an element $g=(g_n)^\omega$ in $\mathcal{MCG}(S)^\omega_e$ has a given property (for example, it is pseudo-Anosov, pure, reducible, etc.) if and only if ω-a.s. g_n has that property (it is pseudo-Anosov, pure, reducible, etc.).

LEMMA 6.6. *Let g be an element in $\mathcal{MCG}(S)^\omega_e$ fixing two distinct points $\boldsymbol{\mu},\boldsymbol{\nu}$ in $\mathcal{AM}$. Then, for every subsurface $\mathbf{U}\in\mathfrak{U}(\boldsymbol{\mu},\boldsymbol{\nu})$, there exists $k\in\mathbb{N}$, with $k\geqslant 1$, such that $g^k\mathbf{U}=\mathbf{U}$.*
In the particular case when g is pure, then k may be taken equal to 1.

Proof. If $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\ \boldsymbol{\nu}) = d > 0$, then, for every $i \in \mathbb{N}$, with $i \geqslant 1$, $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu},\ \boldsymbol{\nu}) = d$. Then there exists $k \geqslant 1$, k smaller than $[\widetilde{\mathrm{dist}}(\boldsymbol{\mu},\ \boldsymbol{\nu})/d] + 1$ such that $g^k\mathbf{U} = \mathbf{U}$.

The latter part of the statement follows from the fact that if g is pure, then any power of it fixes exactly the same subsurfaces as g itself. □

LEMMA 6.7. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ be two distinct points in the same piece. Then $\mathbf{S} \notin \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$.*

Proof. Assume on the contrary that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = 4\epsilon > 0$. By Proposition 5.5 there exist $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $U(P)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\mu}') < \epsilon$ and $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, \boldsymbol{\nu}') < \epsilon$. Then $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu}', \boldsymbol{\nu}') > 2\epsilon > 0$, whence $\lim_\omega(\mathrm{dist}_{C(S)}(\mu_n', \nu_n')) = +\infty$, contradicting the fact that $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are in $U(P)$. □

We now state the result that constitutes the main ingredient in the proof of Theorem 6.2. Its proof will be postponed until after the proof of Theorem 6.2 is completed.

PROPOSITION 6.8. *Let $g_1 = (g_n^1)^\omega, \ldots, g_m = (g_n^m)^\omega$ be pure elements in $\mathcal{MCG}(S)_e^\omega$, such that $\langle g_1, \ldots, g_m\rangle$ is composed only of pure elements and its orbits in $\mathcal{AM}$ are bounded. Then $g_1, \ldots, g_m$ fix a point in $\mathcal{AM}$.*

Proof of Theorem 6.2. Assume that there exists an infinite collection $\Phi = \{\phi_1, \phi_2, \ldots\}$ of pairwise non-conjugate homomorphisms $\phi_n : \Lambda \to \mathcal{MCG}(S)$. Lemma 5.9 implies that, given a finite generating set A of Λ, $\lim_{n\to\infty} d_n = \infty$, where

$$d_n = \inf_{\mu \in \mathcal{M}(S)} \sup_{a\in A} \mathrm{dist}(\phi_n(a)\mu, \mu). \tag{20}$$

Since $\mathcal{M}(S)$ is a simplicial complex, there exists a vertex $\mu_n^0 \in \mathcal{M}(S)$ such that $d_n = \sup_{a\in A} \mathrm{dist}(\phi_n(a)\mu_n^0, \mu_n^0)$. Consider an arbitrary ultrafilter ω and the asymptotic cone $\mathcal{AM} = \mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n^0), (d_n))$. We use the notation that appears before Lemma 6.6.

The infinite sequence of homomorphisms (ϕ_n) defines a homomorphism

$$\phi_\omega :\ \Lambda \longrightarrow \mathcal{MCG}(S)_e^\omega,\ \ \phi_\omega(g) = (\phi_n(g))^\omega.$$

This homomorphism defines an isometric action of Λ on $\mathcal{AM}$ without global fixed point.

NOTATION. When there is no possibility of confusion, we write $g\boldsymbol{\mu}$ instead of $\phi_\omega(g)\boldsymbol{\mu}$ for $g \in \Lambda$ and $\boldsymbol{\mu}$ in $\mathcal{AM}$.

We prove by induction on the complexity of S that if Λ has bounded orbits in $\mathcal{AM}$, then Λ has a global fixed point. When $\xi(S) \leqslant 1$ the asymptotic cone $\mathcal{AM}$ is a complete real tree and the previous statement is known to hold. Assume that we have proved the result for surfaces with complexity at most k, and assume that $\xi(S) = k + 1$.

If Λ does not fix set-wise a piece in the (most refined) tree-graded structure of $\mathcal{AM}$, then by Lemma 3.12 Λ has a global fixed point. Thus we may assume that Λ fixes set-wise a piece P in $\mathcal{AM}$. Then Λ fixes the interior $U(P)$ of P as well by Proposition 5.5.

The point $\boldsymbol{\mu}^0 = \lim_\omega(\mu_n^0)$ must be in the piece P. If not, the projection $\boldsymbol{\nu}$ of $\boldsymbol{\mu}^0$ on P would be moved by less that 1 by all $a \in A$. Indeed, if $a\boldsymbol{\nu} \neq \boldsymbol{\nu}$, then the concatenation of geodesics $[\boldsymbol{\mu}^0, \boldsymbol{\nu}] \sqcup [\boldsymbol{\nu}, a\boldsymbol{\nu}] \sqcup [a\boldsymbol{\nu}, a\boldsymbol{\mu}^0]$ is a geodesic according to [27].

According to Lemma 2.13 there exists a normal subgroup Λ_p in Λ of index at most $N = N(S)$ such that, for every $g \in \Lambda_p$, $\phi_\omega(g)$ is pure and fixes set-wise the boundary components of S.

By Proposition 6.8, Λ_p fixes a point $\boldsymbol{\alpha}$ in $\mathcal{AM}$. Since it also fixes set-wise the piece P, it fixes the unique projection of $\boldsymbol{\alpha}$ to P. Denote this projection by $\boldsymbol{\mu}$.

Assume that Λ acts on $\mathcal{AM}$ without fixed point. It follows that there exists $g \in \Lambda$ such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then $\Lambda_p = g\Lambda_p g^{-1}$ also fixes $g\boldsymbol{\mu}$. Lemma 6.6 implies that Λ_p fixes a subsurface $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$. Since $\boldsymbol{\mu}, g\boldsymbol{\mu}$ are in the piece P, it follows that $\mathbf{U}$ is a proper subsurface of $\mathbf{S}$, by Lemma 6.7. Thus Λ_p must fix a multicurve $\partial\mathbf{U}$.

Let $\boldsymbol{\Delta}$ be a maximal multicurve fixed by Λ_p. Assume that there exists $g \in \Lambda$ such that $g\boldsymbol{\Delta} \neq \boldsymbol{\Delta}$. Then $\Lambda_p = g\Lambda_p g^{-1}$ also fixes $g\boldsymbol{\Delta}$, contradicting the maximality of $\boldsymbol{\Delta}$. We then conclude that all Λ fixes $\boldsymbol{\Delta}$. It follows that the image of ϕ_ω is in $\mathrm{Stab}(\boldsymbol{\Delta})$, hence ω-a.s. $\phi_n(\Lambda) \subset \mathrm{Stab}(\Delta_n)$. Up to taking a subsequence and conjugating, we may assume that $\phi_n(\Lambda) \subset \mathrm{Stab}(\Delta)$ for some fixed multicurve Δ. Let $U_1, \ldots, U_m$ be the subsurfaces and annuli determined by Δ. Hence, we can see ϕ_n as isomorphisms with target $\mathrm{Stab}(\Delta)$ which are pairwise non-conjugate by hypothesis and thus define an isometric action with bounded orbits on the asymptotic cone of $\mathrm{Stab}(\Delta)$. Recall that $\mathrm{Stab}(\Delta)$ is quasi-isometric to $\mathcal{MCG}(U_1) \times \ldots \times \mathcal{MCG}(U_m)$ by Remark 2.27. Hence, the sequence ϕ_n defines an isometric action with bounded orbits on the spaces $\mathcal{AM}(U_j)$. By hypothesis at least one of these actions is without a global fixed point, while the inductive hypothesis implies that each of these actions has a global fixed point, which is a contradiction. $\square$

To prove Proposition 6.8, we first provide a series of intermediate results.

Lemma 6.9. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)^\omega_e$ be a reducible element in $\mathcal{AM}$, and let $\boldsymbol{\Delta} = (\Delta_n)^\omega$ be a multicurve such that if $U_n^1, \ldots, U_n^m$ are the connected components of $S \setminus \Delta_n$ and the annuli with core curve in Δ_n then ω-a.s. g_n is a pseudo-Anosov on $U_n^1, \ldots, U_n^k$ and the identity map on $U_n^{k+1}, \ldots, U_n^m$, and $\Delta_n = \partial U_n^1 \cup \ldots \cup \partial U_n^k$ (the latter condition may be achieved by deleting the boundary between two components onto which g_n acts as identity).*

Then the limit set $Q(\boldsymbol{\Delta})$ appears in the asymptotic cone (that is, the distance from the basepoint μ_n^0 to $Q(\Delta_n)$ is $O(d_n)$), in particular g fixes the piece containing $Q(\boldsymbol{\Delta})$.

If g fixes a piece P, then $U(P)$ contains $Q(\boldsymbol{\Delta})$.

Proof. Consider a point $\boldsymbol{\mu} = \lim_\omega(\mu_n)$ in $\mathcal{AM}$. Let $D_n = \mathrm{dist}_{\mathcal{M}(S)}(\mu_n, \Delta_n)$. Assume that $\lim_\omega D_n/d_n = +\infty$. Let ν_n be a projection of μ_n onto $Q(\Delta_n)$. Note that, for every $i = 1, 2, \ldots, k$, $\mathrm{dist}_{C(U_n^i)}(\mu_n, g_n\mu_n) = \mathrm{dist}_{C(U_n^i)}(\nu_n, g_n\nu_n) + O(1)$. Therefore, when replacing g by some large enough power of it, we may ensure that $\mathrm{dist}_{C(U_n^i)}(\mu_n, g_n\mu_n) > M$, where M is the constant from Lemma 2.18, while we still have that $\mathrm{dist}_{\mathcal{M}(S)}(\mu_n, g_n\mu_n) \leqslant Cd_n$. In the cone $\mathrm{Con}^\omega(\mathcal{M}(S); (\mu_n), (D_n))$ we have that $\boldsymbol{\mu} = g\boldsymbol{\mu}$ projects onto $Q(\boldsymbol{\Delta})$ into $\boldsymbol{\nu} = g\boldsymbol{\nu}$, which is at distance 1. This contradicts Lemma 4.27. It follows that $\lim_\omega D_n/d_n < +\infty$.

Assume now that g fixes a piece P and assume that the point $\boldsymbol{\mu}$ considered above is in $U(P)$. Since the previous argument implies that a hierarchy path joining $\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$ for some large enough k intersects $Q(\partial\mathbf{U}_i)$, where $\mathbf{U}_i = (U_n^i)^\omega$ and $i = 1, 2, \ldots, k$, and $Q(\boldsymbol{\Delta}) \subset Q(\partial\mathbf{U}_i)$, it follows that $Q(\boldsymbol{\Delta}) \subset U(P)$. $\square$

Notation. Given two points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}$ we denote by $\mathfrak{U}(\boldsymbol{\mu},\ \boldsymbol{\nu})$ the set of subsurfaces $\mathbf{U} \subseteq \mathbf{S}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu},\ \boldsymbol{\nu}) > 0$. Note that $\mathfrak{U}(\boldsymbol{\mu},\ \boldsymbol{\nu})$ is non-empty if and only if $\boldsymbol{\mu} \neq \boldsymbol{\nu}$.

Lemma 6.10. *Let $\boldsymbol{\Delta}$ be a multicurve.*

(1) *If $\boldsymbol{\mu}, \boldsymbol{\nu}$ are two points in $Q(\boldsymbol{\Delta})$, then any $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$ has the property that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$.*

(2) *If $\boldsymbol{\mu}$ is a point outside $Q(\boldsymbol{\Delta})$ and $\boldsymbol{\mu}'$ is the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$, then any $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}')$ has the property that $\mathbf{U} \pitchfork \boldsymbol{\Delta}$.*

(3) *Let $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ be as in (2). For every $\boldsymbol{\nu} \in Q(\boldsymbol{\Delta})$ we have that $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \sqcup \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu})$.*

(4) *Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ be two points in $Q(\boldsymbol{\Delta})$. Any geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ joining $\boldsymbol{\mu}, \boldsymbol{\nu}$ is entirely contained in $Q(\boldsymbol{\Delta})$.*

Proof. (1) Indeed if $\mathbf{U} = (U_n)^\omega \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$, then $\lim_\omega(\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n)) = \infty$, according to Lemma 4.15. On the other hand if $\mathbf{U} \pitchfork \boldsymbol{\Delta}$, then $\mathrm{dist}_{C(U_n)}(\mu_n, \nu_n) = O(1)$, as the bases of both μ_n and ν_n contain Δ_n.

(2) This follows immediately from Lemma 4.26.

(3) According to (1) and (2), $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \cap \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu}) = \emptyset$. The triangle inequality implies that, for every $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}')$, either $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}', \boldsymbol{\nu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. But since the former cannot occur, it follows that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Likewise we prove that $\mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. The inclusion $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\mu}') \sqcup \mathfrak{U}(\boldsymbol{\mu}', \boldsymbol{\nu})$ follows from the triangle inequality.

(4) This follows from the fact that, for any point $\boldsymbol{\alpha}$ on a $\widetilde{\mathrm{dist}}$-geodesic joining $\boldsymbol{\mu}, \boldsymbol{\nu}$, $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\nu})$, as well as from (1) and (3). □

LEMMA 6.11. *Let $g \in \mathcal{MCG}(S)^\omega_e$ and $\boldsymbol{\Delta}$ be as in Lemma 6.9. If $\boldsymbol{\mu}$ is fixed by g, then $\boldsymbol{\mu} \in Q(\boldsymbol{\Delta})$.*

Proof. Assume on the contrary that $\boldsymbol{\mu} \notin Q(\boldsymbol{\Delta})$, and let $\boldsymbol{\nu}$ be its projection onto $Q(\boldsymbol{\Delta})$. Then $g\boldsymbol{\nu}$ is the projection of $g\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Corollary 2.29 implies that $g\boldsymbol{\nu} = \boldsymbol{\nu}$. By replacing g with some power of it, we may assume that the hypotheses of Lemma 4.27 hold. On the other hand, the conclusion of Lemma 4.27 does not hold since the geodesic between μ and ν and the geodesic between $g\mu$ and $g\nu$ coincide. This contradiction proves the lemma. □

LEMMA 6.12. *Let $g \in \mathcal{MCG}(S)^\omega_e$ be a pure element such that $\langle g \rangle$ has bounded orbits in $\mathcal{AM}$, and let $\boldsymbol{\mu}$ be a point such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then, for every $k \in \mathbb{Z} \setminus \{0\}$, $g^k\boldsymbol{\mu} \neq \boldsymbol{\mu}$.*

Proof. *Case* 1. Assume that g is a pseudo-Anosov element.

Case 1(a). Assume, moreover, that $\boldsymbol{\mu}$ is in a piece P stabilized by g. Let $\mathbf{U}$ be a subsurface in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$. As $\boldsymbol{\mu}, g\boldsymbol{\mu}$ are both in P, it follows by Lemma 6.7 that $\mathbf{U} \subsetneq \mathbf{S}$.

Assume that the subsurfaces $g^{-i_1}\mathbf{U}, \ldots, g^{-i_k}\mathbf{U}$ are also in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$, where $i_1 < \ldots < i_k$. Let $3\epsilon > 0$ be the minimum of $\widetilde{\mathrm{dist}}_{g^{-i}\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$, with $i = 0, i_1, \ldots, i_k$. Since P is the closure of its interior $U(P)$ (Proposition 5.5), there exists $\boldsymbol{\nu} \in U(P)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < \epsilon$. It follows that $\widetilde{\mathrm{dist}}_{g^{-i}\mathbf{U}}(\boldsymbol{\nu}, g\boldsymbol{\nu}) \geqslant \epsilon$ for $i = 0, i_1, \ldots, i_k$. Then, by Lemma 4.15, $\lim_\omega \mathrm{dist}_{C(g^{-i}U_n)}(\nu_n, g_n\nu_n) = \infty$. Let $\mathfrak{h} = \lim_\omega(\mathfrak{h}_n)$ be a hierarchy path joining $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$. The above implies that ω-a.s. $\mathfrak{h}_n$ intersects $Q(g^{-i_j}\partial U_n)$, hence there exists a vertex v_n^j on the tight geodesic $\mathfrak{t}_n$ shadowed by $\mathfrak{h}_n$ such that $g^{-i_j}U_n \subseteq S \setminus v_n^j$. In particular $\mathrm{dist}_{\mathcal{C}(S)}(g^{-i_j}\partial U_n, v_n^j) \leqslant 1$. Since $\boldsymbol{\nu} \in U(P)$ and g stabilizes $U(P)$, it follows that $g\boldsymbol{\nu} \in U(P)$, whence $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n, g_n\nu_n) \leqslant D = D(g)$ ω-a.s. In particular the length of the tight geodesic $\mathfrak{t}_n$ is at most $D + 2$ ω-a.s.

According to [**16**, Theorem 1.4], there exists $m = m(S)$ such that ω-a.s. g_n^m preserves a bi-infinite geodesic $\mathfrak{g}_n$ in $C(S)$. To denote g^m we write g_1 for the sequence with terms $g_{1,n}$.

For every curve γ let γ' be a projection of it on $\mathfrak{g}_n$. A standard hyperbolic geometry argument implies that, for every $i \geqslant 1$,

$$\mathrm{dist}_{\mathcal{C}(S)}(\gamma, g_{1,n}^{-i}\gamma) \geqslant \mathrm{dist}_{\mathcal{C}(S)}(\gamma', g_{1,n}^{-i}\gamma') + O(1) \geqslant i + O(1).$$

The same estimate holds for γ replaced by ∂U_n. Now assume that the maximal power $i_k = mq + r$, where $0 \leqslant r < m$. Then $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n, g_n^{-mq}\partial U_n) = \mathrm{dist}_{\mathcal{C}(S)}(g_n^{-r}\partial U_n, \partial U_n) \leqslant$

$2(D+2)+\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-r}\nu_n,\nu_n)\leqslant 2(D+2)+rD=D_1$. It follows that $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n,\partial U_n)\geqslant \mathrm{dist}_{\mathcal{C}(S)}(\partial U_n, g_n^{-mq}\partial U_n)-D_1\geqslant q+O(1)-D_1$.

On the other hand, $\mathrm{dist}_{\mathcal{C}(S)}(g_n^{-i_k}\partial U_n,\partial U_n)\leqslant 2+\mathrm{dist}_{\mathcal{C}(S)}(v_n^k,v_n^0)\leqslant D+4$, whence $q\leqslant D+D_1+4+O(1)=D_2$ and $i_k\leqslant m(D_2+1)$. Thus the sequence $i_1,\dots,i_k$ is bounded, and it has a maximal element. It follows that there exists a subsurface $\mathbf{U}$ in $\mathfrak{U}(\boldsymbol{\mu},g\boldsymbol{\mu})$ such that, for every $k>0$, $\widetilde{\mathrm{dist}}_{g^{-k}\mathbf{U}}(\boldsymbol{\nu},g\boldsymbol{\nu})=\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^k\boldsymbol{\nu},g^{k+1}\boldsymbol{\nu})=0$. The triangle inequality in $T_{\mathbf{U}}$ implies that $\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu},g\boldsymbol{\mu})=\mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu},g^k\boldsymbol{\mu})>0$ for every $k\geqslant 1$. It follows that no power g^k fixes $\boldsymbol{\mu}$.

Case 1(b). Assume now that $\boldsymbol{\mu}$ is not contained in any piece fixed by g. By Lemma 3.11 g fixes either the middle cut-piece P or the middle cut-point m of $\boldsymbol{\mu},g\boldsymbol{\mu}$.

Assume that $\boldsymbol{\mu},g\boldsymbol{\mu}$ have a middle cut-piece P, and let $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ be the endpoints of the intersection with P of any arc joining $\boldsymbol{\mu},g\boldsymbol{\mu}$. Then $g\boldsymbol{\nu}=\boldsymbol{\nu}'$ hence $g\boldsymbol{\nu}\neq\boldsymbol{\nu}$. By case 1(a) it then follows that for every $k\neq 0$, $g^k\boldsymbol{\nu}\neq\boldsymbol{\nu}$, and since $[\boldsymbol{\mu},\boldsymbol{\nu}]\sqcup[\boldsymbol{\nu},g^k\boldsymbol{\nu}]\sqcup[g^k\boldsymbol{\nu},g^k\boldsymbol{\mu}]$ is a geodesic, it follows that $g^k\boldsymbol{\mu}\neq\boldsymbol{\mu}$.

We now assume that $\boldsymbol{\mu},g\boldsymbol{\mu}$ have a middle cut-point m, fixed by g. Assume that $\mathfrak{U}(\boldsymbol{\mu},g\boldsymbol{\mu})$ contains a strict subsurface of $\mathbf{S}$. Then the same thing holds for $\mathfrak{U}(\boldsymbol{\mu},m)$. Let $\mathbf{U}\subsetneq\mathbf{S}$ be an element in $\mathfrak{U}(\boldsymbol{\mu},m)$.

If $g^k\boldsymbol{\mu}=\boldsymbol{\mu}$ for some $k\neq 0$, since $g^km=m$, it follows that $g^{kn}(\mathbf{U})=\mathbf{U}$ for some $n\neq 0$, by Lemma 6.6. However, this is impossible, since g is a pseudo-Anosov.

Thus, we may assume that $\mathfrak{U}(\boldsymbol{\mu},g\boldsymbol{\mu})=\{\mathbf{S}\}$, that is, that $\boldsymbol{\mu},g\boldsymbol{\mu}$ are in the same transversal tree.

Let $\mathfrak{g}_n$ be a bi-infinite geodesic in $C(S)$ such that $g_n\mathfrak{g}_n$ is at Hausdorff distance $O(1)$ from $\mathfrak{g}_n$. Let γ_n be the projection of $\pi_{\mathcal{C}(S)}(\mu_n)$ onto $\mathfrak{g}_n$. A hierarchy path $\mathfrak{h}=\lim_\omega(\mathfrak{h}_n)$ joining μ_n and $g_n\mu_n$ contains two points ν_n,ν_n' such that:

(i) the subpath with endpoints μ_n,ν_n is at $C(S)$-distance $O(1)$ from any $C(S)$-geodesic joining $\pi_{\mathcal{C}(S)}(\mu_n)$ and γ_n;
(ii) the subpath with endpoints $g\mu_n,\nu_n'$ is at $C(S)$-distance $O(1)$ from any $C(S)$-geodesic joining $\pi_{\mathcal{C}(S)}(g\mu_n)$ and $g\gamma_n$;
(iii) if $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n,\nu_n')$ is large enough, then the subpath with endpoints ν_n,ν_n' is at $C(S)$-distance $O(1)$ from $\mathfrak{g}_n$;
(iv) $\mathrm{dist}_{\mathcal{C}(S)}(\nu_n',g\nu_n)$ is $O(1)$.

Let $\boldsymbol{\nu}=\lim_\omega(\nu_n)$ and $\boldsymbol{\nu}'=\lim_\omega(\nu_n')$. The last property above implies that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu}',g\boldsymbol{\nu})=0$. Assume that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu},\boldsymbol{\nu}')>0$, hence $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu},g\boldsymbol{\nu})>0$. Let $\mathfrak{h}'$ be a hierarchy subpath with endpoints $\boldsymbol{\nu},g\boldsymbol{\nu}$. Its projection onto $T_{\mathbf{S}}$ and the projection of $g\mathfrak{h}'$ onto $T_{\mathbf{S}}$ have in common only their endpoint. Otherwise there would exist $\boldsymbol{\alpha}$ on $\mathfrak{h}'\cap g\mathfrak{h}'$ with $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\alpha},g\boldsymbol{\mu})>0$, and such that $\mathrm{Cutp}\,\{\boldsymbol{\alpha},g\boldsymbol{\mu}\}$ is in the intersection of $\mathrm{Cutp}\,(\mathfrak{h}')$ with $\mathrm{Cutp}\,(g\mathfrak{h}')$. Consider $\boldsymbol{\beta}\in\mathrm{Cutp}\,\{\boldsymbol{\alpha},g\boldsymbol{\mu}\}$ at equal $\widetilde{\mathrm{dist}}_{\mathbf{S}}$-distance from $\boldsymbol{\alpha},g\boldsymbol{\mu}$. Take α_n,β_n on $\mathfrak{h}_n'$ and α_n',β_n' on $g\mathfrak{h}_n'$, such that $\boldsymbol{\alpha}=\lim_\omega(\alpha_n)=\lim_\omega(\alpha_n')$ and $\boldsymbol{\beta}=\lim_\omega(\beta_n)=\lim_\omega(\beta_n')$. Since α_n,α_n' and β_n,β_n' are at distance $o(d_n)$, it follows that $\mathfrak{h}_n'$ between α_n and β_n and $g\mathfrak{h}_n'$ between α_n' and β_n' share a large domain U_n. Let σ_n and σ_n' be the corresponding points on the two hierarchy subpaths contained in $Q(\partial U_n)$. The projections of $\mathfrak{h}_n'$ and $g\mathfrak{h}_n'$ onto $C(S)$, both tight geodesics, would contain the points $\pi_{\mathcal{C}(S)}(\sigma_n)$ and $\pi_{\mathcal{C}(S)}(\sigma_n')$ at $\mathrm{dist}_{\mathcal{C}(S)}$-distance $O(1)$, while $\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\sigma_n,g\nu_n)=\infty$ and $\lim_\omega\mathrm{dist}_{\mathcal{C}(S)}(\sigma_n',g\nu_n)=\infty$. This contradicts the fact that the projection of $\mathfrak{h}_n'\sqcup g\mathfrak{h}_n'$ is at $\mathrm{dist}_{\mathcal{C}(S)}$-distance $O(1)$ from the geodesic $\mathfrak{g}_n$.

We may thus conclude that the projections of $\mathfrak{h}'$ and $g\mathfrak{h}'$ on $T_{\mathbf{S}}$ intersect only at their endpoints. From this fact one can easily deduce by induction that $\langle g\rangle$ has unbounded orbits in $T_{\mathbf{S}}$, hence in $\mathbb{F}$.

Assume now that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\nu},\boldsymbol{\nu}')=0$ (hence $\boldsymbol{\nu}=m$) and that $\widetilde{\mathrm{dist}}_{\mathbf{S}}(\boldsymbol{\mu},\boldsymbol{\nu})>0$. Let $\boldsymbol{\alpha}$ be the point on the hierarchy path joining $\boldsymbol{\mu},\boldsymbol{\nu}$ at equal $\widetilde{\mathrm{dist}}_{\mathbf{S}}$-distance from its extremities and let

$\mathfrak{h}'' = \lim_\omega(\mathfrak{h}''_n)$ be the subpath of the endpoints $\boldsymbol{\mu}, \boldsymbol{\alpha}$. All the domains of $\mathfrak{h}''_n$ have $C(S)$-distance to $\mathfrak{g}_n$ going to infinity, likewise for the $C(S)$-distance to any geodesic joining $\pi_{\mathcal{C}(S)}(g^k\mu_n)$ and $\pi_{\mathcal{C}(S)}(g^k\nu_n)$ with $k \neq 0$. It follows that $\mathrm{dist}(\boldsymbol{\mu}, g^k\boldsymbol{\mu}) \geqslant \mathrm{dist}(\boldsymbol{\mu}, \boldsymbol{\alpha}) > 0$.

Case 2. Assume now that g is a reducible element, and let $\boldsymbol{\Delta} = (\Delta_n)^\omega$ be a multicurve as in Lemma 6.9. According to the same lemma, $Q(\boldsymbol{\Delta}) \subset U(P)$.

If $\boldsymbol{\mu} \notin Q(\boldsymbol{\Delta})$, then $g^k\boldsymbol{\mu} \neq \boldsymbol{\mu}$ by Lemma 6.11. Assume therefore that $\boldsymbol{\mu} \in Q(\boldsymbol{\Delta})$. The set $Q(\boldsymbol{\Delta})$ can be identified to $\prod_{i=1}^m \mathcal{M}(\mathbf{U}_i)$ and $\boldsymbol{\mu}$ can be therefore identified to $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m)$. If, for every $i \in \{1, 2, \ldots, k\}$, the component of g acting on $\mathbf{U}_i$ would fix $\boldsymbol{\mu}_i$ in $\mathcal{M}(\mathbf{U}_i)$, then g would fix $\boldsymbol{\mu}$. This would contradict the hypothesis on g. Thus, for some $i \in \{1, 2, \ldots, k\}$, the corresponding component of g acts on $\mathcal{M}(\mathbf{U}_i)$ as a pseudo-Anosov and does not fix $\boldsymbol{\mu}_i$. According to the first case, for every $k \in \mathbb{Z} \setminus \{0\}$, the component of g^k acting on $\mathbf{U}_i$ does not fix $\boldsymbol{\mu}_i$ either, hence g^k does not fix $\boldsymbol{\mu}$. □

LEMMA 6.13. *Let $g \in \mathcal{MCG}(S)^\omega_e$ be a pure element and let $\boldsymbol{\mu} = \lim_\omega(\mu_n)$ be a point such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. If g is reducible, then take $\boldsymbol{\Delta} = (\Delta_n)^\omega$ and $U_n^1, \ldots, U_n^m$ as in Lemma 6.9, while if g is pseudo-Anosov, then take $\boldsymbol{\Delta} = \emptyset$ and $\{U_n^1, \ldots, U_n^m\} = \{S\}$, and by convention $Q(\Delta_n) = \mathcal{M}(S)$. Assume that g is such that, for any $\nu_n \in Q(\Delta_n)$, $\mathrm{dist}_{C(U_n^i)}(\nu_n, g_n\nu_n) > D$ ω-a.s. for every $i \in \{1, \ldots, k\}$, where D is a fixed constant, depending only on $\xi(S)$ (this may be achieved, for instance, by replacing g with a large enough power of it).*

Then $\mathfrak{U} = \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ splits as $\mathfrak{U}_0 \sqcup \mathfrak{U}_1 \sqcup g\mathfrak{U}_1 \sqcup \mathfrak{P}$, where:

(i) *$\mathfrak{U}_0$ is the set of elements $\mathbf{U} \in \mathfrak{U}$ such that no $g^k\mathbf{U}$ with $k \in \mathbb{Z} \setminus \{0\}$ is in $\mathfrak{U}$;*
(ii) *$\mathfrak{P}$ is the intersection of $\mathfrak{U}$ with $\{\mathbf{U}^1, \ldots, \mathbf{U}^k\}$, where $\mathbf{U}^j = (U_n^j)^\omega$;*
(iii) *$\mathfrak{U}_1$ is the set of elements $\mathbf{U} \in \mathfrak{U} \setminus \mathfrak{P}$ such that $g^k\mathbf{U} \in \mathfrak{U}$ only for $k = 0, 1$ (hence $g\mathfrak{U}_1$ is the set of elements $\mathbf{U} \in \mathfrak{U} \setminus \mathfrak{P}$ such that $g^k\mathbf{U} \in \mathfrak{U}$ only for $k = 0, -1$).*

Moreover, if either $\mathfrak{U}_0 \neq \emptyset$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$, then the $\langle g \rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Proof. *Case* 1. Assume that g is a pseudo-Anosov with $C(S)$-translation length D, where D is a large enough constant. There exists a bi-infinite axis $\mathfrak{p}_n$ such that $g_n\mathfrak{p}_n$ is at Hausdorff distance $O(1)$ from $\mathfrak{p}_n$. Consider $\mathfrak{h} = \lim_\omega(\mathfrak{h}_n)$ a hierarchy path joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, such that $\mathfrak{h}_n$ shadows a tight geodesic $\mathfrak{t}_n$. Choose two points γ_n and γ'_n on $\mathfrak{p}_n$ that are nearest to $\pi_{\mathcal{C}(S)}(\mu_n)$, and $\pi_{\mathcal{C}(S)}(g\mu_n)$, respectively. Note that $\mathrm{dist}_{\mathcal{C}(S)}(\gamma'_n, g_n\gamma_n) = O(1)$.

Standard arguments concerning the way hyperbolic elements act on hyperbolic metric spaces imply that the geodesic $\mathfrak{t}_n$ is in a tubular neighbourhood with radius $O(1)$ of the union of $C(S)$-geodesics $[\pi_{\mathcal{C}(S)}(\mu_n), \gamma_n] \sqcup [\gamma_n, \gamma'_n] \sqcup [\gamma'_n, g_n\pi_{\mathcal{C}(S)}(\mu_n)]$. Moreover, any point on $\mathfrak{t}_n$ has any nearest point projection on $\mathfrak{p}_n$ at distance $O(1)$ from $[\gamma_n, \gamma'_n] \subset \mathfrak{p}_n$.

Now let $\mathbf{U} = (U_n)^\omega$ be a subsurface in $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$, $\mathbf{U} \subsetneq \mathbf{S}$. Assume that, for some $i \in \mathbb{Z}$, $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) > 0$. This implies that $\lim_\omega \mathrm{dist}_{C(U_n)}(g_n^j\mu_n, g_n^{j+1}\mu_n) = +\infty$ for $j \in \{0, i\}$, according to Lemma 4.15. In particular, by Lemma 2.18, ∂U_n is at $C(S)$-distance at most 1 from a vertex $u_n \in \mathfrak{t}_n$ and $g_n^{-i}\partial U_n$ is at $C(S)$-distance at most 1 from a vertex $v_n \in \mathfrak{t}_n$. It follows from the above that ∂U_n and $g_n^{-i}\partial U_n$ have any nearest point projection on $\mathfrak{p}_n$ at distance $O(1)$ from $[\gamma_n, \gamma'_n] \subset \mathfrak{p}_n$. Let x_n be a nearest point projection on $\mathfrak{p}_n$ of ∂U_n. Then $g_n^{-i}x_n$ is a nearest point projection on $\mathfrak{p}_n$ of $g_n^{-i}\partial U_n$. As both x_n and $g_n^{-i}x_n$ are at distance $O(1)$ from $[\gamma_n, \gamma'_n]$, they are at distance at most $D + O(1)$ from each other. On the other hand $\mathrm{dist}_{\mathcal{C}(S)}(x_n, g_n^{-i}x_n) = |i|D + O(1)$. For D large enough this implies that $i \in \{-1, 0, 1\}$. Moreover, for $i = -1$, ∂U_n projects on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n, while $g_n\partial U_n$ projects on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g_n\gamma_n$. This, in particular, implies that, for D large enough, either $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g\boldsymbol{\mu}, g^2\boldsymbol{\mu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, \boldsymbol{\mu}) > 0$, but not both.

Let $\mathfrak{U}_0 = \mathfrak{U} \setminus (g\mathfrak{U} \cup g^{-1}\mathfrak{U})$. Let $\mathfrak{U}_1 = (\mathfrak{U} \cap g^{-1}\mathfrak{U}) \setminus \{\mathbf{S}\}$ and $\mathfrak{U}_2 = (\mathfrak{U} \cap g\mathfrak{U}) \setminus \{\mathbf{S}\}$. Clearly $\mathfrak{U} = \mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{P}$, where $\mathfrak{P}$ is either $\emptyset$ or $\{\mathbf{S}\}$. Since $g^{-1}\mathfrak{U} \cap g\mathfrak{U}$ is either empty or $\{\mathbf{S}\}$, it follows that $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{P}$ are pairwise disjoint, and $\mathfrak{U}_2 = g\mathfrak{U}_1$.

Assume that $\mathfrak{U}_0$ is non-empty, and let $\mathbf{U}$ be an element in $\mathfrak{U}_0$. Then $d = \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) > 0$ and $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) = 0$ for every $i \in \mathbb{Z} \setminus \{0\}$. Indeed if there exists $i \in \mathbb{Z} \setminus \{0\}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^{i+1}\boldsymbol{\mu}) > 0$, then, by the choice of D large enough, either $i = -1$ or $i = 1$, therefore either $\mathbf{U} \in g\mathfrak{U}_1$ or $\mathbf{U} \in \mathfrak{U}_1$, both contradicting the fact that $\mathbf{U} \in \mathfrak{U}_0$. The triangle inequality then implies that, for every $i \leqslant 0 < j$, $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = d$. Moreover, for every $i \leqslant k \leqslant j$, by applying g^{-k} to the previous equality, we deduce that $\widetilde{\mathrm{dist}}_{g^k\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = d$. Thus, for every $i \leqslant 0 < j$, the distance $\widetilde{\mathrm{dist}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu})$ is at least $\sum_{i\leqslant k\leqslant j} \widetilde{\mathrm{dist}}_{g^k\mathbf{U}}(g^i\boldsymbol{\mu}, g^j\boldsymbol{\mu}) = (j-i)d$. This implies that the $\langle g\rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Assume that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$. Then the distance $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, g\boldsymbol{\mu})$ is at least $|\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) - \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})| = d > 0$. Moreover, since $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^k\boldsymbol{\mu}, g^{k+1}\boldsymbol{\mu}) = 0$ for every $k \geqslant 1$ and $k \leqslant -2$, it follows that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) = \widetilde{\mathrm{dist}}_{\mathbf{U}}(g^{-1}\boldsymbol{\mu}, g\boldsymbol{\mu}) > d$ for every $k, m \geqslant 1$. We then obtain that, for every $\mathbf{V} = g^j\mathbf{U}$ with $j \in \{-k+1, \ldots, m-1\}$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) > d$. Since $\mathbf{U} \subsetneq \mathbf{S}$ and g is a pseudo-Anosov, it follows that if $i \neq j$, then $g^i\mathbf{U} \neq g^j\mathbf{U}$. Then $\widetilde{\mathrm{dist}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) \geqslant \sum_{j=-k+1}^{m-1} \widetilde{\mathrm{dist}}_{g^j\mathbf{U}}(g^{-k}\boldsymbol{\mu}, g^m\boldsymbol{\mu}) \geqslant (k+m-1)d$. Hence the $\langle g\rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Case 2. Assume that g is reducible. Let $\boldsymbol{\nu}$ be the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Consequently $g\boldsymbol{\nu}$ is the projection of $g\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. Lemma 6.10 implies that $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu}) \cup \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu}) \cup \mathfrak{U}(g\boldsymbol{\nu}, g\boldsymbol{\mu})$.

Consider an element $\mathbf{U} \in \mathfrak{U}$, $\mathbf{U} \notin \{\mathbf{U}_1, \ldots, \mathbf{U}_m\}$, and assume that, for some $i \in \mathbb{Z} \setminus \{0\}$, $g^i\mathbf{U} \in \mathfrak{U}$. We prove that $i \in \{-1, 0, 1\}$.

Assume that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu})$. Then, since $\lim_\omega \mathrm{dist}_{C(U_n)}(\mu_n, g\mu_n) = +\infty$, it follows that $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}$ and $\mathbf{U}$ is contained in $\mathbf{U}^j$ for some $j \in \{1, \ldots, k\}$. Either $\mathbf{U} = \mathbf{U}^j \in \mathfrak{P}$ or $\mathbf{U} \subsetneq \mathbf{U}^j$. In the latter case, an argument as in Case 1 implies that for D large enough $i \in \{-1, 0, 1\}$.

Assume that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Then $\mathbf{U} \pitchfork \boldsymbol{\Delta}$, since $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ do not differ inside the subsurfaces $\mathbf{U}^j$, with $j = 1, \ldots, m$. Since $\boldsymbol{\Delta} = \bigcup_{j=1}^k \partial\mathbf{U}^j$, it follows that, for some $j \in \{1, \ldots, k\}$, $\mathbf{U} \pitchfork \partial\mathbf{U}^j$.

We have that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g^i\boldsymbol{\mu}, g^i\boldsymbol{\nu}) > 0$, hence a hierarchy path joining $g_n^i\mu_n$ and $g_n^i\nu_n$ contains a point β_n in $Q(g_n^i\partial U_n)$.

The hypothesis that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) > 0$ implies that either $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$ or $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$. Assume that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$. Then a hierarchy path joining μ_n and ν_n also contains a point β'_n in $Q(g_n^i\partial U_n)$.

For the element $j \in \{1, \ldots, k\}$, such that $\mathbf{U} \pitchfork \partial\mathbf{U}^j$, $\mathrm{dist}_{C(U_n^j)}(\beta_n, \beta'_n) = O(1)$ since both β_n and β'_n contain the multicurve ∂U_n. By properties of projections, $\mathrm{dist}_{C(U_n^j)}(\mu_n, \nu_n) = O(1)$ and $\mathrm{dist}_{C(U_n^j)}(g_n^i\mu_n, g_n^i\nu_n) = O(1)$, which implies that $\mathrm{dist}_{C(U_n^j)}(\beta_n, g_n^i\nu_n) = O(1)$ and $\mathrm{dist}_{C(U_n^j)}(\beta'_n, \nu_n) = O(1)$. It follows that the distance $\mathrm{dist}_{C(U_n^j)}(g_n^i\nu_n, \nu_n)$ has order $O(1)$. On the other hand $\mathrm{dist}_{C(U_n^j)}(g_n^i\nu_n, \nu_n) > |i|D$. For D large enough this implies that $i = 0$.

Assume that $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$. This and the fact that $\widetilde{\mathrm{dist}}_{g\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$ imply as in the previous argument, with $\boldsymbol{\mu}, \boldsymbol{\nu}$ and $\mathbf{U}$ replaced by $g\boldsymbol{\mu}, g\boldsymbol{\nu}$ and $g\mathbf{U}$, respectively, that $i = 1$.

The case when $\widetilde{\mathrm{dist}}_{\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$ is dealt with similarly. In this case it follows that if $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0$, then $i = -1$, and if $\widetilde{\mathrm{dist}}_{g^i\mathbf{U}}(g\boldsymbol{\nu}, g\boldsymbol{\mu}) > 0$, then $i = 0$.

We have thus proved that, for every $\mathbf{U} \in \mathfrak{U}$, $\mathbf{U} \notin \{\mathbf{U}_1, \ldots, \mathbf{U}_m\}$, if for some $i \in \mathbb{Z} \setminus \{0\}$, $g^i\mathbf{U} \in \mathfrak{U}$, then $i \in \{-1, 0, 1\}$. We take $\mathfrak{P} = \mathfrak{U} \cap \{\mathbf{U}_1, \ldots, \mathbf{U}_m\}$ and $\mathfrak{U}' = \mathfrak{U} \setminus \mathfrak{P}$. We define

$\mathfrak{U}_0 = \mathfrak{U}' \setminus (g\mathfrak{U}' \cup g^{-1}\mathfrak{U}')$. Let $\mathfrak{U}_1 = \mathfrak{U}' \cap g^{-1}\mathfrak{U}'$ and $\mathfrak{U}_2 = \mathfrak{U}' \cap g\mathfrak{U}'$. Clearly $\mathfrak{U} = \mathfrak{U}_0 \cup \mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{P}$. Since $g^{-1}\mathfrak{U}' \cap g\mathfrak{U}'$ is empty, $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{P}$ are pairwise disjoint, and $\mathfrak{U}_2 = g\mathfrak{U}_1$.

If $\mathfrak{U}_0 \neq \emptyset$, then a proof as in Case 1 yields that the $\langle g\rangle$-orbit of $\boldsymbol{\mu}$ is unbounded.

Assume that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu}) \neq \widetilde{\mathrm{dist}}_{g\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ for some $\mathbf{U} \in \mathfrak{U}_1$. It follows from the previous argument that $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, g\boldsymbol{\nu})$, hence $g\mathbf{U}$ is in the same set. Without loss of generality we may therefore replace $\boldsymbol{\mu}$ by $\boldsymbol{\nu}$ and assume that $\boldsymbol{\mu} \in Q(\Delta)$. In particular $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ is composed only of subsurfaces that do not intersect Δ. We proceed as in case (1) and prove that the $\langle g\rangle$-orbit of $\boldsymbol{\mu}$ is unbounded. □

LEMMA 6.14. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)^\omega_e$ be a pseudo-Anosov fixing a piece P, such that $\langle g\rangle$ has bounded orbits in $\mathcal{AM}$. Assume that ω-a.s. the translation length of g_n on $C(S)$ is larger than a uniformly chosen constant depending only on $\xi(S)$. Then, for any point $\boldsymbol{\mu}$ in P and for any hierarchy path $\mathfrak{h}$ connecting $\boldsymbol{\mu}$ and its translate $g\boldsymbol{\mu}$, the isometry g fixes the midpoint of $\mathfrak{h}$.*

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in P and $\mathfrak{h} = \lim_\omega(\mathfrak{h}_n)$ a hierarchy path joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, such that $\mathfrak{h}_n$ shadows a tight geodesic $\mathfrak{t}_n$. We may assume that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$, and consider the splitting defined in Lemma 6.13, $\mathfrak{U} = \mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}_0 \sqcup \mathfrak{U}_1 \sqcup g\mathfrak{U}_1$. Note that since $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$ are both in the same piece P, $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ cannot contain $\mathbf{S}$, by Lemma 6.7. As $\langle g\rangle$ has bounded orbits, we may assume that $\mathfrak{U}_0$ is empty, and that $\mathfrak{U} = \mathfrak{U}_1 \cup g\mathfrak{U}_1$. We denote $g\mathfrak{U}_1$ also by $\mathfrak{U}_2$. For every $\mathbf{U} \in \mathfrak{U}$ choose a sequence (U_n) representing it, and define $\mathfrak{U}(n)$, $\mathfrak{U}_1(n)$, $\mathfrak{U}_2(n)$ as the set of U_n corresponding to $\mathbf{U}$ in $\mathfrak{U}, \mathfrak{U}_1, \mathfrak{U}_2$ respectively.

Let α_n be the last point on the hierarchy path $\mathfrak{h}_n$ belonging to $Q(\partial U_n)$ for some $U_n \in \mathfrak{U}_1(n)$. Let $\boldsymbol{\alpha} = \lim_\omega(\alpha_n)$. Assume that $g\boldsymbol{\alpha} \neq \boldsymbol{\alpha}$. For every subsurface $\mathbf{V} = (V_n)^\omega \in \Pi\mathcal{U}/\omega$ such that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha}) > 0$ it follows by the triangle inequality that either $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\mu}) > 0$ or $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g\boldsymbol{\mu}, g\boldsymbol{\alpha}) > 0$. In the first case $\mathbf{V} \in \mathfrak{U}$. If $\mathbf{V} \in \mathfrak{U}_1$, then $\mathbf{V} = (U_n)^\omega$ for one of the chosen sequences (U_n) representing an element in $\mathfrak{U}_1$, whence $\lim_\omega \mathrm{dist}_{C(U_n)}(\alpha_n, g_n\mu_n) = \infty$ and the hierarchy subpath of $\mathfrak{h}_n$ between α_n and $g_n\mu_n$ has a large intersection with $Q(\partial U_n)$. This contradicts the choice of α_n. Thus in this case we must have that $\mathbf{V} \in g\mathfrak{U}_1$.

We now consider the second case, where $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g\boldsymbol{\mu}, g\boldsymbol{\alpha}) > 0$. Since this condition is equivalent to $\widetilde{\mathrm{dist}}_{g^{-1}\mathbf{V}}(\boldsymbol{\mu}, \boldsymbol{\alpha}) > 0$, it follows that $g^{-1}\mathbf{V} \in \mathfrak{U}$. Moreover, ω-a.s. the hierarchy subpath of $\mathfrak{h}_n$ between μ_n and α_n has a large intersection with $Q(g_n^{-1}\partial V_n)$.

Define $\mathfrak{p}_n$ and the points γ_n, γ_n' on $\mathfrak{p}_n$ as in Case 1 of the proof of Lemma 6.13. The argument in that proof shows that for every $\mathbf{U} = (U_n)^\omega \in \mathfrak{U}_1$, ω-a.s. ∂U_n has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n while $g_n\partial U_n$ has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g_n\gamma_n$. In particular α_n has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n whence $g_n^{-1}\partial V_n$ has any nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from γ_n too. For sufficiently large translation length (that is, the constant in the hypothesis of the lemma), this implies that $g_n^{-1}\partial V_n$ cannot have a nearest point projection on $\mathfrak{p}_n$ at $C(S)$-distance $O(1)$ from $g\gamma_n$. Thus ω-a.s. $g_n^{-1}V_n \notin \mathfrak{U}_2(n)$, therefore $g^{-1}\mathbf{V} \notin \mathfrak{U}_2$. It follows that $g^{-1}\mathbf{V} \in \mathfrak{U}_1$, whence $\mathbf{V} \in g\mathfrak{U}_1$.

We have thus obtained that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha}) > 0$ implies that $\mathbf{V} \in g\mathfrak{U}_1$, therefore for every $k \in \mathbb{Z}$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(g^k\boldsymbol{\alpha}, g^{k+1}\boldsymbol{\alpha}) > 0$ implies that $\mathbf{V} \in g^{k+1}\mathfrak{U}_1$. Since the collections of subsurfaces $g^i\mathfrak{U}_1$ and $g^j\mathfrak{U}_1$ are disjoint for $i \neq j$, it follows that $\widetilde{\mathrm{dist}}(g^{-i}\boldsymbol{\alpha}, g^j\boldsymbol{\alpha}) = \sum_{k=-i}^{j-1}\sum_{\mathbf{V}\in g^{k+1}\mathfrak{U}_1} \widetilde{\mathrm{dist}}_{\mathbf{V}}(g^{-i}\boldsymbol{\alpha}, g^j\boldsymbol{\alpha}) = \sum_{k=-i}^{j-1}\sum_{\mathbf{V}\in g^{k+1}\mathfrak{U}_1} \widetilde{\mathrm{dist}}_{\mathbf{V}}(g^k\boldsymbol{\alpha}, g^{k+1}\boldsymbol{\alpha}) = (j+i-1)\widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, g\boldsymbol{\alpha})$. This implies that the $\langle g\rangle$-orbit of $\boldsymbol{\alpha}$ is unbounded, contradicting our hypothesis.

Therefore $\boldsymbol{\alpha} = g\boldsymbol{\alpha}$. From this, the fact that g acts as an isometry on $(\mathcal{AM}, \widetilde{\mathrm{dist}})$, and the fact that hierarchy paths are geodesics in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$, it follows that $\boldsymbol{\alpha}$ is the midpoint of $\mathfrak{h}$. □

Lemma 6.15. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)^\omega_e$ be a pseudo-Anosov such that $\langle g \rangle$ has bounded orbits in $\mathcal{AM}$. Assume that ω-a.s. the translation length of g_n on $C(S)$ is larger than a uniformly chosen constant depending only on $\xi(S)$. Then for any point $\boldsymbol{\mu}$ and for any hierarchy path $\mathfrak{h}$ connecting $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, the isometry g fixes the midpoint of $\mathfrak{h}$.*

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in $\mathcal{AM}$ and assume $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Lemma 3.11, (1), implies that g fixes either the middle cut-point or the middle cut-piece of $\boldsymbol{\mu}, g\boldsymbol{\mu}$. In the former case we are done. In the latter case consider P the middle cut-piece, $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ the entrance and exit points, respectively, of $\mathfrak{h}$ from P. Then $\boldsymbol{\nu}' = g\boldsymbol{\nu} \neq \boldsymbol{\nu}$ and we may apply Lemma 6.14 to g and $\boldsymbol{\nu}$ to complete the argument. □

Lemma 6.16. *Let $g = (g_n)^\omega \in \mathcal{MCG}(S)^\omega_e$ be a pseudo-Anosov. The set of fixed points of g is either empty or it is a convex subset of a transversal tree of $\mathcal{AM}$.*

Proof. Assume that there exists a point $\boldsymbol{\mu} \in \mathcal{AM}$ fixed by g. Let $\boldsymbol{\nu}$ be another point fixed by g. Since g is an isometry permuting pieces, this and property (T_2') implies that g fixes every point in $\mathrm{Cutp}\,\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$. If a geodesic (any geodesic) joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ has a non-trivial intersection $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ with a piece, then $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are also fixed by g. By Lemma 6.7, $\mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ contains a proper subsurface $\mathbf{U} \subsetneq \mathbf{S}$, and by Lemma 6.6, $g\mathbf{U} = \mathbf{U}$, which is impossible.

It follows that any geodesic joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ intersects all pieces in points. This means that the set of points fixed by g is contained in the transversal tree $T_{\boldsymbol{\mu}}$ (as defined in Definition 3.6). It is clearly a convex subset of $T_{\boldsymbol{\mu}}$. □

Lemma 6.17. *Let $g \in \mathcal{MCG}(S)^\omega_e$ be a reducible element such that $\langle g \rangle$ has bounded orbits in $\mathcal{AM}$, and let $\boldsymbol{\Delta} = (\Delta_n)^\omega$ and $\mathbf{U}_1 = (U_n^1)^\omega, \dots, \mathbf{U}_m = (U_n^m)^\omega$ be, respectively, the multicurve and the subsurfaces associated to g as in Lemma 6.9. Assume that, for any $i \in \{1, 2, \dots, k\}$ and for any $\boldsymbol{\nu} \in Q(\boldsymbol{\Delta})$, the distance $\mathrm{dist}_{C(U_i)}(\boldsymbol{\nu}, g\boldsymbol{\nu})$ is larger than some sufficiently large constant D depending only on $\xi(S)$.*

Then, for any point $\boldsymbol{\mu}$, there exists a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting $\boldsymbol{\mu}$ and its translate $g\boldsymbol{\mu}$, such that the isometry g fixes its midpoint.

Proof. Let $\boldsymbol{\mu}$ be an arbitrary point in $\mathcal{AM}$. By means of Lemma 6.9(1), we may reduce the argument to the case when $\boldsymbol{\mu}$ is contained in a piece P fixed set-wise by g. Lemma 6.9 implies that $U(P)$ contains $Q(\boldsymbol{\Delta})$. Let $\boldsymbol{\nu}$ be the projection of $\boldsymbol{\mu}$ onto $Q(\boldsymbol{\Delta})$. According to Lemma 4.27, if D is large enough, then given $\mathfrak{h}_1, \mathfrak{h}_2$ and $\mathfrak{h}_3$ hierarchy paths connecting $\boldsymbol{\mu}, \boldsymbol{\nu}$, $\boldsymbol{\nu}, g\boldsymbol{\nu}$ and $g\boldsymbol{\nu}, g\boldsymbol{\mu}$, respectively, $\mathfrak{h}_1 \sqcup \mathfrak{h}_2 \sqcup \mathfrak{h}_3$ is a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$.

If $\boldsymbol{\nu} = g\boldsymbol{\nu}$, then we are done. If not, we apply Lemma 6.15 to g restricted to each $\mathbf{U}^j$ and we find a point $\boldsymbol{\alpha}$ between $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$, fixed by g. Since both $\boldsymbol{\nu}$ and $g\boldsymbol{\nu}$ are between $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, it follows that $\boldsymbol{\alpha}$ is between $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, hence on a geodesic in $(\mathcal{AM}, \widetilde{\mathrm{dist}})$ connecting them. □

Lemma 6.18. *Let $g \in \mathcal{MCG}(S)^\omega_e$ be a reducible element, and let $\boldsymbol{\Delta} = (\Delta_n)^\omega$ and $\mathbf{U}_1 = (U_n^1)^\omega, \dots, \mathbf{U}_m = (U_n^m)^\omega$ be, respectively, the multicurve and the subsurfaces associated to g as in Lemma 6.9.*

If the set $\mathrm{Fix}(g)$ of points fixed by g contains a point $\boldsymbol{\mu}$, then, when identifying $Q(\boldsymbol{\Delta})$ with $\mathcal{M}(\mathbf{U}_1) \times \dots \times \mathcal{M}(\mathbf{U}_m)$ and correspondingly $\boldsymbol{\mu}$ with a point $(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_m)$, $\mathrm{Fix}(g)$ identifies with $C_1 \times \dots \times C_k \times \mathcal{M}(\mathbf{U}_{k+1}) \times \dots \times \mathcal{M}(\mathbf{U}_m)$, where C_i is a convex subset contained in the transversal tree $T_{\boldsymbol{\mu}_i}$.

Proof. This follows immediately from the fact that $\mathrm{Fix}(g) = \mathrm{Fix}(g(1)) \times \ldots \times \mathrm{Fix}(g(m))$, where $g(i)$ is the restriction of g to the subsurface $\mathbf{U}_i$, and from Lemma 6.16. □

LEMMA 6.19. *Let g be a pure element such that $\langle g \rangle$ has bounded orbits in $\mathcal{AM}$. Let $\boldsymbol{\mu}$ be a point in $\mathcal{AM}$ such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$ and let $\boldsymbol{m}$ be a midpoint of a $\widetilde{\mathrm{dist}}$-geodesic joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, $\boldsymbol{m}$ fixed by g. Then, in the splitting of $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu})$ given by Lemma 6.13, the set $\mathfrak{U}_1$ coincides with $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \setminus \mathfrak{P}$.*

Proof. As $\langle g \rangle$ has bounded orbits, we have that $\mathfrak{U}_0 = \emptyset$, according to the last part of the statement of Lemma 6.13.

Since $\boldsymbol{m}$ is on a geodesic joining $\boldsymbol{\mu}$ and $g\boldsymbol{\mu}$, it follows that $\mathfrak{U}(\boldsymbol{\mu}, g\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \cup \mathfrak{U}(\boldsymbol{m}, g\boldsymbol{\mu})$. From the definition of $\mathfrak{U}_1$ it follows that $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{m}) \setminus \mathfrak{P}$ is contained in $\mathfrak{U}_1$. Also, if an element $\mathbf{U} \in \mathfrak{U}_1$ would be contained in $\mathfrak{U}(\boldsymbol{m}, g\boldsymbol{\mu})$, then it would follow that $g^{-1}\mathbf{U}$ is also in $\mathfrak{U}$, which is a contradiction. □

NOTATION. In what follows, for any reducible element $t \in \mathcal{MCG}(S)^\omega_e$ we denote by $\boldsymbol{\Delta}_t$ the multicurve associated to t as in Lemma 6.9.

LEMMA 6.20. (1) *Let g be a pure element with $\mathrm{Fix}(g)$ non-empty. For every $\boldsymbol{x} \in \mathcal{AM}$ there exists a unique point $\boldsymbol{y} \in \mathrm{Fix}(g)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{x}, \boldsymbol{y}) = \widetilde{\mathrm{dist}}(\boldsymbol{x}, \mathrm{Fix}(g))$.*

(2) *Let g and h be two pure elements not fixing a common multicurve. If $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ are non-empty, then there exists a unique pair of points $\boldsymbol{\mu} \in \mathrm{Fix}(g)$ and $\boldsymbol{\nu} \in \mathrm{Fix}(h)$ such that $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\mathrm{Fix}(g), \mathrm{Fix}(h))$.*

Moreover, for every $\boldsymbol{\alpha} \in \mathrm{Fix}(g)$, $\widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \mathrm{Fix}(h))$, and $\boldsymbol{\nu}$ is the unique point with this property; likewise for every $\boldsymbol{\beta} \in \mathrm{Fix}(h)$, $\widetilde{\mathrm{dist}}(\boldsymbol{\beta}, \boldsymbol{\mu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\beta}, \mathrm{Fix}(g))$ and $\boldsymbol{\mu}$ is the unique point with this property.

Proof. We identify $\mathcal{AM}$ with a subset of the product of trees $\prod_{\mathbf{U} \in \Pi\mathcal{U}/\omega} T_{\mathbf{U}}$. Let g be a pure element with $\mathrm{Fix}(g)$ non-empty. By Lemma 6.6, for any $\mathbf{U}$ such that $g(\mathbf{U}) \neq \mathbf{U}$ we have that the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{U}}$ is a point which we denote by $\boldsymbol{\mu}_{\mathbf{U}}$. If $\mathbf{U}$ is such that $\mathbf{U} \pitchfork \boldsymbol{\Delta}_g$, then the projection of $\mathrm{Fix}(g)$, and indeed of $Q(\boldsymbol{\Delta}_g)$ onto $T_{\mathbf{U}}$ also reduces to a point, by Lemma 6.10, (1). The only surfaces $\mathbf{U}$ such that $g(\mathbf{U}) = \mathbf{U}$ and $\mathbf{U} \not\pitchfork \boldsymbol{\Delta}_g$ are $\mathbf{U}_1, \ldots, \mathbf{U}_k$ and $\mathbf{Y} \subseteq \mathbf{U}_j$ with $j \in \{k+1, \ldots, m\}$, where $\mathbf{U}_1, \ldots, \mathbf{U}_m$ are the subsurfaces determined on $\mathbf{S}$ by $\boldsymbol{\Delta}_g$, g restricted to $\mathbf{U}_1, \ldots, \mathbf{U}_k$ is a pseudo-Anosov and g restricted to $\mathbf{U}_{k+1}, \ldots, \mathbf{U}_m$ is the identity. By Lemma 6.18, the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{U}_i}$ is a convex tree $C_{\mathbf{U}_i}$, when $i = 1, \ldots, k$, and the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{Y}}$ with $\mathbf{Y} \subseteq \mathbf{U}_j$ and $j \in \{k+1, \ldots, m\}$ is $T_{\mathbf{Y}}$.

(1) The point $\boldsymbol{x}$ in $\mathcal{AM}$ is identified to the element $(\boldsymbol{x}_{\mathbf{U}})_{\mathbf{U}}$ in the product of trees $\prod_{\mathbf{U} \in \Pi\mathcal{U}/\omega} T_{\mathbf{U}}$.

For every $i \in \{1, \ldots, k\}$ we choose the unique point $\boldsymbol{y}_{\mathbf{U}_i}$ in the tree $C_{\mathbf{U}_i}$ realizing the distance from $\boldsymbol{x}_{\mathbf{U}_i}$ to that tree. The point $\boldsymbol{y}_{\mathbf{U}_i}$ lifts to a unique point $\boldsymbol{y}_i$ in the transversal subtree C_i.

Let $i \in \{k+1, \ldots, m\}$ and let $\boldsymbol{y}_{\mathbf{U}_i} = (\boldsymbol{x}_{\mathbf{Y}})_{\mathbf{Y}}$ be the projection of $(\boldsymbol{x}_{\mathbf{U}})_{\mathbf{U}}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$. Now the projection of $\mathcal{AM}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ coincides with the embedded image of $\mathcal{M}(\mathbf{U}_i)$, since for every $\boldsymbol{x} \in \mathcal{AM}$ its projection in $T_{\mathbf{Y}}$ coincides with the projection of $\pi_{\mathcal{M}(\mathbf{U}_i)}(\boldsymbol{x})$. Therefore there exists a unique element $\boldsymbol{y}_i \in \mathcal{M}(\mathbf{U}_i)$ such that its image in $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ is $\boldsymbol{y}_{\mathbf{U}_i}$. Note that the point $\boldsymbol{y}_i$ can also be found as the projection of $\boldsymbol{x}$ onto $\mathcal{M}(\mathbf{U}_i)$.

Let $\boldsymbol{z}$ be an arbitrary point in $\mathrm{Fix}(g)$. For every subsurface $\mathbf{U}$ the point $\boldsymbol{z}$ has the property that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{z}, \boldsymbol{x}) \geqslant \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{y}, \boldsymbol{x})$. Moreover, if $\boldsymbol{z} \neq \boldsymbol{y}$, then there exist at least one subsurface $\mathbf{V}$ with $g(\mathbf{V}) = \mathbf{V}$ and $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}_g$ such that $\boldsymbol{z}_{\mathbf{V}} \neq \boldsymbol{y}_{\mathbf{V}}$. By the choice of $\boldsymbol{y}_{\mathbf{V}}$ it follows that

$\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{z}_{\mathbf{V}}, \boldsymbol{x}_{\mathbf{V}}) > \widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{y}_{\mathbf{V}}, \boldsymbol{x}_{\mathbf{V}})$. Therefore $\widetilde{\mathrm{dist}}(\boldsymbol{z}, \boldsymbol{x}) \geqslant \widetilde{\mathrm{dist}}(\boldsymbol{y}, \boldsymbol{x})$, and the inequality is strict if $\boldsymbol{z} \neq \boldsymbol{y}$.

(2) Let $\mathbf{V}_1, \ldots, \mathbf{V}_s$ be the subsurfaces determined on $\mathbf{S}$ by $\boldsymbol{\Delta}_h$, such that h restricted to $\mathbf{V}_1, \ldots, \mathbf{V}_l$ is a pseudo-Anosov and h restricted to $\mathbf{V}_{l+1}, \ldots, \mathbf{V}_s$ is the identity. The projection of $\mathrm{Fix}(h)$ onto $T_{\mathbf{V}_i}$ is a convex tree $C_{\mathbf{V}_i}$, when $i = 1, \ldots, l$, the projection of $\mathrm{Fix}(g)$ onto $T_{\mathbf{Z}}$ with $\mathbf{Z} \subseteq \mathbf{V}_j$ and $j \in \{l+1, \ldots, s\}$ is $T_{\mathbf{Z}}$, and for any other subsurface $\mathbf{U}$ the projection of $\mathrm{Fix}(h)$ is one point $\boldsymbol{\nu}_{\mathbf{U}}$.

For every $i \in \{1, \ldots, k\}$ $\mathrm{Fix}(h)$ projects onto a point $\boldsymbol{\nu}_{\mathbf{U}_i}$ by the hypothesis that g, h do not fix a common multicurve (hence a common subsurface). Consider $\boldsymbol{\mu}_{\mathbf{U}_i}$ the nearest to $\boldsymbol{\nu}_{\mathbf{U}_i}$ point in the convex tree $C_{\mathbf{U}_i}$. This point lifts to a unique point $\boldsymbol{\mu}_i$ in the transversal subtree C_i. Let $i \in \{k+1, \ldots, m\}$. On $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ $\mathrm{Fix}(h)$ projects onto a unique point, since it has a unique projection in each $T_{\mathbf{Y}}$. As pointed out already in the proof of (1), the projection of $\mathcal{AM}$ onto $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ coincides with the embedded image of $\mathcal{M}(\mathbf{U}_i)$. Therefore there exists a unique element $\boldsymbol{\mu}_i \in \mathcal{M}(\mathbf{U}_i)$ such that its image in $\prod_{\mathbf{Y} \subseteq \mathbf{U}_i} T_{\mathbf{Y}}$ is $(\boldsymbol{\nu}_{\mathbf{Y}})_{\mathbf{Y}}$. Note that the point $\boldsymbol{\mu}_i$ can also be found as the unique point which is the projection of $\mathrm{Fix}(h)$ onto $\mathcal{M}(\mathbf{U}_i)$ for $i = k+1, \ldots, m$. We consider the point $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m) \in C_1 \times \ldots \times C_k \times \mathcal{M}(\mathbf{U}_{k+1}) \times \ldots \times \mathcal{M}(\mathbf{U}_m)$. Let $\boldsymbol{\alpha}$ be an arbitrary point in $\mathrm{Fix}(g)$ and let $\boldsymbol{\beta}$ be an arbitrary point in $\mathrm{Fix}(h)$. For every subsurface $\mathbf{U}$ the point $\boldsymbol{\mu}$ has the property that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leqslant \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Moreover, if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$, then there exist at least one subsurface $\mathbf{V}$ with $g(\mathbf{V}) = \mathbf{V}$ and $\mathbf{V} \not\pitchfork \boldsymbol{\Delta}_g$ such that $\boldsymbol{\alpha}_{\mathbf{V}} \neq \boldsymbol{\mu}_{\mathbf{V}}$. By the choice of $\boldsymbol{\mu}_{\mathbf{V}}$ it follows that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\mu}_{\mathbf{V}}, \boldsymbol{\beta}_{\mathbf{V}}) < \widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}_{\mathbf{V}}, \boldsymbol{\beta}_{\mathbf{V}})$. Therefore $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leqslant \widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and the inequality is strict if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$.

We construct similarly a point $\boldsymbol{\nu} \in \mathrm{Fix}(h)$. Then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \leqslant \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\beta}) \leqslant \widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for any $\boldsymbol{\alpha} \in \mathrm{Fix}(g)$ and $\boldsymbol{\beta} \in \mathrm{Fix}(h)$. Moreover, the first inequality is strict if $\boldsymbol{\beta} \neq \boldsymbol{\nu}$, and the second inequality is strict if $\boldsymbol{\alpha} \neq \boldsymbol{\mu}$. □

LEMMA 6.21. *Let $g \in \mathcal{MCG}(S)^{\omega}_e$ be a pure element satisfying the hypotheses from Lemma* 6.13 *and, moreover, assume that $\langle g \rangle$ has bounded orbits, whence $\mathrm{Fix}(g) \neq \emptyset$, by Lemmas* 6.15 *and* 6.17. *Let $\boldsymbol{\mu}$ be an element such that $g\boldsymbol{\mu} \neq \boldsymbol{\mu}$ and let $\boldsymbol{\nu}$ be the unique projection of $\boldsymbol{\mu}$ onto $\mathrm{Fix}(g)$ defined in Lemma* 6.20(1).

Then, for every $k \in \mathbb{Z} \setminus \{0\}$, $\boldsymbol{\nu}$ is on a geodesic joining $\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$.

Proof. By Lemmas 6.15 and 6.17 there exists $\boldsymbol{m}$ middle of a geodesic joining $\boldsymbol{\mu}$ and $g^k\boldsymbol{\mu}$ such that $\boldsymbol{m} \in \mathrm{Fix}(g^k)$. By Lemma 6.12, $\mathrm{Fix}(g^k) = \mathrm{Fix}(g)$. Assume that $\boldsymbol{m} \neq \boldsymbol{\nu}$. Then by Lemma 6.20, (1), $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{m})$. Then $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, g^k\boldsymbol{\mu}) \leqslant \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) + \widetilde{\mathrm{dist}}(\boldsymbol{\nu}, g^k\boldsymbol{\mu}) = 2\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{\nu}) < 2\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, \boldsymbol{m}) = \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, g^k\boldsymbol{\mu})$, which is impossible. □

LEMMA 6.22. *Let $g = (g_n)^{\omega}$ and $h = (h_n)^{\omega}$ be two pure reducible elements in $\mathcal{MCG}(S)^{\omega}_e$, such that they do not both fix a multicurve. If a proper subsurface $\mathbf{U}$ has the property that $h(\mathbf{U}) = \mathbf{U}$, then*:

(1) *$g^m\mathbf{U} \pitchfork \boldsymbol{\Delta}_h$ for $|m| \geqslant N = N(g)$;*

(2) *the equality $h(g^k(\mathbf{U})) = g^k(\mathbf{U})$ can hold only for finitely many $k \in \mathbb{Z}$.*

Proof. (1) Assume, for a contradiction, that $g^m\mathbf{U} \not\pitchfork \boldsymbol{\Delta}_h$ for $|m|$ large. Since $h(\mathbf{U}) = \mathbf{U}$, it follows that $\mathbf{U}$ must overlap a component $\mathbf{V}$ of $\mathbf{S} \setminus \boldsymbol{\Delta}_g$ on which g is a pseudo-Anosov (otherwise $g\mathbf{U} = \mathbf{U}$). If $\boldsymbol{\Delta}_h$ would also intersect $\mathbf{V}$, then the projections of $\Delta_{h,n}$ and of ∂U_n onto the curve complex $C(V_n)$ would be at distance $O(1)$. On the other hand, since $\mathrm{dist}_{C(V_n)}(g^m \partial U_n, \partial U_n) \geqslant |m| + O(1)$, it follows that for $|m|$ large enough $\mathrm{dist}_{C(V_n)}(g^m \partial U_n, \Delta_{h,n}) > 3$, that is, $g^m \partial \mathbf{U}$ would intersect $\boldsymbol{\Delta}_h$, which is a contradiction. Thus $\boldsymbol{\Delta}_h$ does not intersect $\mathbf{V}$. It follows that

$\mathbf{U}$ does not have all boundary components from $\mathbf{\Delta}_h$; thus the only possibility for $h(\mathbf{U}) = \mathbf{U}$ to be achieved is that $\mathbf{U}$ is a finite union of subsurfaces determined by $\mathbf{\Delta}_h$ and subsurfaces contained in a component of $\mathbf{S} \setminus \mathbf{\Delta}_h$ on which h is identity. Since $\mathbf{V}$ intersects $\mathbf{U}$ and not $\mathbf{\Delta}_h$, $\mathbf{V}$ intersects only a subsurface $\mathbf{U}_1 \subseteq \mathbf{U}$ restricted to which h is identity, and $\mathbf{V}$ is in the same component of $\mathbf{S} \setminus \mathbf{\Delta}_h$ as $\mathbf{U}_1$. Therefore $h\mathbf{V} = \mathbf{V}$, and we also had that $g\mathbf{V} = \mathbf{V}$, which is a contradiction.

(2) Assume that $h(g^k(\mathbf{U})) = g^k(\mathbf{U})$ holds for infinitely many $k \in \mathbb{Z}$. Without loss of generality, we may assume that all k are positive integers and that, for all k, $g^k\mathbf{U} \pitchfork \mathbf{\Delta}_h$. Up to taking a subsequence of k, we may assume that there exist $\mathbf{U}_1, \dots, \mathbf{U}_m$ subsurfaces determined by $\mathbf{\Delta}_h$ and $1 \leqslant r \leqslant m$ such that h restricted to $\mathbf{U}_1, \dots, \mathbf{U}_r$ is either a pseudo-Anosov or identity, h restricted to $\mathbf{U}_{r+1}, \dots, \mathbf{U}_m$ is identity, and $g^k(\mathbf{U}) = \mathbf{U}_1 \cup \dots \cup \mathbf{U}_r \cup \mathbf{V}_{r+1}(k) \cup \dots \cup \mathbf{V}_m(k)$, where $\mathbf{V}_j(k) \subsetneq \mathbf{U}_j$ for $j = r+1, \dots, m$. The boundary of $g^k(\mathbf{U})$ decomposes as $\partial' S \sqcup \mathbf{\Delta}'_h \sqcup \partial_k$, where $\partial' S$ is the part of $\partial g^k(\mathbf{U})$ contained in ∂S, $\mathbf{\Delta}'_h$ is the part contained in $\mathbf{\Delta}_h$ and ∂_k is the remaining part (coming from the subsurfaces $\mathbf{V}_j(k)$). Up to taking a subsequence and pre-composing with some g^{-k_0}, we may assume that $\mathbf{U} = \mathbf{U}_1 \cup \dots \cup \mathbf{U}_r \cup \mathbf{V}_{r+1}(0) \cup \dots \cup \mathbf{V}_m(0)$ and that g^k do not permute the boundary components. It follows that $\mathbf{\Delta}'_h = \emptyset$, hence $\partial_k \neq \emptyset$. Take a boundary curve $\gamma \in \partial_0$. Then $\gamma \in \partial \mathbf{V}_j(0)$ for some $j \in \{r+1, \dots, m\}$, and, for every k, $g^k\gamma \in \partial \mathbf{V}_j(k) \subset \mathbf{U}_j$, in particular $g^k\gamma \not\pitchfork \mathbf{\Delta}_h$. An argument as in (1) yields a contradiction. $\square$

LEMMA 6.23. *Let $g = (g_n)^\omega$ and $h = (h_n)^\omega$ be two pure elements in $\mathcal{MCG}(S)^\omega_e$, such that $\langle g, h \rangle$ is composed only of pure elements and its orbits in $\mathcal{AM}$ are bounded. Then g and h fix a point.*

Proof. (1) Assume that g and h do not fix a common multicurve. We argue by contradiction and assume that g and h do not fix a point, and we shall deduce from this that $\langle g, h \rangle$ has unbounded orbits.

Since g and h do not fix a point, by Lemma 6.20(2), $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ do not intersect, therefore the $\widetilde{\mathrm{dist}}$-distance between them is $d > 0$. Let $\boldsymbol{\mu} \in \mathrm{Fix}(g)$ and $\boldsymbol{\nu} \in \mathrm{Fix}(h)$ be the unique pair of points realizing this distance d, according to Lemma 6.20. Possibly by replacing g and h by some powers, we may assume that g, h and all their powers have the property that each pseudo-Anosov components has sufficiently large translation lengths in their respective curve complexes.

(1.a) We prove that, for every $\boldsymbol{\alpha} \in \mathcal{AM}$ and every $\epsilon > 0$, there exists k such that $g^k(\boldsymbol{\alpha})$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$.

Let $\boldsymbol{\mu}_1$ be the unique projection of $\boldsymbol{\alpha}$ on $\mathrm{Fix}(g)$, as defined in Lemma 6.20(1). According to Lemma 6.21, $\boldsymbol{\mu}_1$ is on a geodesic joining $\boldsymbol{\alpha}$ and $p\boldsymbol{\alpha}$ for every $p \in \langle g \rangle \setminus \{\mathrm{id}\}$. Let $\boldsymbol{\nu}_1$ be the unique point on $\mathrm{Fix}(h)$ that is nearest to $p(\boldsymbol{\alpha})$, whose existence is ensured by Lemma 6.20(1).

By Lemma 6.13, $\mathfrak{U}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) = \mathfrak{U}_1^p \sqcup p\mathfrak{U}_1^p \sqcup \mathfrak{P}$. Moreover, by Lemma 6.19, $\mathfrak{U}_1^p = \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\mu}_1) \setminus \mathfrak{P}$, therefore $\mathfrak{U}_1^p$ is independent of the power p. Therefore we shall henceforth denote it simply by $\mathfrak{U}_1$.

Let $\mathbf{U}$ be a subsurface in $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$. If $\mathbf{U}$ is a pseudo-Anosov component of h, then the projection of $\mathrm{Fix}(h)$ onto $T_\mathbf{U}$ is a subtree $C_\mathbf{U}$; the whole set $\mathrm{Fix}(g)$ projects onto a point $\boldsymbol{\mu}_\mathbf{U}$; $\boldsymbol{\nu}_\mathbf{U}$ is the projection of $\boldsymbol{\mu}_\mathbf{U}$ onto $C_\mathbf{U}$; $(\boldsymbol{\nu}_1)_\mathbf{U}$ is the projection of $(p(\boldsymbol{\alpha}))_\mathbf{U}$ onto $C_\mathbf{U}$; and $\boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U}$ are distinct. It follows that the geodesic joining $(\boldsymbol{\mu}_1)_\mathbf{U}$ and $(p(\boldsymbol{\alpha}))_\mathbf{U}$ covers the geodesic joining $\boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U}$, whence $\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \geqslant \widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$.

If $\mathbf{U}$ is a subsurface of an identity component of h, then the projection of $\mathrm{Fix}(h)$ onto $T_\mathbf{U}$ is the whole tree $T_\mathbf{U}$, $\mathrm{Fix}(g)$ projects onto a unique point $\boldsymbol{\mu}_\mathbf{U} = \boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U} = (p(\boldsymbol{\alpha}))_\mathbf{U}$. It follows that the geodesic joining $(\boldsymbol{\mu}_1)_\mathbf{U}$ and $(p(\boldsymbol{\alpha}))_\mathbf{U}$ is the same as the geodesic joining $\boldsymbol{\nu}_\mathbf{U}$ and $(\boldsymbol{\nu}_1)_\mathbf{U}$, whence $\widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) = \widetilde{\mathrm{dist}}_\mathbf{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$.

Thus in both cases $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \geqslant \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) > 0$, in particular $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha}))$. Since g and h do not fix a common subsurface, $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) \cap \mathfrak{P} = \emptyset$, therefore $\mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) \subset \mathfrak{U}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \setminus \mathfrak{P} = p\mathfrak{U}_1$. The last equality holds by Lemma 6.19.

Now consider $\mathbf{V}_1, \dots, \mathbf{V}_r$ subsurfaces in $\mathfrak{U}_1$ such that the sum

$$\sum_{j=1}^{r} (\widetilde{\mathrm{dist}}_{\mathbf{V}_j}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) + \widetilde{\mathrm{dist}}_{g\mathbf{V}_j}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha}))) + \sum_{\mathbf{U} \in \mathfrak{P}} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})))$$

is at least $\widetilde{\mathrm{dist}}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) - \epsilon$.

According to Lemma 6.22, (2), by taking p a large enough power of g we may ensure that $h(p(\mathbf{V}_j)) \neq p(\mathbf{V}_j)$ for every $j = 1, \dots, r$. Then

$$\begin{aligned} \widetilde{\mathrm{dist}}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) = \sum_{\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\nu}, \boldsymbol{\nu}_1) &\leqslant \sum_{\mathbf{U} \in \mathfrak{U}(\boldsymbol{\nu}, \boldsymbol{\nu}_1)} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) \\ &\leqslant \sum_{\mathbf{U} \in p\mathfrak{U}_1, \mathbf{U} \neq p\mathbf{V}_j} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}_1, p(\boldsymbol{\alpha})) = \sum_{\mathbf{U} \in p\mathfrak{U}_1, \mathbf{U} \neq p\mathbf{V}_j} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, p(\boldsymbol{\alpha})) \leqslant \epsilon. \end{aligned}$$

(1.b) In a similar way we prove that, for every $\boldsymbol{\beta} \in \mathcal{AM}$, and every $\delta > 0$ there exists m such that $h^m(\boldsymbol{\beta})$ projects onto $\mathrm{Fix}(g)$ at distance at most δ from $\boldsymbol{\mu}$.

(1.c) We now prove by induction on k that, for every $\epsilon > 0$, there exists a word w in g and h such that:

(i) $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w\boldsymbol{\nu})$ is in the interval $[2kd - \epsilon, 2kd]$;

(ii) $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w\boldsymbol{\nu})$ is in the interval $[(2k-1)d - \epsilon, (2k-1)d]$;

(iii) $w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$.

This will show that the $\boldsymbol{\nu}$-orbit of $\langle g, h \rangle$ is unbounded, contradicting the hypothesis.

Take $k = 1$. Then (1.a) applied to $\boldsymbol{\nu}$ and ϵ implies that there exists a power p of g such that $p\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$. Note that by Lemma 6.21, $\boldsymbol{\mu}$ is the middle of a geodesic joining $\boldsymbol{\nu}$ and $p\boldsymbol{\nu}$, hence $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, p\boldsymbol{\nu}) = 2d$ and $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, p\boldsymbol{\nu}) = d$.

Assume that the statement is true for k, and consider $\epsilon > 0$ arbitrary. The induction hypothesis applied to $\epsilon_1 = \epsilon/16$ produces a word w in g and h. Property (1.b) applied to $\boldsymbol{\beta} = w\boldsymbol{\nu}$ implies that there exists a power h^m such that $h^m w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(g)$ at distance at most $\delta = \epsilon/4$.

The distance $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu})$ is equal to $\widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\nu})$, hence it is in $[2kd - \epsilon_1, 2kd]$. The distance $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu})$ is at most $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu}) + d = (2k+1)d$. Also $\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu}) \geqslant \widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, w\boldsymbol{\nu}) - \widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\mu}) \geqslant 2(\widetilde{\mathrm{dist}}(w\boldsymbol{\nu}, \boldsymbol{\nu}) - \epsilon_1) - (2k-1)d \geqslant 2(2kd - 2\epsilon_1) - (2k-1)d = (2k+1)d - 4\epsilon_1 \geqslant (2k+1)d - \epsilon$.

We apply (1.a) to $\boldsymbol{\alpha} = h^m w\boldsymbol{\nu}$ and ϵ and obtain that, for some k, $g^k h^m w\boldsymbol{\nu}$ projects onto $\mathrm{Fix}(h)$ at distance at most ϵ from $\boldsymbol{\nu}$. Take $w' = g^k h^m w$. We have $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w'\boldsymbol{\nu}) = \widetilde{\mathrm{dist}}(\boldsymbol{\mu}, h^m w\boldsymbol{\nu})$, and the latter is in $[(2k+1)d - \epsilon, (2k+1)d]$.

The distance $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w'\boldsymbol{\nu})$ is at most $\widetilde{\mathrm{dist}}(\boldsymbol{\mu}, w'\boldsymbol{\nu}) + d$, hence at most $(2k+2)d$. Also $\widetilde{\mathrm{dist}}(\boldsymbol{\nu}, w'\boldsymbol{\nu})$ is at least $\widetilde{\mathrm{dist}}(w'\boldsymbol{\nu}, h^m w\boldsymbol{\nu}) - \widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\nu}) \geqslant 2(\widetilde{\mathrm{dist}}(h^m w\boldsymbol{\nu}, \boldsymbol{\mu}) - \delta) - 2kd \geqslant 2((2k+1)d - 4\epsilon_1 - \delta) - 2kd = (2k+2)d - 8\epsilon_1 - 2\delta = (2k+2)d - \epsilon$.

(2) Let $\boldsymbol{\Delta}$ be a multicurve fixed by both g and h, and let $\mathbf{U}_1, \dots, \mathbf{U}_n$ be the subsurfaces determined by $\boldsymbol{\Delta}$. The restrictions of g and h to each $\mathbf{U}_i$, $g(i)$ and $h(i)$, do not fix any multicurve. By (1), $g(i)$ and $h(i)$ fix a point $\boldsymbol{\nu}_i$ in $\mathcal{M}(\mathbf{U}_i)$. It then follows that g and h fix the point $(\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n) \in \mathcal{M}(\mathbf{U}_1) \times \dots \times \mathcal{M}(\mathbf{U}_n) = Q(\boldsymbol{\Delta})$. □

We are now ready to prove Proposition 6.8.

Proof of Proposition 6.8. According to Lemma 3.12, it suffices to prove the following statement: if $g_1, \dots, g_m$ are pure elements in $\mathcal{MCG}(S)^{\omega}_{e}$, such that $\langle g_1, \dots, g_m \rangle$ is composed only of pure elements, its orbits in $\mathcal{AM}$ are bounded and it fixes set-wise a piece P, then $g_1, \ldots, g_m$ fix a point in P. We prove this statement by induction on k. For $k = 1$ and $k = 2$, it follows from Lemma 6.23. Note that if an isometry of a tree-graded space fixes a point x and a piece P, then it fixes the projection of x on P.

Assume by induction that the statement is true for k elements, and consider $g_1, \ldots, g_{k+1}$ pure elements in $\mathcal{MCG}(S)^\omega_e$, such that $\langle g_1, \ldots, g_{k+1}\rangle$ is composed only of pure elements, its orbits in $\mathcal{AM}$ are bounded and it fixes set-wise a piece P.

(1) Assume that $g_1, \ldots, g_{k+1}$ do not fix a common multicurve. By the induction hypothesis $g_1, \ldots, g_{k-2}, g_{k-1}, g_k$ fixes a point $\boldsymbol{\alpha} \in P$, $g_1, \ldots, g_{k-2}, g_{k-1}, g_{k+1}$ fixes a point $\boldsymbol{\beta} \in P$, and $g_1, \ldots, g_{k-2}, g_k, g_{k+1}$ fixes a point $\boldsymbol{\gamma} \in P$. If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are not pairwise distinct, then we are done. Assume therefore that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are pairwise distinct, and let $\boldsymbol{\mu}$ be their unique median point. Since pieces are convex in tree-graded spaces, $\boldsymbol{\mu} \in P$. For $i \in \{1, \ldots, k-2\}$, g_i fixes each of the points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, hence it fixes their median point $\boldsymbol{\mu}$.

Assume that $g_{k-1}\boldsymbol{\mu} \neq \boldsymbol{\mu}$. Then $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup \mathfrak{U}(\boldsymbol{\alpha}, g_{k-1}\boldsymbol{\mu}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \cup g_{k-1}$ $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. Now $\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) \subset \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, and since g_{k-1} fixes both $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$, it fixes every subsurface $\mathbf{U} \in \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, by Lemma 6.6. In particular $g_{k-1}\mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha}) = \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. Hence $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. A similar argument implies that $\mathfrak{U}(\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu}) \subset \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\beta})$. Take $\mathbf{V} \in (\boldsymbol{\mu}, g_{k-1}\boldsymbol{\mu})$. Then $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\alpha})$. In particular $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\beta}, \boldsymbol{\alpha})$, hence each g_i, with $i = 1, 2, \ldots, k-1$ fixes $\mathbf{V}$, since it fixes the points $\boldsymbol{\beta}, \boldsymbol{\alpha}$. Also $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\gamma}, \boldsymbol{\alpha})$, whence $g_k\mathbf{V} = \mathbf{V}$. Finally, as $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\mu}, \boldsymbol{\beta}) \subset \mathfrak{U}(\boldsymbol{\gamma}, \boldsymbol{\beta})$, it follows that $g_{k+1}\mathbf{V} = \mathbf{V}$. This contradicts the hypothesis that $g_1, \ldots, g_{k+1}$ do not fix a common multicurve. Note that $\mathbf{V} \subsetneq \mathbf{S}$ by Lemma 6.7, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the same piece and $\mathbf{V} \in \mathfrak{U}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

We conclude that $g_{k-1}\boldsymbol{\mu} = \boldsymbol{\mu}$. Similar arguments imply that $g_k\boldsymbol{\mu} = \boldsymbol{\mu}$ and $g_{k+1}\boldsymbol{\mu} = \boldsymbol{\mu}$.

(2) Assume that $g_1, \ldots, g_{k+1}$ fix a common multicurve. Let $\boldsymbol{\Delta}$ be this multicurve and let $\mathbf{U}_1, \ldots, \mathbf{U}_m$ be the subsurfaces determined by $\boldsymbol{\Delta}$. According to Lemma 6.9, $Q(\boldsymbol{\Delta}) \subset U(P)$.

The restrictions of $g_1, \ldots, g_{k+1}$ to each $\mathbf{U}_i$, $g_1(i), \ldots, g_{k+1}(i)$, do not fix any multicurve. By Lemma 3.12 either $g_1(i), \ldots, g_{k+1}(i)$ fix a point $\boldsymbol{\nu}_i$ in $\mathcal{M}(\mathbf{U}_i)$ or they fix set-wise a piece P_i in $\mathcal{M}(\mathbf{U}_i)$. In the latter case, by (1) we may conclude that $g_1(i), \ldots, g_{k+1}(i)$ fix a point $\boldsymbol{\nu}_i \in P_i$.

It then follows that $g_1, \ldots, g_{k+1}$ fix the point $(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_n) \in \mathcal{M}(\mathbf{U}_1) \times \ldots \times \mathcal{M}(\mathbf{U}_m) = Q(\boldsymbol{\Delta}) \subset U(P)$. □

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