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Addendum

Median structures on asymptotic cones and homomorphisms into mapping class groups

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Jason Behrstock, Cornelia Druţu and Mark Sapir

The goal of this addendum to [**1**] is to show that our methods together with a result of Bestvina, Bromberg and Fujiwara [**3**, Proposition 5.9] yield a proof of the following theorem.

THEOREM 1. *If a finitely presented group* Γ *has infinitely many pairwise non-conjugate homomorphisms into* $\mathcal{MCG}(S)$, *then* Γ *virtually splits* (*virtually acts non-trivially on a simplicial tree*).

This theorem is a particular case of a result announced by Groves.† From private emails received by the authors, it is clear that the methods used by Groves are significantly different. Note that the same new methods allow us to give another proof of the finiteness of the set of homomorphisms from a group with property (T) to a mapping class group [**1**, Theorem 1.2], which is considerably shorter than our original proof; see Corollary 6 and the discussion following it. Theorem 1.2 in [**1**] may equally be obtained from Theorem 1 and the fact that every group with property (T) is a quotient of a finitely presented group with property (T) (see [**11**, Theorem p. 5]).

The property of the mapping class groups given in Theorem 1 can be viewed as another 'rank 1' feature of these groups. In contrast, note that a recent result of [**8**] shows that the rank 2 lattice $\mathrm{SL}_3(\mathbb{Z})$ contains infinitely many pairwise non-conjugate copies of the triangle group $\Delta(3,3,4) = \langle a, b \mid a^3 = b^3 = (ab)^4 = 1\rangle$. Also, as was pointed out to us by Kassabov, although the group $\mathrm{SL}_3(\mathbb{Z}[x])$ has property (T) (see [**12**]), it has infinitely many pairwise non-conjugate homomorphisms into $\mathrm{SL}_3(\mathbb{Z})$ induced by ring homomorphisms $\mathbb{Z}[x] \to \mathbb{Z}$.

The following proposition contains one of the main auxiliary results in [**3**] and the key ingredient missed in our treatment of groups with many homomorphisms into mapping class groups in [**1**].

PROPOSITION 2 (Bestvina, Bromberg and Fujiwara [**3**, Proposition 5.9]). *There exists an explicitly defined finite index torsion-free subgroup* $\mathcal{BBF}(S)$ *of* $\mathcal{MCG}(S)$ *such that the set of all sub-surfaces of* S *can be partitioned into a finite number of subsets* $C_1, C_2, \dots, C_s$, *each of which is an orbit of* $\mathcal{BBF}(S)$, *and any two sub-surfaces in the same subset overlap and have the same complexity.*

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†Groves first announced a version of this result at MSRI in 2007 (see http://www.math.cornell.edu/~vogtmann/MSRI/groves_daniel415.pdf). More recent announcements by Groves have included stronger versions of Theorem 1.

The proof of this important result, explained to us by Bestvina, is surprisingly simple: the subgroup $\mathcal{BBF}(S)$ is the subgroup of mapping classes from $\mathcal{MCG}(S)$ acting as an identity on the factor $\pi_1(S)/B$ over certain characteristic subgroup B of $\pi_1(S)$ of finite index which is explicitly constructed.

We consider the set of colors $K = \{1, 2, \ldots, s\}$, and we color each sub-surface of S contained in the subset C_i by i. Note that the whole surface S has a color which is different from that of any proper sub-surface.

Recall that for every sequence of sub-surfaces $\mathbf{U}$ from $\Pi\mathcal{U}/\omega$, we defined an $\mathbb{R}$-tree $T_{\mathbf{U}}$ (see [**1**, Notation 4.4]) and that there is an equivariant bi-Lipschitz embedding ψ of $\mathcal{AM}$ into $\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega} T_{\mathbf{U}}$ (see [**1**, Corollary 4.17]). Let C_k be the set of all sub-surfaces of S with the given color $k \in K$. Let π_k be the projection of $\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega} T_{\mathbf{U}}$ onto $\prod_{\mathbf{U}\in\Pi C_k/\omega} T_{\mathbf{U}}$.

REMARK 3. By [**1**, Lemma 2.1], we have that $\prod_{\mathbf{U}\in\Pi\mathcal{U}/\omega} T_{\mathbf{U}}$ can be written as

$$\prod_{k\in K}\prod_{\mathbf{U}\in\Pi C_k/\omega} T_{\mathbf{U}}.$$

In what follows, we use the notion of tree-graded space introduced in [**6**].

THEOREM 4. *Consider an arbitrary color $k \in K$ and the image $T_k = \pi_k\psi(\mathcal{AM})$.*

For every sub-surface $\mathbf{U} \in \Pi C_k/\omega$, consider the tree $T'_{\mathbf{U}} = T_{\mathbf{U}} \times \prod_{\mathbf{V}\in\Pi C_k/\omega\setminus\{\mathbf{U}\}}\{a_{\mathbf{V}}\}$, where $a_{\mathbf{V}}$ is the point in $T_{\mathbf{V}}$ that is the projection of $\partial\mathbf{U}$ to $T_{\mathbf{V}}$.

The space T_k is tree-graded with respect to $T'_{\mathbf{U}}$ and with transversal trees reduced to singletons. In particular, it is an $\mathbb{R}$-tree.

Proof. *Step* 1. We prove by induction on n that, for any finite subset $F \subset \Pi C_k/\omega$ of cardinality n, the projection $\pi_F(\mathcal{AM})$ of $\mathcal{AM}$ onto the finite product $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$ is an $\mathbb{R}$-tree. The case $n = 1$ is obvious, the case $n = 2$ follows from [**1**, Theorem 4.21, (2)], since the sub-surfaces in F pairwise overlap. Assume that the statement is proved for n and consider $F \subset \Pi C_k/\omega$, F of cardinality $n + 1$.

Both $\psi(\mathcal{AM})$ and its projections are geodesic spaces. For $\psi(\mathcal{AM})$ this follows from Proposition 4.18, while for projections it follows from the fact that the distance is ℓ^1. To prove that $\pi_F(\psi(\mathcal{AM}))$ is a real tree, it suffices therefore to prove that it is 0-hyperbolic, that is, for every geodesic triangle its three edges have a common point. By Lemma 4.30, the subset $\pi_F(\psi(\mathcal{AM}))$ is a median, thus it suffices to prove that, for an arbitrary triple of points $\boldsymbol{\nu}, \boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ in $\pi_F(\psi(\mathcal{AM}))$ and every geodesic $\mathfrak{g}$ joining $\boldsymbol{\nu}, \boldsymbol{\rho}$ in $\pi_F(\psi(\mathcal{AM}))$, the median point $\boldsymbol{\mu}$ of the triple is on $\mathfrak{g}$.

Assume that there exist $\mathbf{U}$ and $\mathbf{V}$ such that the projection of $\boldsymbol{\mu}$ on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ is not (v, u). Assume that it is (x, u), with $x \neq v$ (the other case is similar).

Consider the projection on the product $\prod_{\mathbf{Y}\in F\setminus\{\mathbf{V}\}} T_{\mathbf{Y}}$. By the inductive hypothesis, $\pi_F(\psi(\mathcal{AM}))$ projects onto a real tree, in particular, there exists $\boldsymbol{\mu}'$ on $\mathfrak{g}$ such that its projection on $\prod_{\mathbf{Y}\in F\setminus\{\mathbf{V}\}} T_{\mathbf{Y}}$ coincides with that of $\boldsymbol{\mu}$. In particular, $\pi_{\mathbf{U}}(\mu') = \pi_{\mathbf{U}}(\mu) = x$. This implies that the projection on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ of both $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}$ is (x, v) (the unique point with first coordinate x). This implies that all coordinates of $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}$ are equal, thus the two points coincide.

Assume now that, for every pair $\mathbf{U}$ and $\mathbf{V}$ in F, the projection of $\boldsymbol{\mu}$ on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ is (v, u). Fix such a pair. By the inductive hypothesis and an argument as above, there exists $\boldsymbol{\mu}_1 \in \mathfrak{g}$ such that its projection on $\prod_{\mathbf{Y}\in F\setminus\{\mathbf{U}\}} T_{\mathbf{Y}}$ coincides with that of $\boldsymbol{\mu}$. Similarly, there exists $\boldsymbol{\mu}_2 \in \mathfrak{g}$ such that its projection on $\prod_{\mathbf{Y}\in F\setminus\{\mathbf{V}\}} T_{\mathbf{Y}}$ coincides with that of $\boldsymbol{\mu}$. Then on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ the point $\boldsymbol{\mu}_1$ projects onto some (x, u) and $\boldsymbol{\mu}_2$ projects onto some (v, y). This implies that there exists some $\boldsymbol{\mu}'$ on $\mathfrak{g}$ between $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ projecting on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ in (v, u). Note that, for every $\mathbf{Y} \in F \setminus \{\mathbf{U}, \mathbf{V}\}$, the projection of $\boldsymbol{\mu}'$ coincides with that of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, hence with that of $\boldsymbol{\mu}$. It follows that $\boldsymbol{\mu}' = \boldsymbol{\mu}$.

We now prove by induction on n that, for any finite $F \subset \Pi C_k/\omega$ of cardinality n, the projection $\pi_F(\mathcal{AM})$ of $\mathcal{AM}$ onto the finite product $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$ is tree-graded with respect to the trees $T_{\mathbf{U}}^F = T_{\mathbf{U}} \times \prod_{\mathbf{V}\in F\setminus\{\mathbf{U}\}}\{a_{\mathbf{V}}\}$, where $a_{\mathbf{V}}$ is the projection of $\partial\mathbf{U}$ to $T_{\mathbf{V}}$. It only remains to prove that $\pi_F(\mathcal{AM})$ is complete and that it is covered by $T_{\mathbf{U}}^F$. Both statements are proved simultaneously when proving that $\pi_F(\mathcal{AM})$ equals the union $\bigcup_{\mathbf{U}\in F} T_{\mathbf{U}}^F$. Clearly, the union is contained in $\pi_F(\mathcal{AM})$. Conversely, consider a point $x = (x_1, \ldots, x_{n+1})$ in $\pi_F(\mathcal{AM})$. The inductive hypothesis applied to $(x_1, \ldots, x_n)$ and $(x_2, \ldots, x_{n+1})$ implies that, for each n-tuple, there exists $\mathbf{U} \in F$ such that, for every $\mathbf{V} \neq \mathbf{U}$, the corresponding coordinate is $\pi_{\mathbf{V}}(\mathbf{U})$, that is, the point in $T_{\mathbf{V}}$ which is the projection of $\partial\mathbf{U}$ to $T_{\mathbf{V}}$. Assume that in $(x_1, \ldots, x_n)$, the surface $\mathbf{U}$ corresponds to the first coordinate, and that in $(x_2, \ldots, x_{n+1})$ the surface $\mathbf{U}'$ corresponds to the last coordinate. The projection (x_1, x_{n+1}) of x on $T_{\mathbf{U}} \times T_{\mathbf{U}'}$ is either of the form $(\pi_{\mathbf{U}}(\mathbf{U}'), x_{n+1})$ or $(x_1, \pi_{\mathbf{U}'}(\mathbf{U}))$. In the first case, x is in $\prod_{\mathbf{V}\in F\setminus\{\mathbf{U}'\}}\{\pi_{\mathbf{V}}(\mathbf{U}')\} \times T_{\mathbf{U}'}$ and in the second case, x is in $T_{\mathbf{U}} \times \prod_{\mathbf{V}\in F\setminus\{\mathbf{U}\}}\{\pi_{\mathbf{V}}(\mathbf{U})\}$.

Step 2. We now prove the statements on T_k. First, we prove that T_k is a real tree, using an approximation argument similar to that in the proof that $\mathcal{AM}$ is a median space [**1**, Theorem 4.25]. Since T_k is a complete geodesic space, it suffices to prove that it is zero hyperbolic. Thus, it suffices to prove that, for every triple $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ and $\boldsymbol{\mu}$ being its median point, $\boldsymbol{\mu}$ is on any geodesic $\mathfrak{g}$ joining $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\psi(\mathcal{AM})$.

Assume that the distance from $\boldsymbol{\mu}$ to $\mathfrak{g}$ is $\varepsilon > 0$. Take a finite set of surfaces F such that the projections of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\mu}$ in $\prod_{\mathbf{U}\notin F} T_{\mathbf{U}}$ compose a set of diameter $\varepsilon/4$. Since the projection on the Cartesian product $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$ is a tree, the projection of $\mathfrak{g}$ contains that of $\boldsymbol{\mu}$, hence there exists $\boldsymbol{\mu}'$ on $\mathfrak{g}$ with the same projection as $\boldsymbol{\mu}$ in $\prod_{\mathbf{U}\in F} T_{\mathbf{U}}$.

Then the distance from $\boldsymbol{\mu}'$ to $\boldsymbol{\mu}$ is

$$\sum_{\mathbf{U}\notin F} \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}', \boldsymbol{\mu}) \leqslant \sum_{\mathbf{U}\notin F} [\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}', \boldsymbol{\alpha}) + \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, \boldsymbol{\mu})] \leqslant \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

The tree T_k is complete. Consider two points $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathcal{AM}$. There exists a countable family $\mathcal{C} \subset \Pi C_k/\omega$ equal to the set of sub-surfaces $\{\mathbf{U}; \mathrm{dist}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu}) > 0\}$. Let $\mathfrak{h}$ be a hierarchical path joining $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be the endpoints of a minimal sub-arc $\mathfrak{h}_{\mathbf{U}}$ on $\mathfrak{h}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Assume that there exists $\mathbf{V} \neq \mathbf{U}, \mathbf{V} \in \mathcal{C}$ such that $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) > 0$. Then, by projecting $\mathfrak{h}$ on $T_{\mathbf{U}} \times T_{\mathbf{V}}$ and using the tree-graded structure of the projection of $\mathcal{AM}$, we obtain that the arc $\mathfrak{h}_{\mathbf{U}}$ has a strict sub-arc of endpoints $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ such that $\widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\alpha}', \boldsymbol{\beta}') = \widetilde{\mathrm{dist}}_{\mathbf{U}}(\boldsymbol{\mu}, \boldsymbol{\nu})$. This contradicts the minimality of $\mathfrak{h}_{\mathbf{U}}$. It follows that, for every $\mathbf{V} \neq \mathbf{U}, \mathbf{V} \in \mathcal{C}$, $\widetilde{\mathrm{dist}}_{\mathbf{V}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0$. Hence, $\mathfrak{h}_{\mathbf{U}}$ is entirely contained in a factor $T_{\mathbf{U}} \times \prod_{\mathbf{V}\in\mathcal{C},\mathbf{V}\neq\mathbf{U}}\{a_{\mathbf{V}}\}$. Since, given any sub-surface $\mathbf{V} \neq \mathbf{U}$, the arc $\mathfrak{h}_{\mathbf{U}}$ contains points with a first coordinate distinct from the projection of $\mathbf{V}$ on $T_{\mathbf{U}}$, it follows that $a_{\mathbf{V}}$ is the projection of $\mathbf{U}$ on $T_{\mathbf{V}}$. Hence, $\mathfrak{h}_{\mathbf{U}}$ is contained in the tree $T'_{\mathbf{U}}$, and the arcs $\mathfrak{h}_{\mathbf{U}}$ with $\mathbf{U} \in \mathcal{C}$ cover $\mathfrak{h}$ up to a subset of zero measure. □

As an immediate consequence of Theorem 4, we obtain the following, which also immediately follows from the main result of [**3**].

Corollary 5. *There exists an equivariant embedding of $\mathcal{AM}$ into a finite product of $\mathbb{R}$-trees.*

Now, let Γ have infinitely many pairwise non-conjugate homomorphisms into $\mathcal{MCG}(S)$. Theorem 4 and Proposition 2 imply that Γ has a finite index subgroup Γ' that acts on the

$\mathbb{R}$-trees T_k for each $k \in K$; further, since the global action is non-trivial (that is, without a global fixed point) at least one of the actions on a factor tree is non-trivial.

Corollary 5 and the standard argument of Bestvina [**2**] and Paulin [**10**] imply the following corollary.

COROLLARY 6. *If a finitely generated group Λ has infinitely many pairwise non-conjugate homomorphisms into the group $\mathcal{MCG}(S)$, then Λ has a subgroup of index at most $|K|$ which is not an $F\mathbb{R}$-group (that is, it acts non-trivially on an $\mathbb{R}$-tree).*

Since in a group with property (T) every subgroup of finite index has property $F\mathbb{R}$ (see [**10**]), Corollary 6.3 of [**1**] follows from Corollary 6.

It is still unknown if every finitely generated group acting non-trivially on an $\mathbb{R}$-tree also acts non-trivially on a simplicial tree. In order to obtain such an action in our case, we apply the theorem of Bestvina and Feighn below.

DEFINITION 7. Given an action of a group on an $\mathbb{R}$-tree, an arc $\mathfrak{g}_0$ is called *stable* if the stabilizer of every non-trivial sub-arc of $\mathfrak{g}_0$ is the same as the stabilizer of $\mathfrak{g}_0$.

The action is called *stable* if every arc $\mathfrak{g}$ contains a non-trivial stable sub-arc $\mathfrak{g}_0$.

THEOREM 8 (Bestvina–Feighn [**4**, Theorem 9.5]). *Let G be a finitely presented group with a non-trivial, minimal and stable action on an $\mathbb{R}$-tree T. Then either (1) G splits over an extension E-by-cyclic subgroup where E is the stabilizer of a non-trivial arc of T, or (2) T is a line. In the second case, G has a subgroup, of index at most 2 that is the extension of the kernel of the action by a finitely generated free abelian group.*

In order to show the stability of the action, as in [**7**], we describe stabilizers of pairs of points and of tripods in T_k.

Lemmas 9 and 11 describing stabilizers of arcs and tripods have similar proofs as Lemmas 5.14 and 5.15 in the main text.

LEMMA 9. *There exists a constant $N = N(S)$, such that if $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are distinct points in $\mathcal{AM}$ that are not in the same piece, then the stabilizer $\mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$ is the extension of a finite subgroup of cardinality at most N by an abelian group.*

Proof. By hypothesis, for every representatives (μ_n) and (ν_n) of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively, the following is satisfied:

$$\lim_{\omega} \mathrm{dist}_{\mathcal{C}(S)}(\mu_n, \nu_n) = \infty. \tag{1}$$

Let $\boldsymbol{g} = (g_n)^\omega$ be an element in $\mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$. Then

$$\delta_n(\boldsymbol{g}) = \max(\mathrm{dist}(\mu_n, g_n\mu_n), \mathrm{dist}(\nu_n, g_n\nu_n))$$

satisfies $\delta_n(\boldsymbol{g}) = o(d_n)$. Let $\mathfrak{q}_n$ be a hierarchy path joining μ_n and ν_n and let $\bar{\mu}_n$ and $\bar{\nu}_n$ be points on $\mathfrak{q}_n$ at distance εd_n from μ_n and ν_n, respectively. By hypothesis, for ε small enough $\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(\bar{\mu}_n, \bar{\nu}_n) = \infty$. Thus, there exist $\tilde{\mu}_n$ and $\tilde{\nu}_n$ on $\mathfrak{q}_n$ between $\bar{\mu}_n$ and $\bar{\nu}_n$ and at respective $\mathcal{C}(S)$-distance 3 from them. Denote by $\mathfrak{q}_n'$ the sub-arc of $\mathfrak{q}_n$ between $\tilde{\mu}_n$ and $\tilde{\nu}_n$.

Divide $\mathfrak{q}'_n$ into three consecutive sub-arcs that shadow geodesics in $\mathcal{C}(S)$ of equal length $\mathrm{dist}_{\mathcal{C}(S)}(\tilde{\mu}_n, \tilde{\nu}_n)/3$. Let us show that there exists a point $x = (x_n)^\omega$ on the first part and a point $y = (y_n)^\omega$ on the third part which are at distance $O(1)$ from $g\mathfrak{p}'$ (the points do not depend on g).

All large domains on $\mathfrak{q}'_n$ are ω-almost surely large domains for $g_n\mathfrak{q}_n$. Suppose that the whole surface S is the only large domain of a part $\mathfrak{p}_n$ of $\mathfrak{q}'_n$ of size $O(d_n)$. Then we can take a projection of $\mathfrak{g}_n$ and $g_n\mathfrak{g}_n$ to the curve complex $\mathcal{C}(S)$ and deduce from the hyperbolicity of $\mathcal{C}(S)$ that the geodesics $\mathfrak{p}'_n$ and $g_n\mathfrak{p}'_n$ are at $\mathcal{C}(S)$-distance $O(1)$ ω-almost surely (a.s.). Thus, in that case we can take points $(x_n)^\omega$ and $(y_n)^\omega$ arbitrarily.

Suppose that such a large domain in $\mathfrak{p}_n$ cannot be found ω-a.s. Note that the distance between the entry points of $\mathfrak{g}'_n$ and $g_n\mathfrak{g}'_n$ into large domains $S' \subset S$ are at $\mathcal{C}(S)$-distance 1. Thus, in this case, we can take $(x_n)^\omega$ and $(y_n)^\omega$ to be the entrance points of the geodesic into large domains.

Obviously, $\lim_\omega \mathrm{dist}_{\mathcal{C}(S)}(x_n, y_n) = \infty$.

For every $\boldsymbol{g} = (g_n)^\omega \in \mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$, we define a sequence of translation numbers. Since x_n is ω-almost surely at distance $O(1)$ from a point x'_n on $g_n q_n$, define $\ell_x(g_n)$ as $(-1)^\epsilon \mathrm{dist}_{\mathcal{C}(S)}(x_n, g_n x_n)$, where $\epsilon = 0$, if x'_n is nearer to $g_n\mu_n$ than $g_n x_n$ and $\epsilon = 1$, otherwise.

Let $\ell_x\colon \mathrm{stab}(\mu, \nu) \to \Pi\mathbb{R}/\omega$ be defined by $\ell_x(\boldsymbol{g}) = (\ell_x(g_n))^\omega$. It is easy to see that ℓ_x is a quasi-morphism, that is,

$$|\ell_x(\boldsymbol{gh}) - \ell_x(\boldsymbol{g}) - \ell_x(\boldsymbol{h})| \leqslant_\omega O(1). \tag{2}$$

It follows that $|\ell_x([\boldsymbol{g}, \boldsymbol{h}])| \leqslant_\omega O(1)$.

The above and a similar argument for y_n imply that, for every commutator, $\boldsymbol{c} = \lim_\omega (c_n)$, in the stabilizer of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, $\mathrm{dist}_{\mathcal{C}(S)}(x_n, c_n x_n)$ and $\mathrm{dist}_{\mathcal{C}(S)}(y_n, c_n y_n)$ are at most $O(1)$. Bowditch's acylindricity result [**5**, Theorem 1.3; Lemma 2.1] imply that the set of commutators of $\mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$ has uniformly bounded cardinality, say, N. Then any finitely generated subgroup G of $\mathrm{stab}(\boldsymbol{\mu}, \boldsymbol{\nu})$ has conjugacy classes of cardinality at most N, that is, G is an FC-group [**9**]. By [**9**], the set of all torsion elements of G is finite, and the derived subgroup of G is finite of cardinality $\leqslant N(S)$ (by Lemma 2.13). □

LEMMA 10 [**7**, Lemma 2.20, (2)]. *Let $\mathbb{F}$ be a tree-graded space. For every non-trivial geodesic $\mathfrak{g}$ in the tree obtained by collapsing non-trivial pieces, $T = \mathbb{F}/\approx$, there exists a non-trivial geodesic $\mathfrak{p}$ in $\mathbb{F}$ such that its projection on T is $\mathfrak{g}$, and such that given an isometry ϕ of $\mathbb{F}$ permuting the pieces, the isometry $\tilde{\phi}$ of T induced by ϕ fixes $\mathfrak{g}$ pointwise if and only if ϕ fixes the set of cut-points* Cutp $(\mathfrak{p})$ *pointwise.*

The quotient tree $\mathcal{AM}/\approx$ is described in [**1**, Lemma 3.8].

LEMMA 11. *Let $\widetilde{\boldsymbol{\mu}}_1, \widetilde{\boldsymbol{\mu}}_2$ and $\widetilde{\boldsymbol{\mu}}_3$ be three points in the quotient tree $\mathcal{AM}/\approx$ which form a non-trivial tripod. Then the stabilizer* $\mathrm{stab}(\widetilde{\boldsymbol{\mu}}_1, \widetilde{\boldsymbol{\mu}}_2, \widetilde{\boldsymbol{\mu}}_3)$ *in $\mathcal{MCG}(S)^\omega_b$ is a finite subgroup of cardinality at most $N = N(S)$.*

Proof. For every $i \in \{1,2,3\}$ let $\mathfrak{g}_i$ denote the geodesic joining $\widetilde{\boldsymbol{\mu}}_j$ and $\widetilde{\boldsymbol{\mu}}_k$ in T, where $\{i,j,k\} = \{1,2,3\}$, and let $\mathfrak{p}_i$ denote a geodesic in $\mathcal{AM}$ associated to $\mathfrak{g}_i$ by Lemma 10. By eventually replacing the endpoints of $\mathfrak{p}_i$ with cut-points in their interiors, we may assume that the three geodesics $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ compose a triangle in $\mathcal{AM}$ of vertices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Note that the elements in $\mathrm{stab}(\widetilde{\boldsymbol{\mu}}_1, \widetilde{\boldsymbol{\mu}}_2, \widetilde{\boldsymbol{\mu}}_3)$ fix pointwise all the cut-points of all the geodesics $\mathfrak{p}_i$. Since the set of cut-points does not change, we may replace the three geodesics by three paths $\mathfrak{h}^i$, each of which is a ultralimit of a sequence of hierarchy paths, $\lim_\omega \left(\mathfrak{h}^i_n\right)$, with the property that the endpoints of $\mathfrak{h}^1_n, \mathfrak{h}^2_n$ and $\mathfrak{h}^3_n$ are in the set of vertices of a triangle, μ^1_n, μ^2_n and μ^3_n. Each $\mathfrak{h}^i_n$

projects onto a geodesic γ_n^i in the curve complex $\mathcal{C}(S)$, and according to [1, Lemma 4.15], we also have $\lim_\omega \left(\text{length}(\gamma_n^i)\right) = \infty$.

By hyperbolicity of $\mathcal{C}(S)$, for every $a > 0$ there exists $b > 0$ such that, for any triple of points $x, y, z \in \mathcal{C}(S)$, the intersection of the three a-tubular neighborhoods of geodesics $[x, y]$, $[y, z]$ and $[z, x]$ is a set $C_a(x, y, z)$ of diameter at most b. In particular, for every n, the three a-tubular neighborhoods of the geodesics γ_n^1, γ_n^2 and γ_n^3 intersect in a set C_n of diameter at most b. Fix an $\epsilon > 0$ and consider a sub-path $\mathfrak{k}_n^1$ of $\mathfrak{h}_n^1$ such that the limit path $\mathfrak{k}_1 = \lim_\omega \left(\mathfrak{k}_n^1\right)$ has endpoints at $\widetilde{\text{dist}}$-distance ϵ and 2ϵ from $\boldsymbol{\mu}_2$. Consider a (sufficiently) large proper domain Y_n^2 for $\mathfrak{k}_n^1$. If no proper large domain exists for $\mathfrak{k}_n^1$ (that is, the only large domain for this hierarchy path is S), then pick instead a marking ρ_n^1 on $\mathfrak{k}_n^1$. Since we started with a non-trivial tripod, for ϵ small enough the sub-arc $\mathfrak{k}_1$ is at positive $\widetilde{\text{dist}}$-distance from $\mathfrak{h}_2$, hence Y_n^2 is ω-almost surely not a large domain of $\mathfrak{h}_n^2$ (or, in the second case, ρ_n^1 is not at uniformly bounded $\mathcal{C}(S)$-distance from $\mathfrak{h}_n^2$). Therefore, Y_n^2 is a large domain of $\mathfrak{h}_n^3$ (respectively, ρ_n^1 is at $\mathcal{C}(S)$-distance $O(1)$ from $\mathfrak{h}_n^3$). Let $g = (g_n)^\omega$ be an element of $\text{stab}(\widetilde{\boldsymbol{\mu}}_1, \widetilde{\boldsymbol{\mu}}_2, \widetilde{\boldsymbol{\mu}}_3)$. Consider any geodesic quadrangle with two of the opposite edges being $\mathfrak{h}_n^1$ and $g_n\mathfrak{h}_n^1$. Since $\mathfrak{k}_1$ is at positive $\widetilde{\text{dist}}$-distance both from $\boldsymbol{\mu}_2$ and from $\boldsymbol{\mu}_3$, the domain Y_n^2 (or the marking ρ_n^1) cannot be at uniformly bounded $C(S)$-distance from the edges $[\mu_n^2, g_n\mu_n^2]$ and $[\mu_n^3, g_n\mu_n^3]$ of the quadrangle. Thus, Y_n^2 can only be a large domain of $\mathfrak{h}_n^1$ and $g_n\mathfrak{h}_n^1$ (respectively, only these two edges contain points at $\mathcal{C}(S)$-distance $O(1)$ from ρ_n^1). A similar argument shows that Y_n^2 is a large domain of (or ρ_n^1 is at $\mathcal{C}(S)$-distance $O(1)$ from) $g_n\mathfrak{h}_n^3$.

In a similar manner, we take a sub-path $\mathfrak{k}_n^2$ of $\mathfrak{h}_n^2$ such that the limit path $\mathfrak{k}_2 = \lim_\omega \left(\mathfrak{k}_n^2\right)$ has endpoints at $\widetilde{\text{dist}}$-distance ϵ and 2ϵ from $\boldsymbol{\mu}_1$; we fix Y_n^1 a proper large domain for $\mathfrak{k}_n^2$ (or a marking ρ_n^1 on $\mathfrak{k}_n^2$ if no such domain exists). Then we show that Y_n^1 is also a large domain for $\mathfrak{h}_n^3$, $g_n\mathfrak{h}_n^2$ and $g_n\mathfrak{h}_n^3$ (respectively, ρ_n^1 is at $\mathcal{C}(S)$-distance $O(1)$ from these paths). Likewise, we find a large domain Y_n^3 for $\mathfrak{h}_n^1$ and $\mathfrak{h}_n^2$ and their translations by g_n (or a marking ρ_n^3 at $C(S)$-distance $O(1)$ from all these paths).

Let $\widehat{\mathfrak{h}}_n^1$ be the sub-arc of $\mathfrak{h}_n^1$ between the sub-arcs corresponding to the domains Y_n^2 and Y_n^3 (respectively, the sub-arc between the markings ρ_n^2 and ρ_n^3), and $\widehat{\gamma}_n^1$ its projection into the complex of curves. Note that $\widehat{\gamma}_n^1$ is a sub-arc of γ_n^1. Likewise consider $\widehat{\mathfrak{h}}_n^i$ and $\widehat{\gamma}_n^i$ for $i = 2, 3$. The set C_n equals also the intersection of the three a-tubular neighborhoods of the geodesics $\widehat{\gamma}_n^1, \widehat{\gamma}_n^2$ and $\widehat{\gamma}_n^3$. Indeed, it clearly contains this intersection. On the other hand, the existence of a point in C_n not in the intersection would imply, up to reindexing, the existence of a point in $\gamma_n^1 \setminus \widehat{\gamma}_n^1$ at finite $\mathcal{C}(S)$-distance from both γ_n^2 and γ_n^3. All elements in $\gamma_n^1 \setminus \widehat{\gamma}_n^1$ are projections in $\mathcal{C}(S)$ of sub-arcs of $\mathfrak{h}_n^1$ with limits at $\widetilde{\text{dist}}$-distance at most 2ϵ from either $\boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_3$. For ϵ small enough these limits are therefore at positive $\widetilde{\text{dist}}$-distance from either $\mathfrak{h}_2$ or $\mathfrak{h}_3$, hence the ultralimit of the $\mathcal{C}(S)$-distance of the corresponding sequence of sub-arcs of $\mathfrak{h}_n^1$ either to $\mathfrak{h}_n^2$ or to $\mathfrak{h}_n^3$ is ∞.

The translation g_nC_n is the intersection of the three a-tubular neighborhoods of the geodesics $g_n\widehat{\gamma}_n^1, g_n\widehat{\gamma}_n^2$ and $g_n\widehat{\gamma}_n^3$. For every i, on the path $g_n\mathfrak{h}_n^i$ the two large domains Y_n^j and $g_nY_n^j$ occur such that the corresponding sub-arcs have limits at $\widetilde{\text{dist}}$ zero. Then with an argument as above, it can be proved that g_nC_n is also the intersection of three a-tubular neighborhoods of geodesics of $\mathcal{C}(S)$ joining the projections of Y_n^1, Y_n^2 and Y_n^3. It follows that C_n and g_nC_n are at the Hausdorff distance at most $D = D(S)$.

According to the above, there exist λ_n satisfying $\lim_\omega (\lambda_n) = \infty$ and points α_n on γ_n^1 at distance at least $2\lambda_n$ from the projections of the domains $Y_n^2, Y_n^3, g_nY_n^2$ and $g_nY_n^3$ and such that $g_n\alpha_n$ is at distance $O(1)$ from α_n. We pick β_n on γ_n^1 at distance λ_n from α_n. Then $g_n\beta_n$ is on $g_n\gamma_n^1$ at distance λ_n from $g_n\alpha_n$.

Since β_n is on a geodesic between α_n and the projection of Y_n^2, say, and both endpoints are at distance $O(1)$ from $g_n\gamma_n^1$, it follows that there exists β_n' on $g_n\gamma_n$ at distance $O(1)$ from β_n. It follows that β_n' is at distance $\lambda_n + O(1)$ from $g_n\alpha_n$, hence it is at distance $O(1)$ from $g_n\beta_n$. We have thus obtained α_n and β_n at $\mathcal{C}(S)$-distance λ_n such that $g_n\alpha_n$ is at $\mathcal{C}(S)$-distance $O(1)$ from α_n, and $g_n\beta_n$ is at $\mathcal{C}(S)$-distance $O(1)$ from β_n. It now follows from Bowditch's acylindricity result [**1**, Lemma 2.1; **5**, Theorem 1.3] that $\mathrm{stab}(\widetilde{\boldsymbol{\mu}}_1, \widetilde{\boldsymbol{\mu}}_2, \widetilde{\boldsymbol{\mu}}_3)$ has uniformly bounded cardinality. □

LEMMA 12. *Let $\mathcal{BBF}(S)^\omega_b$ be the subset in $\Pi_b\mathcal{MCG}(S)/\omega$ composed of elements $(x_i)^\omega$ with $x_i \in \mathcal{BBF}(S)$ ω-almost surely. Then $\mathcal{BBF}(S)^\omega_b$ is a torsion-free subgroup of index $|\mathcal{MCG}(S)/\mathcal{BBF}(S)|$ in $\Pi_b\mathcal{MCG}(S)/\omega$. Moreover, $\mathcal{BBF}(S)^\omega_b$ acts on each T_k faithfully.*

Proof. Only the last statement requires a proof. An element $g_\omega = (g_n)^\omega$ in $\Pi_b\mathcal{MCG}(S)/\omega$, which acts by fixing T_k pointwise must fix pointwise $T_{\mathbf{U}}$ for each $\mathbf{U} \in \Pi C_k/\omega$. In particular, for each $U \in C_k$ the mapping class g_n fixes ω-almost surely its boundary ∂U. Each C_k contains a pair of sub-surfaces whose boundaries fill the surface, and the only mapping classes which fix a pair of filling curves are those of finite order (uniformly bounded by the complexity of S). Hence, only finite-order elements of $\Pi_b\mathcal{MCG}(S)/\omega$ can be in the kernel of the homomorphism $\mathcal{BBF}(S)^\omega_b \to \mathrm{Isom}(T_k)$. Since $\mathcal{BBF}(S)^\omega_b$ is torsion-free, the proof is complete. □

COROLLARY 13. *A finitely generated $F\mathbb{R}$ group Λ cannot have infinitely many pairwise non-conjugate homomorphisms into the group $\mathcal{BBF}(S)$.*

Let $\mathbf{U} = (U_i)^\omega$ be an element of $\Pi C_k/\omega$ and let $T'_{\mathbf{U}}$ be the corresponding sub-tree in T_k.

LEMMA 14. (1) *The stabilizer in $\mathcal{BBF}(S)^\omega_b$ of a non-trivial arc in $T'_{\mathbf{U}}$ has a homomorphism onto a (finite of cardinality at most $N = N(S)$)-by-abelian subgroup A of $\Pi_b\mathcal{MCG}(U_i)/\omega$. The kernel W of that homomorphism acts identically on $T'_{\mathbf{U}}$.*

(2) *The stabilizer in $\mathcal{BBF}(S)^\omega_b$ of a non-trivial tripod in $T'_{\mathbf{U}}$ has a homomorphism onto a finite of cardinality at most $N = N(S)$ subgroup of $\Pi_b\mathcal{MCG}(U_i)/\omega$; the kernel of that homomorphism is W.*

Proof. Let g be an element in $\mathcal{BBF}(S)^\omega_b$ stabilizing a non-trivial arc $\mathfrak{h}$ in $T'_{\mathbf{U}}$. Then g stabilizes $\mathbf{U}$. Indeed, we have $gT'_{\mathbf{U}} = T'_{g\mathbf{U}}$. If $g\mathbf{U} \neq \mathbf{U}$, then $T'_{\mathbf{U}}$ and $T'_{g\mathbf{U}}$ intersect at more than one point (since they both contain $\mathfrak{h}$), which is impossible since these trees are the pieces in a tree-graded structure. Therefore, the stabilizer of $\mathfrak{h}$ in $\mathcal{BBF}(S)^\omega_b$ must stabilize $\mathbf{U}$. Hence, there exists a homomorphism from that stabilizer to $\mathcal{MCG}_b(\mathbf{U})$ whose kernel fixes $T'_{\mathbf{U}}$ pointwise. By Lemma 9 the image A of that homomorphism is (finite of cardinality at most $N = N(S)$)-by-abelian.

If instead of the stabilizer of an arc in $T'_{\mathbf{U}}$ we consider the stabilizer of a tripod, then the argument is similar, except that we use Lemma 11 instead of 9. □

LEMMA 15. *Let Λ be a finitely generated group with infinitely many pairwise non-conjugate homomorphisms into $\mathcal{MCG}(S)$. Then Λ contains a subgroup Λ' of index at most $|K|$ which acts on each of the limit trees T_k. Moreover, each of the actions of Λ' on T_k is stable.*

Proof. That Λ contains a subgroup Λ' of index at most $|K|$ which acts on each of the trees T_k follows immediately from Corollary 5. We now prove that these actions are stable.

By Theorem 4, the tree T_k is a tree-graded space with pieces of the trees $T'_{\mathbf{U}}$ and with all the transversal trees consisting of singletons. Hence, every geodesic $\mathfrak{g}$ in T_k is covered, up to a subset of measure zero, by (countably many) non-trivial arcs in trees $T'_{\mathbf{U}}$.

Consider an arbitrary $\mathbf{U} \in \Pi\mathcal{U}/\omega$ and the intersection of Λ' with the stabilizer of $T'_{\mathbf{U}}$ in $\mathcal{MCG}_b^\omega(S)$, denoted by $\Lambda_{\mathbf{U}}$. In view of Lemma 14, in order to prove stability, it suffices to prove that stabilizers in $\Lambda_{\mathbf{U}}$ of non-trivial arcs in $T'_{\mathbf{U}}$ satisfy the ascending chain condition. Consider the homomorphism $\pi\colon \Lambda_{\mathbf{U}} \to \Pi_b\mathcal{MCG}(U_i)/\omega$ defined in Lemma l4. The stabilizer in $\Lambda_{\mathbf{U}}$ of a non-trivial arc $\mathfrak{h}$ in $T'_{\mathbf{U}}$ is the inverse image by π of the stabilizer of $\mathfrak{h}$ in $\pi(\Lambda_{\mathbf{U}})$. Thus, it is enough to prove that stabilizers of arcs in $\pi(\Lambda_{\mathbf{U}})$ satisfy the ascending chain condition. According to Lemma 14, the stabilizers of arcs in $\pi(\Lambda_{\mathbf{U}})$ are (finite of cardinality at most $N(S)$)-by-abelian, and stabilizers of tripods are finite of cardinality at most $N(S)$. According to [**7**, Lemma 2.35] an arc with stabilizer in $\pi(\Lambda_{\mathbf{U}})$ of order larger than $(N + 1)!$ is stable. (Note that the hypothesis in Lemma 2.35 that the group acting be finitely generated is not needed in the proof.) The ascending chain condition is obviously satisfied on the set of stabilizers of sub-arcs of order at most $(N + 1)!$. □

Now Theorem 1 follows from Theorem 8 and Lemma 15.

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Jason Behrstock
Lehman College
City University of New York
New York, NY 10469
USA

jason.behrstock@lehman.cuny.edu

Cornelia Druţu
Mathematical Institute
24-29 St Giles
Oxford
OX1 3LB
United Kingdom

drutu@maths.ox.ac.uk

Mark Sapir
Department of Mathematics
Vanderbilt University
Nashville, TN 37240
USA

m.sapir@vanderbilt.edu