Journal of Topology and Analysis
Vol. 11, No. 3 (2019) 499–534
© World Scientific Publishing Company
DOI: 10.1142/S1793525319500225

Combinatorial higher dimensional isoperimetry and divergence

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Received 27 June 2016
Revised 4 April 2017
Accepted 15 September 2017
Published 3 November 2017

In this paper we provide a framework for the study of isoperimetric problems in finitely generated groups, through a combinatorial study of universal covers of compact simplicial complexes. We show that, when estimating filling functions, one can restrict to simplicial spheres of particular shapes, called "round" and "unfolded", provided that a bounded quasi-geodesic combing exists. We prove that the problem of estimating higher dimensional divergence as well can be restricted to round spheres. Applications of these results include a combinatorial analogy of the Federer–Fleming inequality for finitely generated groups.

Keywords: Non-positive curvature; isoperimetric function; divergence.

AMS Subject Classification: Primary: 20F65, 28A75, 58E35; Secondary: 20F69, 53C23

1. Introduction

The *k-dimensional isoperimetric* (*or filling*) *function* of a space X, denoted in this paper Iso_k, measures the smallest volume of a $(k+1)$-dimensional ball needed to fill a k-dimensional sphere of a given area. There is a whole range of *filling functions*, from the ones mentioned above, to filling functions of the form Iso_V, measuring how copies of a given manifold ∂V are filled by copies of a manifold V, where V is an arbitrary $(k+1)$-dimensional connected compact sub-manifold with boundary of $\mathbb{R}^{k+1}$. The notion of "volume" that is being used also varies.

A particularly significant type of filling function, especially in the presence of non-positive curvature, are the *divergence functions*. These functions measure the volume of a filling of a sphere, where the filling is required to avoid a ball of large size. These functions, in some sense, describe the spread of geodesics, and the filling near the boundary at infinity. See, e.g., [1, 5, 6, 16, 20, 21, 27].

Traditionally, the topic of filling (of spheres, hypersurfaces, cycles etc.) belongs to Riemannian geometry and geometric measure theory. More recently, it has made its way in the study of infinite groups, in which the most appropriate framework is that of simplicial complexes and simplicial maps; this setting arises naturally in the context of groups whose Eilenberg–MacLane space has a finite $(k+1)$-skeleton. In a simplicial setting, analytic arguments and tools are no longer available and analogous tools must be established anew.

A main difficulty in obtaining estimates for higher dimensional filling functions, whether in the context of Riemannian geometry, geometric measure theory, or elsewhere, comes from the fact that, unlike in the 1-dimensional case, the knowledge of the volume of a hypersurface to fill does not bring with it any knowledge of its shape, or even, in particular, its diameter. This difficulty does not occur in the case of one-dimensional filling, hence in group theory significant results have already been found in that case. Isoperimetric functions have been used to characterize a number of interesting classes of finitely presented groups. For instance, a group is Gromov hyperbolic if and only if its 1-dimensional isoperimetric function is subquadratic, and this occurs if and only if this function is linear [26, 34, 35]. Also, it has been shown that automatic groups have k-dimensional isoperimetric functions that are at most polynomial for every k, and that in the particular case when $k = 1$ their isoperimetric functions are always at most quadratic [17]. Recently there has been important progress on isoperimetric functions for lattices in higher rank semisimple groups, see [8, 33, 45].

In this paper we prove that, under certain conditions, the study of the minimal filling of simplicial spheres by simplicial balls can be restricted to classes of spheres with particular shapes, which we call *round* and *unfolded*. A k-dimensional hypersurface $\mathfrak{h}$ is called η-round, for some $\eta > 0$, if its diameter is at most $\eta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$. We defer the precise definition of an unfolded hypersurface to Definition 4.9, but, roughly speaking, a k-dimensional hypersurface is unfolded at scale ρ if within distance at most ρ of every point the hypersurface looks like a k-dimensional disk (as opposed to, say, looking like a k-dimensional cylindrical surface that is long and thin; note that such a cylindrical surface could be spiraling around a point, so the roundedness condition alone does not rule out its existence).

Convention 1.1. Throughout this paper we work in the context of an $(n+1)$-dimensional simplicial complex X which is n–connected, and which is the universal cover of a compact simplicial complex.

We prove that an arbitrary sphere of dimension $k \leq n$ in X has a partition into spheres with particular shapes, such that the sum of the volumes of the spheres in the partition is bounded by a multiple of the volume of the initial sphere. Here a *partition* consists of the following: finitely many spheres (more generally, hypersurfaces) $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ compose a partition of a sphere $\mathfrak{h}$ if by filling all of them one obtains a ball filling $\mathfrak{h}$ (see Definition 3.6). The hypersurfaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ are called *contours of the partition.*

We prove the following:

Theorem 4.4. *Consider an integer* $2 \leq k \leq n$. *Assume that* X *satisfies an isoperimetric inequality at most Euclidean for* $k-1$, *if* $k \neq 3$, *or an inequality of the form* $\mathrm{Iso}_V(x) \leq Bx^3$ *for every compact closed surface bounding a handlebody* V, *if* $k = 2$ *(where* B *is a positive constant independent of* V*).*

Then for every $\varepsilon > 0$ *there exists a constant* $\eta > 0$ *such that every* k*-dimensional sphere* $\mathfrak{h}$ *has a partition with contours* $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ *that are* η*-round hypersurfaces, and contours* $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ *that are hypersurfaces of volume and filling volume zero such that*

(1) $\sum_{i=1}^{n} \mathrm{Vol}(\mathfrak{h}_i) \leq 2 \cdot 6^{k+1} \mathrm{Vol}(\mathfrak{h})$.
(2) $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ *and* $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ *are contained in the tubular neighborhood* $\mathcal{N}_R(\mathfrak{h})$, *where* $R = \varepsilon \mathrm{Vol}(\mathfrak{h})^{1/k}$.

For $k = 2$ *all the hypersurfaces* $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ *and* $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ *are spheres.*

If moreover X *has a bounded quasi-geodesic combing then for every* $k \geq 3$ *as well,* $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ *are* $\kappa\eta$*-round spheres, where* κ *depends only on the constants of the quasi-geodesic combing.*

An immediate consequence of the above is that a simplicial complex with a bounded quasi-geodesic combing always satisfies an isoperimetric inequality which is at most Euclidean. Further, an estimate for the filling of round spheres yields one for arbitrary spheres.

The existence of bounded quasi-geodesic combing implies cone-type inequalities in the sense of (10) in a straightforward way, see e.g., [17, Sec. 10] and Lemma 3.7; in dimension 1 these immediately imply a Euclidean filling function, but applying these inequalities in higher dimensions to obtain Euclidean filling functions requires significantly more elaborate arguments, which we carry out here. Along the same lines, Gromov previously studied homological filling functions by Lipschitz chains. In this context, Gromov proved that such homological fillings are at least Euclidean for Hadamard manifolds, and, more generally, for complete Riemannian manifolds satisfying a cone-type inequality (see for instance (10)) and for Banach spaces [25]. Gromov's results were extended to Hadamard spaces in [39], and then to complete metric spaces satisfying cone-type inequalities in [41] via another homological version of filling functions using integral currents.

From our result on the partition into round hypersurfaces previously described, we can deduce the following.

Theorem 4.8. *Assume that X has a bounded quasi-geodesic combing.*

(1) (Federer–Fleming inequality for groups). *For every $k \geq 1$, the k-th isoperimetric inequality of X is at most Euclidean. For $k = 2$, moreover, the supremum of the 2-dimensional filling functions $\mathrm{Iso}_V(x)$ modelled over all handlebodies V is at most Bx^3, where $B > 0$ is independent of V.*
(2) *Assume that for some $k \geq 2$, for some $\eta > 0$ large enough, and for some $A' > 0$ it is known that every η-round k-sphere of volume at most $A'x^k$ has filling volume at most Bx^α with $\alpha \in [k, k+1)$ for some $B > 0$. Then $\mathrm{Iso}_k(x) \leq \xi B x^\alpha$, where ξ is a universal constant.*

S. Wenger informed us that he believes that the Federer–Fleming inequality for combable groups can alternatively be proven using an analytic approach as in [39] and [43]; the details of such an approach appear to be delicate.

We further generalize Theorem 4.8 to show that in the study of higher dimensional divergence one may also restrict the study to round spheres. For a definition of higher dimensional divergence we refer to Definition 5.1. We prove the following.

Theorem 5.3. *Assume that X is a simplicial complex of dimension n endowed with a bounded quasi-geodesic combing. For every $\varepsilon > 0$ there exists $\eta > 0$ such that the following holds. Consider the restricted divergence function $\mathrm{Div}_k^r(x, \delta)$, obtained by taking the supremum only over k-dimensional spheres that are η-round, of volume at most $2Ax^k$ and situated outside balls of radius x.*

Assume that $\mathrm{Div}_k^r(x, \delta) \leq Br^\beta$ for some $\beta \geq k+1$ and $B > 0$ universal constant. Then the general divergence function $\mathrm{Div}_k(x, \delta(1-\varepsilon))$ is at most $B'r^\beta$ for some $B' > 0$ depending on B, ε, η and X.

In the theorem above, the notation Div^r stands for "restricted divergence". To avoid heavy notation we have omitted to specify the parameter η determining the restricted divergence.

In fact we prove a more general version of the theorem above (see Sec. 5).

Theorem 5.3 is a powerful tool which we expect will be widely used. Indeed, in [7] the authors use this to study the higher dimensional divergence of the mapping class group.

When a bounded quasi-geodesic combing exists, another useful restriction to impose is to restrict one's study to fillings of spheres that are both round and unfolded. Note that the condition of roundedness only forbids that a sphere stretches too much towards infinity, but it does not guarantee that the sphere does not contain many long and thin "fingers" (which may eventually be spiralling, so that their diameter satisfies the condition imposed by roundedness). It is the condition of "unfoldedness" that requires from a k-dimensional sphere to be shaped like a Euclidean k-dimensional sphere. In such a setting we prove the following result.

Theorem 4.14. *Let X be a simplicial complex with a bounded quasi-geodesic combing.*

(1) *Let $k \geq 2$ be an integer. If every k-dimensional sphere of volume at most Ax^k that is η-round and ε-unfolded at scale δx, in the sense of Definition* 4.9, *has filling volume at most Bx^α with $\alpha \geq k$, then $\mathrm{Iso}_k(x) \leq Cx^\alpha$, where $C = C(\eta, \varepsilon, \delta)$.*
(2) *If every (closed) surface of volume at most Ax^2 that is η-round and ε-unfolded at scale δx has filling volume at most Bx^α with $\alpha \geq 2$ and B independent of the genus, then $\mathrm{Iso}_V(x) \leq Cx^\alpha$, for every handlebody V, where $C = C(\eta, \varepsilon, \delta)$.*

Unlike the case of reduction to round spheres, Theorem 4.14 is not based on a partition as in Theorem 4.8. Instead, for this result, for an arbitrary sphere, we produce a partition with a uniformly bounded number of contours that are spheres round and unfolded and where the remaining contours are spheres whose respective area is at most ε times the area of the initial sphere, with $\varepsilon > 0$ small.

In previous work of other authors, partial results have been obtained concerning decompositions of spheres into special types of spheres which yield good limits in the asymptotic cones. The main results in this direction are those in the one-dimensional case by P. Papasoglu in [36, 37], concerning N-connectedness by T. Riley in [38], and for integral currents by S. Wenger in [41, 42].

Related filling functions. There have been several versions of higher dimensional isoperimetric and divergence fillings which have been considered in other works. Most prior studies, starting with that of Brady–Farb [10] considered the case of CAT(0) manifolds and their generalizations, or of groups acting on such non-positively curved spaces, e.g., [1, 29, 32, 40]. Different notions of filling sometimes yield the same function [15], but in general have no relation, as shown in [2].

The subject of geometric measure theory has a long history of studying filling functions, see [18]. One approach in this direction, that of filling functions in metric spaces, has been carried out in some contexts using integral currents, cf. [4, 41, 42]. Indeed, part of our initial plan had been to apply an inductive strategy as described in [41, 42], but it turns out that working in the simplicial setting creates significant differences which do not allow such a strategy to work directly. Also, there is no established connection between this simplicial filling of minimal volume, and the minimal filling volume of integral currents by currents, beyond the obvious remark that simplicial complexes are currents. In response to an early draft of this paper, we were informed by S. Wenger that, since simplicial complexes are currents, he believes that using the tools in [39] and [43], he can prove the filling function defined in terms of simplicial balls filling simplicial spheres is at most the filling function of integral currents by integral currents, of the corresponding dimensions (up to an affine rescaling). This appears to be far from obvious and, accordingly, we believe that clarifying this inequality would be of independent interest.

For readers familiar with the analytic setting, we note some of the sources of difficulties that needed to be surmounted to prove results about simplicial filling functions, as we do in this paper.

- Unlike for integral currents, where a minimal surface-type theory is at work, a ball that realizes the minimal filling volume for a given sphere typically has singularities. Hence, in the neighborhood of certain vertices of the filling ball the volume may be zero, the volume may not grow with the distance to the point, etc.
- A main tool in the setting of integral currents is the slicing theorem of Ambrosio–Kirchheim. For its use, it is essential that the intersections of integral currents with the level subsets of differentiable functions are integral currents. In such a setting, the homological nature of the filling is essential.

  Slicing arguments cannot be used in our setting. Therefore, for instance, instead of working with intersections of filling balls with balls $B(v, r)$ centered in given vertices, we must use different objects, the domains $\mathfrak{c}(v, r)$. For these, the position of the boundary $\partial\mathfrak{c}(v, r)$ with respect to the boundary sphere $S(v, r)$ is unclear, also $\mathfrak{c}(v, r)$ is strictly smaller than the intersection with $B(v, r)$.
- While in the integral current setting after the decomposition into, say, round parts, there is no need to control the shape of the newly obtained currents, after our decomposition it is in general not true that the result is a collection of simplicial spheres. Additional arguments must be made to decompose the hypersurfaces thus obtained into spheres and hypersurfaces of volume zero. For this, the Federer–Fleming inequality is not sufficient on its own, the existence of a bounded quasi-geodesic combing is needed, as explained in detail in the proofs of our main theorems.

Outline. The plan of the paper is as follows. In Sec. 2 we recall some basic notions and establish notation which we will use in the paper. In Sec. 3 we recall a few facts about filling functions and we prove an estimate of the filling radius in terms of the filling function. Section 4 is devoted to the proof that in a simplicial complex with bounded combing the study of the filling function can be reduced to round and unfolded spheres. In Sec. 5, we prove that in the presence of a combing, the study of divergence can likewise be restricted to round spheres.

2. Preliminaries

2.1. *General terminology and notation*

We begin with standard notions and notation used in the study of quasi-isometry invariants. Consider a constant $C \geq 1$ and an integer $k \geq 1$. Given two functions f, g which both map $\mathbb{R}_+$ to itself, we write $f \preceq_{C,k} g$ if

$$f(x) \leq Cg(Cx + C) + Cx^k + C \quad \text{for all } x \in \mathbb{R}_+.$$

We write $f \asymp_{C,k} g$ if and only if $f \preceq_{C,k} g$ and $g \preceq_{C,k} f$. Two functions $\mathbb{R}_+ \to \mathbb{R}_+$ are said to be *k-asymptotically equal* if there exists $C \geq 1$ s.t. $f \asymp_{C,k} g$. This is an equivalence relation.

When at least one of the two functions f, g involved in the relations above is an n-dimensional isoperimetric or divergence function, we automatically consider only relations where $k = n$, therefore k will no longer appear in the subscript of the relation. When irrelevant, we do not mention the constant C either and likewise remove the corresponding subscript.

Given f and g real-valued functions of one real variable, we write $f = O(g)$ to mean that there exists a constant $L > 0$ such that $f(x) \leq Lg(x)$ for every x; in particular $f = O(1)$ means that f is bounded, and $f = g + O(1)$ means that $f - g$ is bounded. The notation $f = o(g)$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

In a metric space (X, dist), the *open R-neighborhood* of a subset A, i.e. $\{x \in X : \mathrm{dist}(x, A) < R\}$, is denoted by $\mathcal{N}_R(A)$. In particular, if $A = \{a\}$ then $\mathcal{N}_R(A) = B(a, R)$ is the open R-ball centered at a. We use the notations $\overline{\mathcal{N}}_R(A)$ and $\bar{B}(a, R)$ to designate the corresponding *closed neighborhood* and *closed ball* defined by non-strict inequalities. We make the convention that $B(a, R)$ and $\bar{B}(a, R)$ are the empty set for $R < 0$ and any $a \in X$.

Fix two constants $L \geq 1$ and $C \geq 0$. A map $\mathfrak{q}\colon\ Y \to X$ is said to be

- (L, C)-*quasi-Lipschitz* if

$$\mathrm{dist}(\mathfrak{q}(y), \mathfrak{q}(y')) \leq L\mathrm{dist}(y, y') + C, \quad \text{for all } y, y' \in Y;$$

- an (L, C)-*quasi-isometric embedding* if moreover

$$\mathrm{dist}(\mathfrak{q}(y), \mathfrak{q}(y')) \geq \frac{1}{L}(y, y') - C \quad \text{for all } y, y' \in Y;$$

- an (L, C)-*quasi-isometry* if it is an (L, C)-quasi-isometric embedding $\mathfrak{q}\colon\ Y \to X$ satisfying the additional assumption that $X \subset \mathcal{N}_C(\mathfrak{q}(Y))$.
- an (L, C)-*quasi-geodesic* if it is an (L, C)-quasi-isometric embedding defined on an interval of the real line;
- a *bi-infinite* (L, C)-*quasi-geodesic* when defined on the entire real line.

In the last two cases the terminology is extended to the image of $\mathfrak{q}$. When the constants L, C are irrelevant, they are not mentioned.

We call $(L, 0)$-quasi-isometries (quasi-geodesics) L-*bi-Lipschitz maps* (*paths*). If an (L, C)-quasi-geodesic $\mathfrak{q}$ is L-Lipschitz then $\mathfrak{q}$ is called an (L, C)-*almost geodesic*. Every (L, C)-quasi-geodesic in a geodesic metric space is at bounded (in terms of L, C) distance from an $(L + C, C)$-almost geodesic with the same end points, see e.g. [14, Proposition 8.3.4]. Therefore, without loss of generality, we assume in this text that all quasi-geodesics are in fact almost geodesics, in particular that they are continuous.

Given two subsets $A, B \subset \mathbb{R}$, a map $f\colon\ A \to B$ is said to be *coarsely increasing* if there exists a constant D such that for each a, b in A satisfying $a + D < b$, we have that $f(a) \leq f(b)$. Similarly, we define *coarsely decreasing* and *coarsely monotonic* maps. A map between quasi-geodesics is coarsely monotonic if it defines a coarsely monotonic map between suitable nets in their domain.

A metric space is called

- *proper* if all its closed balls are compact;
- *cocompact* if there exists a compact subset K in X such that all the translations of K by isometries of X cover X;
- *periodic* if it is geodesic and for fixed constants $L \geq 1$ and $C \geq 0$ the image of some fixed ball under (L, C)-quasi-isometries of X covers X;
- a *Hadamard space* if X is geodesic, complete, simply connected and satisfies the CAT(0) condition;
- a *Hadamard manifold* if moreover X is a smooth Riemannian manifold.

2.2. *Combinatorial terminology*

The usual setting for defining an n-dimensional filling function is that of an n-connected space X of dimension $n+1$; of particular interest is when X is the universal cover of a compact CW-complex K, with fundamental group G. By the Simplicial Approximation Theorem, cf. [28, Theorem 2.C.2], K is homotopy equivalent to a finite simplicial complex K' of the same dimension; hence we may assume that both X and K are simplicial.

In this paper we use the standard terminology related to simplicial complexes as it appears in [28]. In the setting of isoperimetry problems, this terminology is used as such in [37], and it is used in a slightly more general but equivalent form (i.e. the cells need not be simplices but rather polyhedra with a uniformly bounded number of faces) in [11, p. 153], [13], and [38, §2.3]. Note that when we speak of simplicial complexes in what follows, we always mean their topological realization. Throughout the paper, we assume that all simplicial complexes are connected.

Given an n-dimensional simplicial complex $\mathcal{C}$, we call the closed simplices of dimension n the *chambers* of $\mathcal{C}$. A *gallery* in $\mathcal{C}$ is a finite sequence of chambers such that two consecutive chambers share a face of dimension $n-1$.

Given a simplicial map $f\colon X \to Y$, where X, Y are simplicial complexes, X of dimension n, we call *f-non-collapsed chambers in X* the chambers whose images by f stay of dimension n. We denote by X_{Vol} the set of f-non-collapsed chambers. We define the *volume of f* to be the (possibly infinite) cardinality of X_{Vol}.

Recall that a group G is *of type $\mathcal{F}_k$* if it admits an Eilenberg–MacLane space $K(G,1)$ whose k-skeleton is finite.

Proposition 2.1. ([3, Proposition 2]) *If a group G acts cellularly on a CW-complex X, with finite stabilizers of points and such that $X^{(1)}/G$ is finite then G is finitely generated and quasi-isometric to X. Moreover, if X is n-connected and $X^{(n+1)}/G$ is finite then G is of type $\mathcal{F}_{n+1}$.*

Conversely, it is easily seen that for a group of type $\mathcal{F}_{n+1}$ one can define an $(n+1)$-dimensional n-connected simplicial complex X on which G acts properly discontinuously by simplicial isomorphisms, with trivial stabilizers of vertices, such that X/G has finitely many cells. Any two such complexes X, Y are quasi-isometric, and the quasi-isometry, which can initially be seen as a bi-Lipschitz map between two subsets of vertices, can be easily extended to a simplicial map $X \to Y$ [3, Lemma 12].

A group is *of type* $\mathcal{F}_\infty$ if and only if it is of type $\mathcal{F}_k$ for every $k \in \mathbb{N}$. It was proven in [17, Theorem 10.2.6] that every combable group is of type $\mathcal{F}_\infty$.

3. Higher Dimensional Isoperimetric Functions

3.1. *Definitions and properties*

There exist several versions of filling functions, measuring how spheres can be filled with balls or, given a manifold pair $(M, \partial M)$, how a copy of ∂M can be filled with a copy of M, or how a cycle can be filled with a chain. The meaning of 'sphere', 'manifold' or 'cycle' also varies, from the measure theoretical notion of integral current [4, 39, 40] to that of (singular) cellular map [9, 13, 38] or of Lipschitz map defined on the proper geometric object. For a comparison between the various versions of filling functions we refer to [22–24].

In the setting of finitely generated groups the most frequently used approach is to refer to a proper cocompact action of the group on a CW-complex. More precisely, the nth dimensional filling function is defined for groups that are *of type* F_{n+1}, that is groups having a classifying space with finite $(n+1)$-skeleton. One can define the nth dimensional filling function using the $(n+1)$-skeleton of the classifying space, or any other $(n+1)$-dimensional complex on which the group G acts properly discontinuously cocompactly. This is due to the quasi-isometry invariance of filling functions proved in [3]. Since a finite $(n+1)$-presentation of a group composed only of simplices can always be found, it suffices to restrict to simplicial complexes. In what follows we therefore define filling functions for simplicial complexes with a cocompact action.

A simplicial complex X may be endowed with a "large scale metric structure" by assuming that all edges have length one and taking the shortest path metric on the 1-skeleton $X^{(1)}$. We say that a metric space Y (or, another simplicial complex Z) is (L, C)-*quasi-isometric to* X if Y (respectively $Z^{(1)}$) is (L, C)-quasi-isometric to $X^{(1)}$.

Convention 3.1. For the rest of the section, we fix a simplicial complex X in which the filling problem is to be considered. We assume that X is the universal cover of a compact simplicial complex K with fundamental group G, that it has dimension $n+1$ and it is n-connected. We will consider fillings in X up to dimension n.

Convention 3.2. Throughout the paper, when we speak of manifolds we always mean *manifolds with a simplicial-complex structure.*

We denote by V an arbitrary m-dimensional connected compact sub-manifold of $\mathbb{R}^m$, where $m \geq 2$ is an integer and V is smooth or piecewise linear, and with boundary. We denote its boundary, by ∂V. Unless otherwise stated, the standing assumption is that ∂V is connected. We denote its interior by $\mathrm{Int}\,(V)$.

Given V as above, a *V-domain in X* is a simplicial map $\mathfrak{d}$ of $\mathcal{D}$ to $X^{(m)}$, where $\mathcal{D}$ is a simplicial structure on V. When the manifold V is irrelevant we simply call $\mathfrak{d}$ a *domain of dimension m* (somewhat abusively, since it might have its entire image inside $X^{(m-1)}$); we also abuse notation by using $\mathfrak{d}$ to denote both the map and its image.

A *∂V-hypersurface in X* is a simplicial map $\mathfrak{h}$ of $\mathcal{M}$ to $X^{(m-1)}$, where $\mathcal{M}$ is a simplicial structure of the boundary ∂V. Again, we abuse notation by letting $\mathfrak{h}$ also denote the image of the above map, and we also call both $\mathfrak{h}$ and its image a *hypersurface of dimension $m-1$*.

According to the terminology introduced in Sec. 2.2, $\mathcal{D}_{\mathrm{Vol}}$, respectively $\mathcal{M}_{\mathrm{Vol}}$, is the set of $\mathfrak{d}$-non-collapsed chambers (respectively $\mathfrak{h}$-non-collapsed chambers). The *volume* of $\mathfrak{d}$ (respectively $\mathfrak{h}$) is the cardinality of $\mathcal{D}_{\mathrm{Vol}}$, respectively $\mathcal{M}_{\mathrm{Vol}}$. Given a vertex v we write $v \in \mathcal{D}_{\mathrm{Vol}}$ (or $v \in \mathcal{M}_{\mathrm{Vol}}$) to signify that v is a vertex in a non-collapsed chamber.

We sometimes say that the domain $\mathfrak{d}$ is *modelled on V*, or that it is a *V-domain*, and $\mathfrak{h}$ is *modelled on ∂V*, or that it is a *∂V-hypersurface*. When V is a closed ball in $\mathbb{R}^m$, we call $\mathfrak{d}$ an *m-dimensional ball* and $\mathfrak{h}$ an *$(m-1)$-dimensional sphere*.

We say that a domain $\mathfrak{d}$ *fills* a hypersurface $\mathfrak{h}$ if this pair corresponds to a $(k+1)$-dimensional connected compact smooth submanifold with boundary V in $\mathbb{R}^{k+1}$ satisfying $\mathcal{D} \cap \partial V = \mathcal{M}$ and $\mathfrak{d}|_{\mathcal{M}} = \mathfrak{h}$, possibly after pre-composing $\mathfrak{h}$ with a simplicial equivalence of $\mathcal{M}$.

The *filling volume* of the hypersurface $\mathfrak{h}$, $\mathrm{FillVol}(\mathfrak{h})$, is the minimum of all the volumes of domains filling $\mathfrak{h}$. If no domain filling $\mathfrak{h}$ exists then we set $\mathrm{FillVol}(\mathfrak{h}) = \infty$.

Remark 3.3. These notions are equivalent to the ones defined in [9, 44] using admissible maps, as well as to the ones in [3], and those in [11, p. 153], [13], [38, §2.3] that are using more polyhedra than just simplices.

Indeed, every domain and hypersurface as above is an admissible map with the same volume. Conversely, consider an admissible map $f\colon\, W \to X$ defined on an m-dimensional domain or boundary of a domain, i.e. a continuous map $f\colon\, W \to X^{(m)}$, such that $f^{-1}(X^{(m)} \setminus X^{(m-1)})$ is a disjoint union $\bigsqcup_{i\in I} B_i$ of open m-dimensional balls, each mapped by f homeomorphically onto an m-simplex of X. Recall that the volume of f is the cardinality of I (by compactness of W, this is finite).

The submanifold with boundary $W \setminus \bigsqcup_{i\in I} B_i$ admits a triangulation. We apply the Cellular Approximation Theorem [28, Theorem 4.8] to the restriction $f\colon\, W \setminus \bigsqcup_{i\in I} B_i \to X^{(m-1)}$ and obtain that it is homotopy equivalent to a simplicial map $\bar{f}\colon\, W \setminus \bigsqcup_{i\in I} B_i \to X^{(m-1)}$. Due to the homotopy equivalence with f it follows that for every $i \in I$, $\bar{f}(\partial B_i)$ and $f(\partial B_i)$ coincide as sets. We may then extend $\bar{f}$ to a simplicial map $\hat{f}\colon\, W \to X^{(m)}$, with the same volume as f, homotopy equivalent to f, and such that the sets $\hat{f}(B_i)$ and $f(B_i)$ coincide for every $i \in I$.

In filling problems, when dealing with extensions of maps from boundaries to domains, one may use an argument as above and the version of the Cellular Approximation Theorem ensuring that if a continuous map between CW-complexes is cellular on a sub-complex A then it is homotopic to a cellular map by a homotopy which is stationary on A [28, Theorem 4.8].

For more on equivalent definitions of (filling) volumes and functions we refer to [22–24].

We now define a notion of filling radius.

Definition 3.4. Given a hypersurface $\mathfrak{h}\colon \mathcal{M} \to X$ and a domain $\mathfrak{d}\colon \mathcal{D} \to X$ filling it, the *radius* $\mathrm{Rad}(\mathfrak{d})$ *of the domain* $\mathfrak{d}$ is the minimal R such that $\mathfrak{d}(\mathcal{D}^{(1)})$ is in the closed tubular neighborhood of radius R of $\mathfrak{h}(\mathcal{M}^{(1)})$.

The *filling radius* $\mathrm{FillRad}(\mathfrak{h})$ *of the hypersurface* $\mathfrak{h}$ is the infimum of all the filling radii of domains realizing $\mathrm{FillVol}(\mathfrak{h})$.

We recall two standard results that we will use later. The first is an immediate consequence of Alexander duality and the second is the Jordan–Schoenflies Theorem.

Proposition 3.5. (1) *Given M a k-dimensional compact connected smooth or piecewise linear sub-manifold without boundary of $\mathbb{R}^{k+1}$ (or $\mathbb{S}^{k+1}$), its complement in $\mathbb{R}^{k+1}$ (respectively $\mathbb{S}^{k+1}$) has two connected components.*

(2) *When $k = 1$ and M is a simple closed curve, there exists a homeomorphism of $\mathbb{R}^2$ transforming M into the unit circle.*

Definition 3.6. Let $\mathfrak{h}\colon \mathcal{M} \to X$ be a k-dimensional ∂V-hypersurface.

A *partition* of $\mathfrak{h}$ is a finite family of k-dimensional hypersurfaces $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_q$, where $\mathfrak{h}_i\colon \mathcal{M}_i \to X$ are simplicial maps defined on simplicial structures $\mathcal{M}_i$ on boundaries ∂V_i, with the following properties.

(1) there exist simplicial structures $\mathcal{D}$ on V which agrees with $\mathcal{M}$ on ∂V, respectively $\mathcal{D}_i$ on V_i which agrees with $\mathcal{M}_i$ on ∂V_i, and simplicial maps $\sigma_i : \mathcal{D}_i \to \mathcal{D}$ which are homeomorphisms on $\mathcal{D}_i \cap \mathrm{Int}\, V_i$, local homeomorphisms on $\mathcal{M}_i$, and may identify distinct simplices on $\mathcal{M}_i$ of codimension at least 1.

(2) V can be written as a (set-wise) union of $\sigma_i(\mathcal{D}_i), i = 1, 2, \dots, q$; the sets $\sigma_i(\mathcal{D}_i \cap \mathrm{Int}\, V_i), i = 1, 2, \dots, q$, are pairwise disjoint;

(3) there exists a domain filling $\mathfrak{h}$, i.e. a simplicial map φ of $\mathcal{D}$ into X such that $\varphi \circ \sigma_i|_{\mathcal{M}_i} = \mathfrak{h}_i$ for every $i \in \{1, 2, \dots, q\}$.

Each of the hypersurfaces $\mathfrak{h}_i$ is called *a contour of the partition.*

When $q = 2$, we say that $\mathfrak{h}$ is obtained from $\mathfrak{h}_1$ *adjoined with* $\mathfrak{h}_2$.

When discussing problems of filling we will often assume the existence of a combing, as defined below.

We say that a simplicial complex X has a *bounded* (L, C)*-quasi-geodesic combing*, where $L \geq 1$ and $C \geq 0$, if for every $x \in X^{(1)}$ there exists a way to assign to every element $y \in X^{(1)}$ an (L, C)-quasi-geodesic $\mathfrak{q}_{xy}$ connecting y to x in $X^{(1)}$, such that

$$\operatorname{dist}(\mathfrak{q}_{xy}(i), \mathfrak{q}_{xa}(i)) \leq L\operatorname{dist}(y, a) + L,$$

for all $x, y, a \in X^{(1)}$ and $i \in \mathbb{R}$. Here the quasi-geodesics are assumed to be extended to $\mathbb{R}$ by constant maps.

The result below is well known in various contexts (CW-complexes, Riemannian geometry etc), see for instance [17, Theorems 10.2.1, 10.3.5 and 10.3.6]. We give a sketch of proof here for the sake of completeness.

Lemma 3.7. *Let X be a simplicial complex with a bounded (L, C)-quasi-geodesic combing. For every hypersurface $\mathfrak{h}$,*

$$\operatorname{FillVol}(\mathfrak{h}) \preceq \operatorname{Vol}(\mathfrak{h}) \operatorname{diam}(\mathfrak{h}).$$

Proof. Consider an arbitrary hypersurface, $\mathfrak{h} : \mathcal{M} \to X$, where $\mathcal{M}$ is a simplicial structure of the boundary ∂V. Recall that the *cut locus*, Cut, of V relative to its boundary, is the closure of the set of points in V that have at least two distinct shortest paths in V joining them to ∂V. Let $p\colon V \to \mathrm{Cut}$ denote the normal map. Note that V is homeomorphic to the mapping cylinder of $p|_{\partial V}$ [25, §3.1.A''].

Fix a vertex x_0 in $\mathfrak{h}(\mathcal{M})$. The map $\mathfrak{h}$ can be extended to a quasi-Lipschitz map on V (with an appropriate simplicial structure) as follows: the whole set Cut is sent onto x_0 and for every vertex $v \in \mathcal{M}$ the geodesic $[v, p(v)]$ is sent to the quasi-geodesic in the combing joining x_0 and $\mathfrak{h}(v)$.

The extension can be transformed into a simplicial map as in [3, Lemma 12]. Note that [3, Lemma 12] (unnecessarily) assumes that the complex X is what they call m-Dehn, but do not use this hypothesis in the proof. Nonetheless, by [3, Theorem 1, p. 92], the simplicial complex X that we use satisfies the m-Dehn condition.

We have thus obtained a domain filling $\mathfrak{h}$ of volume $\preceq \operatorname{diam}(\mathfrak{h}) \operatorname{Vol}(\mathfrak{h})$. $\square$

Definition 3.8. The *kth isoperimetric function*, also known as the *kth filling function*, of a simplicial complex X is the function $\mathrm{Iso}_k\colon \mathbb{R}_+^* \to \mathbb{R}_+ \cup \{\infty\}$ such that $\mathrm{Iso}_k(x)$ is the supremum of the filling volume $\operatorname{FillVol}(\mathfrak{h})$ over all k-dimensional spheres $\mathfrak{h}$ of volume at most Ax^k.

The *kth filling radius* of the simplicial complex X is the function $\mathrm{Rad}_k\colon \mathbb{R}_+^* \to \mathbb{R}_+$ such that $\mathrm{Rad}_k(x)$ is the supremum of all filling radii $\operatorname{FillRad}(\mathfrak{h})$ over all $k$dimensional spheres $\mathfrak{h}$ of volume at most Ax^k.

In what follows the constant $A > 0$ from Definition 3.8 is fixed, but not made precise. Note that the two filling functions corresponding to two different values of A are equivalent in the sense of the relation $\asymp$ defined in Sec. 2.1.

Remark 3.9. In dealing with isoperimetric functions some authors use an alternate formulation where the volume is bounded above by Ar, instead of Ar^k; this yields an equivalent notion (although the functions differ by a power of k), but with the alternative definition one must modify the equivalence relation to allow an additive term which is a multiple of r instead of r^k. We use the present definition because it yields a formulation consistent with the standard definition of the divergence function that we use in this paper, see also [10, 29, 32, 40].

Remark 3.10. Note that when considering the k-dimensional isoperimetric function (and, as we shall see below, the k-dimensional divergence function), we have that $x^k \asymp x$. Accordingly, in this case we often represent our function by x, as the property of having a *linear filling* means the same both under our choice of normalizing the volume and under the alternative choice.

We may generalize the functions above, using instead of the sphere and its filling with a ball, a hypersurface and its filling with a domain, both modelled on a $(k+1)$-dimensional submanifold with boundary V in $\mathbb{R}^{k+1}$. We then define as above the filling function and radius, denoted Iso_V and Rad_V, respectively.

Proposition 3.11. ([9]) *Assume that V has dimension $k+1$ at least* 4*.*

(1) *Assume that ∂V is connected.*

For every hypersurface $\mathfrak{h} : \partial V \to X$ there exists a simplicial map $f : \mathbb{B}^k \to \partial V$ defined on the k-dimensional unit ball, whose image $\mathcal{B}$ contains all the chambers that contribute to the volume of $\mathfrak{h}$, and a ball $\mathfrak{b} : \mathbb{B}^k \to X^{(k-1)}$ filling the sphere $\mathfrak{h} \circ f|_{\mathbb{S}^{k-1}}$.

Therefore $\mathfrak{s}$ and $\mathfrak{h}'$ compose a partition of $\mathfrak{h}$, where $\mathfrak{s}$ is the sphere composed by the ball $\mathfrak{h} \circ f$ and the ball $\mathfrak{b}$, and $\mathfrak{h}'$ is the hypersurface with image in $X^{(k-1)}$ obtained by adding $\mathfrak{h}|_{\partial V \setminus \mathcal{B}}$ to the ball $\mathfrak{b} : \mathbb{B}^k \to X^{(k-1)}, \mathfrak{h}'$ with filling volume zero. In particular

$$\mathrm{FillVol}(\mathfrak{h}) \leq \mathrm{FillVol}(\mathfrak{s}). \tag{1}$$

(2) *If either ∂V is connected or $\mathrm{Iso}_k(x)$ is super-additive then the following inequality holds*:

$$\mathrm{Iso}_V(x) \leq \mathrm{Iso}_k(x).$$

Proof. The proof is identical to the proof in [9, Remark 2.6, (4)]. □

Using the terminology in the end of Definition 3.6, we can express the result above by stating that *every hypersurface $\mathfrak{h}$ of dimension $k \geq 3$ is obtained by adjoining to a k-dimensional sphere a hypersurface with image in $X^{(k-1)}$.*

Remark 3.12. A stronger result than Proposition 3.11 is Corollary 1 in [24] which removes the hypothesis that ∂V be connected. We do not need that generality to obtain the results of this paper.

By Proposition 3.11, we may assume without loss of generality that the hypersurfaces of dimension $k \neq 2$ are defined on simplicial structures of the k-sphere, while for $k = 2$ they are defined on a simplicial structure of a surface.

Lemma 3.13. *Given a hypersurface $\mathfrak{h} : \mathcal{M} \to X^{(m-1)}$, consider $\mathfrak{d} : \mathcal{D} \to X^{(m)}$ a domain filling $\mathfrak{h}$.*

Every domain $\mathfrak{d}' : \mathcal{D}' \to X^{(m)}$ filling $\mathfrak{h}$, with the same volume and image as $\mathfrak{d}$ and a minimal number of chambers in $\mathcal{D}'$ has the property that every chamber of $\mathcal{D}'$ is either non-collapsed or it has a vertex in the boundary, $\mathcal{M}^{(0)}$.

Proof. Consider $\mathfrak{d}' : \mathcal{D}' \to X^{(m)}$ filling $\mathfrak{h}$, with the same volume and image as $\mathfrak{d}$ and a minimal number of chambers in $\mathcal{D}'$.

Let $\{v, w\}$ be endpoints of an edge that are not in $\mathcal{M}^{(0)}$. If $\mathfrak{d}'$ sends both v and w to one vertex, then by retracting the link of $\{v, w\}$ to the link of v one obtains a new domain filling $\mathfrak{h}$, with the same volume and image as $\mathfrak{d}$ and a smaller number of chambers in its domain than $\mathfrak{d}'$.

We may therefore deduce that every chamber in $\mathcal{D}'$ is either non-collapsed or it has a vertex in $\mathcal{M}^{(0)}$. □

Theorem 3.14. (Theorem 1 and Corollary 3 in [3]) *Let X_1 and X_2 be two n-connected locally finite CW-complexes such that for each $i \in \{1, 2\}$ a group G_i acts on X_i cellularly and such that $X_i^{(n+1)}/G_i$ has finitely many cells.*

Let V be an arbitrary $(n+1)$-dimensional connected compact submanifold of $\mathbb{R}^{n+1}$, where $n \geq 1, V$ smooth or piecewise linear, and with connected boundary.

If X_1 is quasi-isometric to X_2 then

$$\mathrm{Iso}_V^{X_1} \asymp \mathrm{Iso}_V^{X_2}.$$

In particular, for every $1 \leq k \leq n$

$$\mathrm{Iso}_k^{X_1} \asymp \mathrm{Iso}_k^{X_2}.$$

Theorem 3.14 allows us to define n-dimensional filling functions for groups of type $\mathcal{F}_{n+1}$, up to the equivalence relation $\asymp$.

Definition 3.15. Let G be a group acting properly discontinuously by simplicial isomorphisms on an n-connected simplicial complex such that $X^{(n+1)}/G$ has finitely many cells. For every $1 \leq k \leq n$, the *kth isoperimetric function* of G is defined to be the kth isoperimetric function of X.

Likewise, the *isoperimetric function of G modelled on V* is defined to be Iso_V^X.

According to Theorem 3.14 any pair of choices of simplicial complexes as in Definition 3.15 yield filling functions which are $\asymp$-equivalent, thus the definition is consistent. This definition is also equivalent to the definitions appearing in [3, 9, 13, 38].

3.2. *Filling radius estimates provided by filling functions*

We begin by defining a simplicial object which, in all arguments on simplicial complexes, is meant to replace the intersection of a submanifold with a ball in a Riemannian manifold.

Consider a simplicial map $\mathfrak{c}: \mathcal{C} \to X$ representing either a domain $\mathfrak{d}: \mathcal{D} \to X$ modelled on a submanifold V of $\mathbb{R}^{k+1}$, or a hypersurface $\mathfrak{h}: \mathcal{M} \to X$ modelled on ∂V. Let v be a vertex of $\mathcal{C}$ and let $r > 0$.

Notation 3.16. We denote by $\mathcal{C}(v,r)$ the maximal sub-complex of $\mathcal{C}$ composed of chambers that can be connected to v by a gallery whose 1-skeleton is entirely contained in $\mathfrak{c}^{-1}(\bar{B}(\mathfrak{c}(v),r))$. Here $\bar{B}(\mathfrak{c}(v),r)$ represents the closed ball centered in $\mathfrak{c}(v)$ with respect to the distance dist on $X^{(1)}$. Let $\partial\mathcal{C}(v,r)$ denote the boundary of this subcomplex.

When $\mathfrak{c}$ is a sphere (or a ball), modulo some slight modifications preserving the volume, its restriction to $\mathcal{C}(v,r)$ is either a domain or the whole sphere (or ball), while its restriction to $\partial\mathcal{C}(v,r)$ is a hypersurface. The same is true when $\mathfrak{c}$ is a domain and r is strictly less than the distance in $X^{(1)}$ between $\mathfrak{c}(v)$ and $\mathfrak{c}(\partial\mathcal{C}^{(1)})$.

Indeed, in case $\mathcal{C}(v,r)$ is not a whole sphere, given an arbitrary point p in it, a small neighborhood of p is either an open set in $\mathbb{R}^{k+1}$, respectively $\mathbb{R}^k$, or is homeomorphic to a half-ball in the same space, or, for a point p in the interior of a simplex of codimension at least 1 contained in the boundary, it may have some other shape. It suffices to cut along all these latter simplices of codimension at least 1 to obtain a complex $\mathcal{C}(v,r)^{\mathrm{cut}}$ which is modelled on a smooth compact submanifold with boundary in $\mathbb{R}^{k+1}$, respectively $\mathbb{R}^k$. If $\mathcal{G}_r^{\mathrm{cut}}: \mathcal{C}(v,r)^{\mathrm{cut}} \to \mathcal{C}(v,r)$ is the map gluing back along the simplices of codimension ≥ 1 where the cutting was done, then the restriction of $\mathfrak{c}$ to $\mathcal{C}(v,r)$ must be pre-composed with $\mathcal{G}_r^{\mathrm{cut}}$.

Notation 3.17. We denote by $\mathfrak{c}(v,r)$ and by $\partial\mathfrak{c}(v,r)$ the domain, respectively the hypersurface, defined by restricting $\mathfrak{c}$ to $\mathcal{C}(v,r)$, respectively to $\partial\mathcal{C}(v,r)$, and pre-composing it with $\mathcal{G}_r^{\mathrm{cut}}$.

A relation can be established between the filling radius and the filling function. Note that for $k = 1$ and $\alpha = 2$ this relation was first proved in [36, Proposition p. 799].

Proposition 3.18. *Let $k \geq 1$ be an integer.*

If $k \neq 2$ then assume that $\mathrm{Iso}_k(x) \leq Bx^\alpha$ for $\alpha \geq k$ and some constant $B > 0$; while if $k = 2$ then assume that for every compact closed surface ∂V bounding a 3-dimensional handlebody V in $\mathbb{R}^3, \mathrm{Iso}_V(x) \leq Bx^\alpha$, where $\alpha \geq 2$ and $B > 0$ are independent of V.

Consider an arbitrary connected hypersurface $\mathfrak{h}: \mathcal{M} \to X$ of dimension k such that $\mathrm{FillVol}(\mathfrak{h}) \geq 1$. For every filling domain $\mathfrak{d}: \mathcal{D} \to X$ realizing $\mathrm{FillVol}(\mathfrak{h})$ and *such that $\mathcal{D}$ has a minimal number of chambers, the following holds.*

(1) *If* $\alpha = k$ *then* $\mathrm{Rad}(\mathfrak{d}) \leq C \ln \mathrm{FillVol}(\mathfrak{h})$, *where* $C = C(A, B, k)$.
(2) *If* $\alpha > k$ *then* $\mathrm{Rad}(\mathfrak{d}) \leq D[\mathrm{FillVol}(\mathfrak{h})]^{\frac{\alpha-k}{\alpha}}$, *where* $D = D(A, B, k, \alpha)$.
(3) *If* $\alpha > k$ *and if for* $\epsilon > 0$ *small enough and for* x *larger than some* $x_0, \mathrm{Iso}_k(x) \leq \epsilon x^\alpha$ *(respectively, for* $k = 2, \mathrm{Iso}_V(x) \leq \epsilon x^\alpha$ *and this holds for* V *as above) then either* $\mathrm{Rad}(\mathfrak{d}) \leq B x_0^\alpha$ *or*

$$\mathrm{Rad}(\mathfrak{d}) \leq L\epsilon[\mathrm{FillVol}(\mathfrak{h})]^{\frac{\alpha-k}{\alpha}} + \frac{1}{(L\epsilon)^{\frac{\alpha}{\alpha-k}}},$$

where $L = L(A, k, \alpha)$.

Proof. Consider an arbitrary vertex v of $\mathcal{D}_{\mathrm{Vol}} \backslash \mathcal{M}$, and a positive integer

$$i < \mathrm{dist}(\mathfrak{d}(v),\, \mathfrak{h}(\mathcal{M}^{(1)})). \tag{2}$$

The filling domain $\mathfrak{d}_i = \mathfrak{d}(v, i)$ realizes the filling volume of the hypersurface $\mathfrak{h}_i = \partial \mathfrak{d}(v, i)$, by the minimality of the volume of $\mathfrak{d}$.

Since v is in $\mathcal{D}_{\mathrm{Vol}} \backslash \mathcal{M}$, $\mathrm{Vol}(\mathfrak{d}_i) \geq 1$ for every $i \geq 1$. By the isoperimetric inequality this implies that $\mathrm{Vol}(\mathfrak{h}_i) \geq 1$.

When $k = 1$, by Proposition 3.5(1), we have that $\mathcal{D}(v, i)$ is homeomorphic to a disk with holes having pairwise disjoint interiors. In particular, $\mathcal{D}(v, i)$ is contained in a simplicial disk $\mathcal{D}_i \subset \mathcal{D}$ whose boundary $\mathcal{S}_i$ is inside a connected component of $\partial \mathcal{D}(v, i)$. It follows that

$$\mathrm{Vol}(\mathfrak{d}_i) \leq \mathrm{Vol}(\mathfrak{d}|_{\mathcal{D}_i}) \leq B\left(\frac{\mathrm{Vol}(\mathfrak{d}|_{\mathcal{S}_i})}{A}\right)^{\frac{\alpha}{k}} \leq B\left(\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A}\right)^{\frac{\alpha}{k}}.$$

Assume now that $k \geq 2$. By Proposition 3.5(2), if $\partial \mathcal{D}(v, i)$ is composed of several closed connected k-dimensional submanifolds $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_q$ then $\mathcal{D}(v, i)$ is the intersection of connected components of $\mathcal{D} \backslash \mathfrak{s}_i$, one component for each $i \in \{1, 2, \ldots, q\}$. For one $i \in \{1, 2, \ldots, q\}$ the connected component does not contain the boundary $\mathcal{M}$. Let $\mathcal{D}_i$ be that connected component. It is a domain modelled on a manifold V such that ∂V is connected and has a simplicial structure isomorphic to $\mathfrak{s}_i$. Then

$$\mathrm{Vol}(\mathfrak{d}_i) \leq \mathrm{Vol}(\mathfrak{d}|_{\mathcal{D}_i}) \leq \mathrm{Iso}_V\left(\left(\frac{\mathrm{Vol}(\mathfrak{d}|_{\mathfrak{s}_i})}{A}\right)^{\frac{1}{k}}\right) \leq \mathrm{Iso}_V\left(\left(\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A}\right)^{\frac{1}{k}}\right).$$

If $k \geq 3$, then $\mathrm{Iso}_V((\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A})^{\frac{1}{k}}) \leq \mathrm{Iso}_k((\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A})^{\frac{1}{k}})$ by Proposition 3.11.
If $k = 2$, then $\mathrm{Iso}_V((\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A})^{\frac{1}{k}}) \leq B(\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A})^{\frac{\alpha}{k}}$ by hypothesis.
Thus, in all cases we obtained that

$$\mathrm{Vol}(\mathfrak{d}_i) \leq B\left(\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A}\right)^{\frac{\alpha}{k}}. \tag{3}$$

Since i is strictly less than $\mathrm{dist}(\mathfrak{d}(v), \mathfrak{h}(\mathcal{M}^{(1)}))$, the volume of $\mathfrak{d}_{i+1}$ is at least the volume of $\mathfrak{d}_i$ plus $(\frac{1}{k+1})$th of the volume of $\mathfrak{h}_i$.

Indeed, every codimension one face in $\mathfrak{h}_i$ is contained in two chambers Δ, Δ', such that Δ is in $\mathcal{D}(v,i)$ and Δ' is not. If Δ' is collapsed then $\mathfrak{d}(\Delta') = \mathfrak{d}(\Delta \cap \Delta')$, whence Δ' is in $\mathcal{D}(v,i)$ too. This contradicts the fact that the codimension one face $\Delta \cap \Delta'$ is in the boundary of $\mathcal{D}(v,i)$. It follows that Δ' is not collapsed, and it is in $\mathcal{D}(v,i+1) \backslash \mathcal{D}(v,i)$.

Whence

$$\begin{aligned} \mathrm{Vol}(\mathfrak{d}_{i+1}) &\geq \mathrm{Vol}(\mathfrak{d}_i) + C' \,\mathrm{Vol}(\mathfrak{d}_{i+1})^{\frac{k}{\alpha}} \\ &\geq \mathrm{Vol}(\mathfrak{d}_i) + C' \,\mathrm{Vol}(\mathfrak{d}_i)^{\frac{k}{\alpha}}, \quad \text{where } C' = \frac{A}{(k+1)B^{\frac{k}{\alpha}}}. \end{aligned} \tag{4}$$

Part (1). Assume that $k = \alpha$. Then the above gives $\mathrm{Vol}(\mathfrak{d}_{i+1}) \geq (1+C')\,\mathrm{Vol}(\mathfrak{d}_i)$, hence by induction

$$\mathrm{Vol}(\mathfrak{d}_{i+1}) \geq (1+C')^i. \tag{5}$$

In (5), when we choose i maximal we have that $i+1$ satisfies the inequality opposite to that in (6), hence

$$i \geq \mathrm{dist}(\mathfrak{d}(v),\, \mathfrak{h}(\mathcal{M}^{(1)})) - 1, \tag{6}$$

while $\mathrm{Vol}(\mathfrak{d}_{i+1})$ is at most $\mathrm{Vol}(\mathfrak{d}) = \mathrm{FillVol}(\mathfrak{h})$. We thus obtain that

$$\frac{\ln \mathrm{FillVol}(\mathfrak{h})}{\ln(1+C')} \geq \mathrm{dist}(\mathfrak{d}(v),\, \mathfrak{h}(\mathcal{M}^{(1)})) - 1,$$

and by taking the supremum over all vertices v we obtain the inequality in (1).

Part (2). Assume that $\alpha > k$. We prove by induction on $i \leq \mathrm{dist}(v, \mathcal{M})$ that

$$\mathrm{Vol}(\mathfrak{d}_i) \geq D' i^{\frac{\alpha}{\alpha-k}} \text{ for } D' \text{ small enough.} \tag{7}$$

The statement is obvious for $i = 1$, and if we assume it for i then

$$\mathrm{Vol}(\mathfrak{d}_{i+1}) \geq D' i^{\frac{\alpha}{\alpha-k}} + C D'^{\frac{k}{\alpha}} i^{\frac{k}{\alpha-k}}.$$

Thus it suffices to prove that

$$D'[(i+1)^{\frac{\alpha}{\alpha-k}} - i^{\frac{\alpha}{\alpha-k}}] \leq C D'^{\frac{k}{\alpha}} i^{\frac{k}{\alpha-k}}.$$

A standard application of the Mean Value Theorem, as we illustrate in more detail in the next part, proves that the latter holds if D' is small enough compared to C.

Part (3). Assume $\alpha > k$ and moreover that for every $x \geq x_0$, $\mathrm{Iso}_k(x) \leq \epsilon x^\alpha$ (respectively $\mathrm{Iso}_V(x) \leq \epsilon x^\alpha$ for every surface, when $k = 2$).

For $i \geq i_0 = B x_0^\alpha$ we have that $x_i = (\frac{\mathrm{Vol}(\mathfrak{h}_i)}{A})^{\frac{1}{k}}$ is at least x_0. Thus the domain $\mathfrak{d}_i$ filling $\mathfrak{h}_i$ and realizing the filling volume has

$$\mathrm{Vol}(\mathfrak{d}_i) \leq \epsilon x_i^\alpha.$$

This implies that for $i \geq i_0$

$$\mathrm{Vol}(\mathfrak{d}_{i+1}) \geq \mathrm{Vol}(\mathfrak{d}_i) + C_\epsilon \,\mathrm{Vol}(\mathfrak{d}_i)^{\frac{k}{\alpha}}, \quad \text{where } C_\epsilon = \frac{A}{(k+1)\epsilon^{\frac{k}{\alpha}}}. \tag{8}$$

Let $D = \frac{\mu}{\epsilon^{\frac{k}{\alpha-k}}}$, where $\mu = \left(\frac{A(\alpha-k)}{\alpha 2^{\frac{k}{\alpha-k}}(k+1)}\right)^{\frac{\alpha}{\alpha-k}}$.

Consider j_0 large enough so that $\mathrm{Vol}(\mathfrak{d}_{j_0+1}) \geq D$. We can take j_0 to be the integer part of D. We prove by induction that for every $i \geq j_0 + 1$

$$\mathrm{Vol}(\mathfrak{d}_i) \geq D(i - j_0)^{\frac{\alpha}{\alpha-k}}.$$

Assume that the statement is true for i. According to (8),

$$\mathrm{Vol}(\mathfrak{d}_{i+1}) \geq D(i - j_0)^{\frac{\alpha}{\alpha-k}} + C_\epsilon D^{\frac{k}{\alpha}}(i - j_0)^{\frac{k}{\alpha-k}}.$$

The right-hand side of the inequality is larger than $D(i+1-j_0)^{\frac{\alpha}{\alpha-k}}$ if

$$D[(i+1-j_0)^{\frac{\alpha}{\alpha-k}} - (i-j_0)^{\frac{\alpha}{\alpha-k}}] \leq C_\epsilon D^{\frac{k}{\alpha}}(i-j_0)^{\frac{k}{\alpha-k}}. \tag{9}$$

We may apply the Mean Value Theorem to bound the left-hand side of (9) from above and write

$$\begin{aligned} D[(i+1-j_0)^{\frac{\alpha}{\alpha-k}} - (i-j_0)^{\frac{\alpha}{\alpha-k}}] &\leq D\frac{\alpha}{\alpha-k}(i+1-j_0)^{\frac{k}{\alpha-k}} \\ &\leq D\frac{\alpha}{\alpha-k}2^{\frac{k}{\alpha-k}}(i-j_0)^{\frac{k}{\alpha-k}}. \end{aligned}$$

Thus the inequality (9) holds true if

$$D\frac{\alpha}{\alpha-k}2^{\frac{k}{\alpha-k}} \leq C_\epsilon D^{\frac{k}{\alpha}}.$$

The value chosen for D implies that we have equality. $\square$

Two types of filling function estimates, listed below, play an important part in the theory.

A simplicial complex X is said to *satisfy a cone-type inequality for* k, where $k \geq 1$ is an integer, if for every k-dimensional sphere $\mathfrak{h}\colon \mathcal{M} \to X$ its filling volume satisfies the inequality:

$$\mathrm{FillVol}(\mathfrak{h}) \preceq \mathrm{Vol}(\mathfrak{h})\mathrm{diam}(\mathfrak{h}). \tag{10}$$

In the above inequality the diameter $\mathrm{diam}(\mathfrak{h})$ is the diameter of $\mathfrak{h}(\mathcal{M}^{(1)})$ measured with respect to the metric of the 1-skeleton $X^{(1)}$.

A simplicial complex X is said to *satisfy an isoperimetric inequality at most Euclidean* (*respectively Euclidean*) *for* k if $\mathrm{Iso}_{k,X}(x) \preceq x^{k+1}$ (respectively $\mathrm{Iso}_{k,X}(x) \asymp x^{k+1}$).

An immediate consequence of Proposition 3.18 is the following.

Corollary 3.19. *Let $k \geq 1$ be an integer and assume that X satisfies an isoperimetric inequality at most Euclidean for k, if $k \neq 2$, or an inequality of the form $\mathrm{Iso}_V(x) \leq Bx^3$ for every compact closed surface bounding a handlebody V, if $k = 2$.*

Then the filling radius described in Definition 3.8 *is bounded from above by an affine function of x.*

4. Filling Reduced to Round Unfolded Spheres

4.1. *Partition into round hypersurfaces*

Among the k-dimensional hypersurfaces, there is a particular type for which the cone-type inequality (10) implies an isoperimetric inequality at most Euclidean.

Definition 4.1. A k-dimensional hypersurface $\mathfrak{h}$ is called *η-round* for a constant $\eta > 0$ if $\mathrm{diam}(\mathfrak{h}) \leq \eta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$.

Note that for $k = 1$ the hypersurfaces, i.e. the closed curves, are always round. In what follows we extend, for $k \geq 2$, a result from Riemannian geometry to the setting of simplicial complexes, more precisely we prove that every sphere has a partition into round hypersurfaces.

Proposition 4.2. (partition into round hypersurfaces) *Consider an integer $k \geq 2$, and X a simplicial complex of dimension at least $k+1$. If $k \neq 3$ then assume that* $\mathrm{Iso}_{k-1}(x) \leq Bx^k$ *for some constant $B > 0$. If $k = 3$ then assume that for every handlebody V in $\mathbb{R}^3, \mathrm{Iso}_V(x) \leq Bx^3$, where $B > 0$ is independent of V.*

Then for every $\epsilon > 0$ there exists a constant $\eta > 0$ such that every k-dimensional sphere $\mathfrak{h}$ has a partition with contours $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ and $\mathfrak{r}$ such that $\mathfrak{h}_i$ are η-round hypersurfaces for every $i \in \{1, 2, \dots, n\}, \mathfrak{r}$ is a disjoint union of hypersurfaces obtained from k-dimensional spheres adjoined with hypersurfaces of volume and filling volume zero, and

(1) $\sum_{i=1}^{n} \mathrm{Vol}(\mathfrak{h}_i) \leq 2\mathrm{Vol}(\mathfrak{h})$;
(2) $\mathrm{Vol}(\mathfrak{r}) \leq \theta \mathrm{Vol}(\mathfrak{h})$, *where* $\theta = 1 - \frac{1}{6^{k+1}}$.
(3) $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ *and* $\mathfrak{r}$ *are entirely contained in the neighborhood* $\mathcal{N}_R(\mathfrak{h})$, *with* $R = \varepsilon \mathrm{Vol}(\mathfrak{h})^{1/k}$.

Remark 4.3. In Proposition 4.2, for $k = 2$ all the hypersurfaces $\mathfrak{h}_i$ are spheres, and $\mathfrak{r}$ is a disjoint union of spheres. For $k \geq 3$, following Proposition 3.11 one can only say that each $\mathfrak{h}_i$ is obtained from a k-dimensional sphere by adjoining it with a hypersurface of volume and of filling volume zero, and the same for all the components of $\mathfrak{r}$.

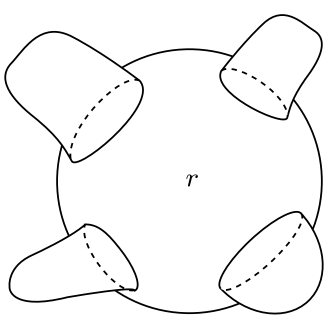

Fig. 1. A sphere partitioned into round spheres and a sphere, r, of smaller area.

Proof. Let $\mathfrak{h}\colon \mathcal{M} \to X$ be a k-dimensional sphere. We denote by $\lambda > 0$ a fixed small constant to be determined during the argument.

If the volume of $\mathfrak{h}$ is zero then we simply take $\mathfrak{r} = \mathfrak{h}$. In what follows we therefore assume that $\mathrm{Vol}(\mathfrak{h}) \geq 1$. For an arbitrary vertex $y \in \mathcal{M}_{\mathrm{Vol}}$ we define $r_0(y)$ to be the maximum of $r \geq 1$ such that $\mathfrak{h}(y, r)$ has volume at least λr^k. Since $\mathfrak{h}(y, 1)$ contains at least one chamber, if we assume that $\lambda \leq \frac{1}{2^k}$, we ensure that for every y the radius $r_0(y)$ is at least 2. Due to the fact that the volume of $\mathfrak{h}(y, r)$ is always an integer, a maximal radius $r_0(y)$ as above exists.

Step 1. Let r_1 be the maximum of the $r_0(y)$ for $y \in \mathcal{M}_{\mathrm{Vol}}$. As all the points y are vertices, there are finitely many of them, hence finitely many values $r_0(y)$ to consider, and one can speak of maximum. Let $y_1 \in \mathcal{M}_{\mathrm{Vol}}$ be such that $r_1 = r_0(y_1)$. Then consider $Y_2 = \mathcal{M}_{\mathrm{Vol}} \backslash \mathcal{M}(y_1, 6r_1)$, the maximum r_2 of the $r_0(y)$ for $y \in Y_2$, and $y_2 \in Y_2$ such that $r_2 = r_0(y_2)$. Assume that we have found inductively $y_1, \ldots, y_j$ and in $Y_{j+1} = \mathcal{M}_{\mathrm{Vol}} \backslash \bigcup_{i=1}^{j} \mathcal{M}(y_i, 6r_i)$ consider the maximal radius $r_0(y)$ denoted by r_{j+1} and a point $y_{j+1} \in Y_{j+1}$ such that $r_0(y_{j+1}) = r_{j+1}$.

We thus find a sequence $y_1, \ldots, y_N$ of vertices and a non-increasing sequence $r_1 \geq r_2 \geq \cdots \geq r_N$ of radii, and we clearly have that for $i \neq j$ the sets $\mathcal{M}(y_i, 2r_i)$ and $\mathcal{M}(y_j, 2r_j)$ do not contain a common chamber. For N large enough we have that $\mathcal{M}_{\mathrm{Vol}} \backslash \bigcup_{i=1}^{N} \mathcal{M}(y_i, 6r_i)$ is empty. For each i, either $\mathrm{Vol}(\mathfrak{h}(y_i, 6r_i))$ equals $\mathrm{Vol}(\mathfrak{h}(y_i, r_i))$ or it is strictly larger than $\mathrm{Vol}(\mathfrak{h}(y_i, r_i))$. In the latter case, we can write:

$$\mathrm{Vol}(\mathfrak{h}(y_i, 6r_i)) \leq \lambda 6^k r_i^k \leq 6^k \mathrm{Vol}(\mathfrak{h}(y_i, r_i)).$$

In both cases we can write:

$$\mathrm{Vol}(\mathfrak{h}) \leq \sum_{i=1}^{N} \mathrm{Vol}(\mathfrak{h}(y_i, 6r_i)) \leq 6^k \sum_{i=1}^{N} \mathrm{Vol}(\mathfrak{h}(y_i, r_i)).$$

We may therefore conclude that the union of the domains $\{\mathfrak{h}(y_i, r_i)\}_{1 \leq i \leq N}$ contains at least $\frac{1}{6^k}$ of the volume of $\mathfrak{h}$.

If for some j we have that $\mathcal{M}_{\mathrm{Vol}} \subset \mathcal{M}(y_j, 6r_j)$ then $\lambda r_j^k \leq \mathrm{Vol}(\mathfrak{h}) \leq \lambda(6r_j)^k$ and this may be seen as a particular case of the above, with the set of domains $\{\mathfrak{h}(y_i, r_i)\}_{1 \leq i \leq N}$ replaced by the singleton set $\{\mathfrak{h}(y_j, r_j)\}$.

In what follows we assume that for every i, $\mathcal{M}_{\mathrm{Vol}}$ is not contained in any $\mathcal{M}(y_i, 6r_i)$.

Step 2. Fix $i \in \{1, 2, \ldots, N\}$ and define the function $\mathcal{V}_i(r) = \mathrm{Vol}(\mathfrak{h}(y_i, r))$. By the definition of r_i we have that $\mathcal{V}_i(r_i) \geq \lambda r_i^k$ while $\mathcal{V}_i(r_i + 1) \leq \lambda(r_i + 1)^k$.

Assume that $\mathrm{Vol}(\partial \mathfrak{h}(y_i, r_i)) > 0$. We may write that

$$\mathcal{V}_i(r_i + 1) \geq \mathcal{V}_i(r_i) + \frac{1}{k}\mathrm{Vol}(\partial \mathfrak{h}(y_i, r_i)),$$

whence, according to the Mean Value theorem,

$$\mathrm{Vol}(\partial \mathfrak{h}(y_i, r_i)) \leq \lambda k^2 (r_i + 1)^{k-1} \leq C \lambda r_i^{k-1},$$

where $C = 2^{k-1} k^2$.

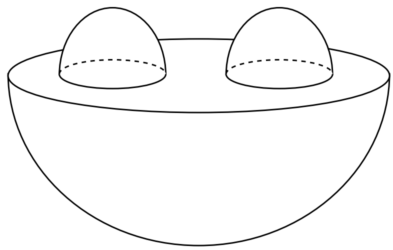

Fig. 2. The sphere $\mathfrak{h}_i$ in the case $k = 2$.

If $\partial\mathfrak{h}(y_i, r_i)$ is empty, i.e. $\mathfrak{h} = \mathfrak{h}(y_i, r_i)$ then the inequality above is automatically satisfied.

If $k = 2$, then $\mathcal{M}(y_i, r_i)$ is either a disk with holes with disjoint interiors or it is the whole sphere, i.e. $\partial\mathcal{M}(y_i, r_i)$ is either empty or a union of circles. We fill the $\mathfrak{h}$-image of each circle in X with a disk of area quadratic in the length of the circle, and transform $\mathfrak{h}(y_i, r_i)$ into a sphere $\mathfrak{h}_i$ of area at most $\mathcal{V}_i(r_i) + C^2\lambda^2 r_i^2$.

The added area is at most $C^2\lambda^2 r_i^2$, which for λ small enough is at most $\frac{1}{6}\lambda r_i^2 \leq \frac{1}{6}\mathcal{V}_i(r_i)$. Therefore the sphere $\mathfrak{h}_i$ has area at most $(1 + \frac{1}{6})\mathcal{V}_i(r_i)$.

On the other hand, we consider the remainder of the complex $\mathcal{M}\setminus\bigcup_{i=1}^N \mathcal{M}(y_i, r_i)$ to which we add $\bigcup_{i=1}^N \partial\mathcal{M}(y_i, r_i)$. We fill in X the $\mathfrak{h}$-image of each circle in $\bigcup_{i=1}^N \partial\mathcal{M}(y_i, r_i)$ with a disk of area quadratic in the length of the circle, and transform $\mathfrak{h}|_{\mathcal{M}\setminus\bigcup_{i=1}^N \mathcal{M}(y_i, r_i)}$ into a disjoint union of 2-spheres, denoted by $\mathfrak{r}$.

If $k \geq 3$ then $\mathcal{M}(y_i, r_i)$ has boundary composed of $(k-1)$-dimensional hypersurfaces and by using a similar argument, the hypothesis on the $(k-1)$-filling and Proposition 3.11(2), when $k \geq 4$, we transform each $\mathfrak{h}(y_i, r_i)$ into a hypersurface $\mathfrak{h}_i$ of volume at most $(1+\frac{1}{6})\mathcal{V}_i(r_i)$, and the remaining complex $\mathcal{M}\setminus\bigcup_{i=1}^N \mathcal{M}(y_i, r_i)$ into a disjoint union composed of a k-sphere and of k-hypersurfaces. Proposition 3.11(1), allows to write each of these hypersurfaces as spheres of the same volume adjoined with k-hypersurfaces contained in $X^{(k-1)}$ with volume and filling volume zero. We again denote this union by $\mathfrak{r}$.

The hypersurface $\mathfrak{h}_i$ has volume $\asymp r_i^k$ and diameter $\preceq r_i$, since $\mathfrak{h}(y_i, r_i)$ has diameter at most $2r_i$, and for the filling domain of each component of $\partial\mathfrak{h}(y_i, r_i)$ the radius is $\preceq r_i$ by Proposition 3.18.

In both cases $k = 2$ and $k \geq 3$ we obtain the following which completes the proof of part (1):

$$\sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}_i) \leq \left(1 + \frac{1}{6}\right) \sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)) \leq 2 \sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)).$$

Since for every $i \neq j$ the sets of chambers in $\mathfrak{h}(y_i, r_i) \cap \mathcal{M}_{\mathrm{Vol}}$ and respectively in $\mathfrak{h}(y_j, r_j) \cap \mathcal{M}_{\mathrm{Vol}}$ are disjoint,

$$\sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)) \leq \mathrm{Vol}(\mathfrak{h}).$$

The union $\mathfrak{r}$ is obtained by replacing the domains $\mathfrak{h}(y_i, r_i)$ with unions of filling domains of their boundary components. It therefore has volume at most $\mathrm{Vol}(\mathfrak{h}) - \sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)) + \frac{1}{6}\sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)) = \mathrm{Vol}(\mathfrak{h}) - \frac{5}{6}\sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i))$. We combine this with the fact that $\sum_{i=1}^N \mathrm{Vol}(\mathfrak{h}(y_i, r_i)) \geq \frac{1}{6^k}\mathrm{Vol}(\mathfrak{h})$ and obtain that $\mathrm{Vol}(\mathfrak{r})$ is at most $\mathrm{Vol}(\mathfrak{h}) - \frac{5}{6^{k+1}}\mathrm{Vol}(\mathfrak{h})$, which implies (2).

To prove (3) it suffices to note that both $\mathfrak{r}$ and $\mathfrak{h}_i$ are obtained by adding minimal volume domains filling hypersurfaces corresponding to connected components of each $\partial\mathcal{M}(y_i, r_i)$. Each of these components has area at most $C\lambda r_i^{k-1}$. This, the hypothesis on the filling in dimension $k-1$ and Proposition 3.18, (2), implies that $\mathfrak{h}_i$ is contained in $\mathcal{N}_{R_i}(\mathfrak{h})$, where $R_i \leq D'\lambda^{\frac{1}{k-1}} r_i$. On the other hand $\lambda r_i^k \leq \mathrm{Vol}(\mathfrak{h})$, whence $r_i \leq \lambda^{-\frac{1}{k}}\mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$. It follows that $R_i \leq D'\lambda^{\frac{1}{k(k-1)}}\mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$, and the latter bound is at most $\varepsilon\mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$, for λ small enough. □

By iterating Proposition 4.2 we obtain that every sphere has a partition composed of round hypersurfaces and hypersurfaces of volume and filling volume zero. This is a key fact which allows one to reduce the filling problem to filling round hypersurfaces, and ultimately to filling round spheres.

Theorem 4.4. *Consider an integer $k \geq 2$, and X a simplicial complex of dimension at least $k+1$. If $k \neq 3$ then assume that $\mathrm{Iso}_{k-1}(x) \leq Bx^k$ for some constant $B > 0$. If $k = 3$ then assume that for every 3-dimensional handlebody V, $\mathrm{Iso}_V(x) \leq Bx^3$, where $B > 0$ is independent of V.*

Then for every $\varepsilon > 0$ there exists a constant $\eta > 0$ such that every k-dimensional sphere $\mathfrak{h}$ has a partition with contours $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ that are η-round hypersurfaces, and contours $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ that are hypersurfaces of volume and filling volume zero such that

(1) $\sum_{i=1}^n \mathrm{Vol}(\mathfrak{h}_i) \leq 2 \cdot 6^{k+1}\mathrm{Vol}(\mathfrak{h})$.
(2) $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ *and* $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ *are contained in the tubular neighborhood* $\mathcal{N}_R(\mathfrak{h})$, *where* $R = \varepsilon\mathrm{Vol}(\mathfrak{h})^{1/k}$.

For $k = 2$ all the hypersurfaces $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ and $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ are spheres.

If moreover X has a bounded quasi-geodesic combing then for every $k \geq 3$ as well, $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ are $\kappa\eta$-round spheres, where κ depends only on the constants of the quasi-geodesic combing.

Remark 4.5. Theorem 4.4 is true also for k-dimensional hypersurfaces $\mathfrak{h}$. Indeed, for $k \geq 3$ the statement follows from Theorem 4.4 for spheres and Proposition 3.11. For $k = 2$ the same argument works and yields a decomposition with $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ and $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ surfaces. The only difference is that, in the proof of Proposition 4.2, $\mathcal{M}(y_i, r_i)$ are subsurfaces with boundary, and after their boundary circles are filled with disks they become closed surfaces $\mathfrak{h}_i$. The remaining $\mathfrak{r}$, obtained from $\mathfrak{h}$ after performing this operation for all i, is likewise a disjoint union of surfaces.

Proof. Proposition 4.2 allows to find, for an arbitrary $\varepsilon' > 0$ a constant $\eta > 0$ such that for every $i \geq 1$, an arbitrary k-dimensional sphere $\mathfrak{h}$ admits a partition with contours $\mathfrak{h}_1^i, \dots, \mathfrak{h}_{n_i}^i, \mathfrak{r}_i$ such that $\mathfrak{h}_1^i, \dots, \mathfrak{h}_{n_i}^i$ are η-round, moreover:

(1) $\mathrm{Vol}(\mathfrak{r}_i) \leq \theta^i \mathrm{Vol}(\mathfrak{h})$;
(2) $\sum_{j=1}^{n_i} \mathrm{Vol}(\mathfrak{h}_j^i) \leq 2 \sum_{\ell=0}^{i-1} \theta^\ell \,\mathrm{Vol}(\mathfrak{h})$;
(3) $\mathfrak{h}_1^i, \dots, \mathfrak{h}_{n_i}^i, \mathfrak{r}_i$ are contained in $\mathcal{N}_{R_i}(\mathfrak{h})$, where $R_i = \varepsilon'(1 + \theta + \cdots + \theta^{i-1})$,

where $\theta = 1 - \frac{1}{6^{k+1}}$.

Indeed, the above can be proved by induction on $i \geq 1$, where the conclusion of Proposition 4.2 yields the initial case $i = 1$.

Assume that we found the required partition for i. We apply Proposition 4.2 to the disjoint union of k-spheres composing $\mathfrak{r}_i$, once the k-dimensional hypersurfaces of volume and filling volume zero are removed.

We obtain $\mathfrak{k}_1, \dots, \mathfrak{k}_m$ η-round hypersurfaces and $\mathfrak{r}_{i+1}$ disjoint union of k-spheres and of k-dimensional hypersurfaces of volume and filling volume zero such that

(1) $\sum_{i=1}^{m} \mathrm{Vol}(\mathfrak{k}_i) \leq 2\mathrm{Vol}(\mathfrak{r}_i) \leq 2\theta^i \mathrm{Vol}(\mathfrak{h})$;
(2) $\mathrm{Vol}(\mathfrak{r}_{i+1}) \leq \theta \mathrm{Vol}(\mathfrak{r}_i) \leq \theta^{i+1} \mathrm{Vol}(\mathfrak{h})$;
(3) $\mathfrak{k}_1, \dots, \mathfrak{k}_m$ and $\mathfrak{r}_{i+1}$ are contained in the tubular neighborhood of $\mathfrak{r}_i$ of radius $\varepsilon' \mathrm{Vol}(\mathfrak{r}_i)^{1/k}$.

We then consider the set of hypersurfaces

$$\left\{\mathfrak{h}_1^{i+1}, \dots, \mathfrak{h}_{n_{i+1}}^{i+1}\right\} = \left\{\mathfrak{h}_1^i, \dots, \mathfrak{h}_{n_i}^i, \mathfrak{k}_1, \dots, \mathfrak{k}_m\right\}.$$

For large enough i, $\mathrm{Vol}(\mathfrak{r}_i) = 0$, thus we obtain the required partition. According to Remark 4.3, for $k = 2$ the hypersurfaces $\mathfrak{h}_i$ are spheres, and $\mathfrak{r}$ is a disjoint union of spheres.

For $k \geq 3$, according to Proposition 3.11 each $\mathfrak{h}_i$ has a partition composed of a k-dimensional sphere $\mathfrak{s}_i$ of dimension k and a hypersurface $\mathfrak{h}_i'$ of volume and of filling volume zero. Moreover, this partition is obtained by filling a sphere σ_i of dimension $k-1$ and volume zero on $\mathfrak{h}_i$ with a ball $\mathfrak{b}_i$ also contained in $X^{(k-1)}$. The existing bounded quasi-geodesic combing allows to construct $\mathfrak{b}_i$ so that its diameter is at most a constant κ (depending on the quasi-geodesic constants) times the diameter of σ_i. This means that if $\mathfrak{h}_i$ is η-round then $\mathfrak{s}_i$ is $\kappa\eta$-round. $\square$

Corollary 4.6. *Consider an integer $k \geq 2$, and a simplicial complex X of dimension at least $k+1$. If $k \neq 3$ then assume that $\mathrm{Iso}_{k-1}(x) \leq Bx^k$ for some constant $B > 0$. If $k = 3$ then assume that for every 3-dimensional handlebody V, $\mathrm{Iso}_V(x) \leq Bx^3$, where $B > 0$ is independent of the handlebody.*

If for some $\eta > 0$ large enough, and some $A' > 0$, all the η-round k-hypersurfaces of volume at most $A'x^k$ have filling volume at most Cx^α with $\alpha \geq k$, then $\mathrm{Iso}_k(x) \leq \xi C x^\alpha$, where ξ is a universal constant.

Remark 4.7. In the above, for $k = 2$ it suffices to require that for every round *k-sphere* of volume at most Ax^k the filling volume is at most Bx^α.

Proof. We consider the set of contours of a partition as provided by Theorem 4.4,

$$\{\mathfrak{h}_1, \ldots, \mathfrak{h}_n, \mathfrak{r}_1, \ldots, \mathfrak{r}_m\}.$$

The problem of filling $\mathfrak{h}$ is thus reduced to the problem of filling the round hypersurfaces $\mathfrak{h}_i$. These are filled by a volume $\leq B\sum_{j=1}^n Vol(\mathfrak{h}_i)^{\frac{\alpha}{k}} \leq B\xi Vol(\mathfrak{h})^{\frac{\alpha}{k}}$. □

This allows to prove a Federer–Fleming type inequality for simplicial complexes, hence for groups of finite type.

Theorem 4.8. *Assume that the simplicial complex X has a bounded quasi-geodesic combing.*

(1) (Federer–Fleming inequality for groups). *For every $k \geq 1, \mathrm{Iso}_k(x) \preceq x^{k+1}$. Moreover for $k = 2$ the supremum of* $\mathrm{Iso}_V(x)$ *over all handlebodies V is $\preceq x^3$.*
(2) *Assume that for some $k \geq 2$ it is known that every round k-sphere of volume at most Ax^k has filling volume at most Bx^α with $\alpha \in [k, k+1)$. Then* $\mathrm{Iso}_k(x) \leq \xi Bx^\alpha$, *where ξ is a universal constant.*

Proof. (1) For $k = 1$ the cone filling inequality implies the quadratic filling inequality. For $k = 2$ consider an arbitrary surface $\mathfrak{h}$ modelled on ∂V in $\mathbb{R}^3$. It can be cut into round surfaces as in proposition 4.2. Indeed, with the notations in the proof of that Proposition, the $\mathfrak{h}(y_i, r_i)$ are modelled on sub-surfaces with boundary of ∂V. By filling the circles composing the boundary of each $\mathfrak{h}(y_i, r_i)$, the surface $\mathfrak{h}$ is cut into round surfaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$, and a disjoint union of surfaces $\mathfrak{r}$ with $\mathrm{Vol}(\mathfrak{r}) \leq \theta \mathrm{Vol}(\mathfrak{h})$. By iterating this decomposition, as in Theorem 4.4, we reduce the problem of filling $\mathfrak{h}$ to that of filling round surfaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$ with $\sum_{j=1}^q \mathrm{Vol}(\mathfrak{h}_j) \leq 2\sum_{\ell=0}^\infty \theta^\ell \mathrm{Vol}(\mathfrak{h})$. Lemma 3.7 allows us to deduce that $\mathrm{FillVol}(\mathfrak{h}_i) \leq B(\mathrm{Vol}(\mathfrak{h}_i))^{\frac{3}{2}}$, where B depends only on the constants of the combing and on η; whence $\mathrm{FillVol}(\mathfrak{h}) \leq B'(\mathrm{Vol}(\mathfrak{h}))^{\frac{3}{2}}$, where B' depends only on B and on λ.

By inducting on k, Theorem 4.6 and Lemma 3.7 allow us to deduce the result for all $k \geq 3$.

(2) follows from (1) and Corollary 4.6. □

4.2. *Filling estimates deduced from those on round unfolded spheres*

In this section we explain how, in the search for filling estimates, the study can be restricted to spheres that are round and unfolded. The latter condition requires that, for a k-sphere of volume at most Ax^k and of diameter at most x, the intersection of the sphere with balls of radius δx centred in each of the sphere's points looks more or less like a k-disk. More precisely, we would like to avoid that, at scale δx, the sphere looks like a long and thin k-dimensional cylindrical surface, and therefore we put the condition that the area enclosed in a ball $B(x,r), r \in (0, \delta x)$, around a point x on the sphere is at least εr^k, for some $\varepsilon > 0$. We begin by defining the "folded part" of a hypersurface, the part that we would like to remove.

Definition 4.9. Given a k-dimensional hypersurface $\mathfrak{h} : \mathcal{M} \to X$, its ε-*folded part at scale* ρ is the set

$$\mathrm{Folded}_\varepsilon(\mathfrak{h}, \rho) = \left\{ v \in \mathcal{M}_{\mathrm{Vol}} \mid \exists\, r \in [1, \rho] \text{ such that } \mathrm{Vol}(\mathfrak{h}(v, r)) \leq \frac{1}{2 \cdot 12^k} \varepsilon r^k \right\}.$$

A hypersurface $\mathfrak{h}$ with $\mathrm{Folded}_\varepsilon(\mathfrak{h}, \rho)$ empty is called ε-*unfolded at scale* ρ. Whenever the parameters are irrelevant, we shall simply say that a hypersurface is *unfolded*.

In the definition above, $\frac{1}{2 \cdot 12^k}\varepsilon$ should be seen as a small positive constant. The multiplicative coefficient $\frac{1}{2 \cdot 12^k}$ has no real relevance for the definition, it is there only to avoid adding a similar multiplicative coefficient in some of the inequalities that follow.

Recall that $\mathcal{M}_{\mathrm{Vol}}$ denotes the set of $\mathfrak{h}$-non-collapsed chambers, and that $v \in \mathcal{M}_{\mathrm{Vol}}$ means that v is a vertex of one of these chambers.

The following result describes how a proportion of the ε-folded part of a k-dimensional sphere can be removed by cutting out round hypersurfaces of volume of order at most $\varepsilon \rho^k$. This will allow to completely remove the folded part via iteration.

Proposition 4.10. (removal of folded parts) *Consider an integer $k \geq 2$. If $k \neq 3$ then assume that* $\mathrm{Iso}_{k-1}(x) \leq Bx^k$ *for some constant $B > 0$. If $k = 3$ then assume that for every handlebody V in $\mathbb{R}^3$,* $\mathrm{Iso}_V(x) \leq Bx^3$, *where $B > 0$ is independent of the handlebody.*

For every $\varepsilon \in (0, 1)$ small enough and every $\rho > 1$ the following holds.

Assume that $k \geq 3$. Then every k-dimensional sphere $\mathfrak{h}$ has a partition with contours $\mathfrak{h}_1, \dots, \mathfrak{h}_q$ and $\mathfrak{r}$, where $\mathfrak{h}_i$ are hypersurfaces and $\mathfrak{r}$ is a disjoint union of k-dimensional spheres adjoined with hypersurfaces of volume and filling volume zero, such that

(1) $0 < \mathrm{Vol}(\mathfrak{h}_i) \leq 2\varepsilon \frac{\rho^k}{6^k}$;
(2) $\mathrm{diam}(\mathfrak{h}_i) \leq \frac{\sigma}{\varepsilon^{\frac{1}{k}}} \mathrm{Vol}(\mathfrak{h}_i)^{\frac{1}{k}}$, *where σ depends on k and the filling constants A, B;*
(3) $\sum_{i=1}^q \mathrm{Vol}(\mathfrak{h}_i) \geq \frac{1}{2 \cdot 12^k}\, \mathrm{card}\, \mathrm{Folded}_\varepsilon(\mathfrak{h}, \rho)$;
(4) $\mathrm{Vol}(\mathfrak{r}) + \frac{1}{2} \sum_{i=1}^q \mathrm{Vol}(\mathfrak{h}_i) \leq \mathrm{Vol}(\mathfrak{h})$.

Assume $k = 2$. Then all the above is true for $\mathfrak{h}_1, \dots, \mathfrak{h}_q$ spheres and $\mathfrak{r}$ a disjoint union of spheres.

Moreover $\mathfrak{h}$ can be taken to be a surface, in which case the above is true for $\mathfrak{r}$ a disjoint union of surfaces.

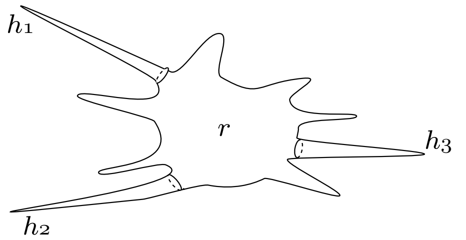

Fig. 3. A sphere partitioned into folded spheres h_1, h_2, and h_3 and a sphere, r, with a smaller folded set.

Proof. We assume that $\mathrm{Folded}_\varepsilon(\mathfrak{h}, \rho) \neq \emptyset$, otherwise we take $\mathfrak{r} = \mathfrak{h}$ and no $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$. We use the notation introduced in Notations 3.16 and 3.17. For every $y \in \mathrm{Folded}_\varepsilon(\mathfrak{h}, \rho)$ consider $\mathcal{V}_y(r) = \mathrm{Vol}(\mathfrak{h}(y, r))$. Consider

$$R_*(y) = \inf \left\{ r \in [1, \rho] \,|\, \mathcal{V}_y(r) < \frac{1}{2 \cdot 12^k} \varepsilon r^k \right\}$$

and

$$r_*(y) = \sup\{ r \in [1, R_*(y)] \,|\, \mathcal{V}_y(r) > \varepsilon r^k \}.$$

We assume that $\varepsilon < 1$, thus for $r = 1$ we know that

$$\mathrm{Vol}(\mathfrak{h}(y, r)) > \varepsilon r^k. \qquad \square$$

Note that $\varepsilon r_*(y)^k \leq \frac{1}{2 \cdot 12^k} \varepsilon R_*(y)^k$, whence $r_*(y) < \frac{R_*(y)}{12}$.

Lemma 4.11. *The radius* $r(y) = r_*(y) + 1$ *satisfies the following*:

(L1) $\mathcal{V}_y(6r(y)) \leq 12^k \mathcal{V}_y(r(y))$;
(L2) $r(y) < \frac{R_*(y)}{6} \leq \frac{\rho}{6}$;
(L3) $\frac{1}{2 \cdot 12^k} \varepsilon r(y)^k \leq \mathcal{V}_y(r(y)) \leq \varepsilon r(y)^k$;
(L4) $\mathrm{Vol}(\partial \mathfrak{h}(y, r(y))) \leq C_k \varepsilon^{\frac{1}{k}} \mathcal{V}_y(r(y))^{\frac{k-1}{k}}$, *where* $C_k = k(k+1)3^{k-1}$.

Proof. In what follows, for simplicity we write r_*, R_* instead of $r_*(y), R_*(y)$.

The inequality

$$\mathcal{V}_y(12 r_*) > 12^k \mathcal{V}_y(r_*),$$

would contradict the maximality of r_*. Therefore we can write

$$\mathcal{V}_y(6r(y)) \leq \mathcal{V}_y(12 r_*) \leq 12^k \mathcal{V}_y(r_*) \leq 12^k \mathcal{V}_y(r(y)).$$

The inequality (L2) follows from the fact that $r(y) = r_*(y) + 1 < \frac{R_*(y)}{12} + 1 < \frac{R_*(y)}{6} \leq \frac{\rho}{6}$. The latter inequalities are true because $1 \leq r_*(y) < \frac{R_*(y)}{12}$, and $R_*(y) \in [1, \rho]$.

Property (L3) follows from the maximality of r_*, and from the fact that $r(y) < \frac{R_*(y)}{6} < R_*(y)$.

We prove (L4). We argue by contradiction and assume the inequality opposite to the one in (L4). We can write that

$$\mathcal{V}_y(r(y)+1) \geq \mathcal{V}_y(r(y)) + \frac{1}{k+1}\mathrm{Vol}(\partial\mathfrak{h}(y,r)) > \varepsilon r_*^k + k\varepsilon^{\frac{1}{k}} 3^{k-1}\varepsilon^{\frac{k-1}{k}} r_*^{k-1}.$$

The right-hand side equals $\varepsilon r_*^k + k\varepsilon(3r_*)^{k-1}$, and the latter is larger than $\varepsilon(r(y)+1)^k$, by a standard application of the Mean Value Theorem, combined with the fact that $r_* \geq 1$. This contradicts the maximality of r_*, hence property (L4) is true as well. □

Proof of Proposition 4.10 continued. Consider r_1 to be the maximum of all $r(y)$ with $y \in \mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)$ and $y_1 \in \mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)$ such that $r(y_1) = r_1$. Then consider r_2 to be the maximum of all the $r(y)$ with $y \in \mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)\backslash\mathcal{M}(y_1, 6r_1)$ and y_2 a point in the previous set such that $r(y_2) = r_2$. Inductively, we find vertices $y_1, \ldots, y_q$ and radii $r_1, \ldots, r_q$ and define r_{q+1} as the maximum of all the $r(y)$ with $y \in \mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)\backslash\bigcup_{i=1}^q \mathcal{M}(y_i, 6r_i)$ and y_{q+1} as a point such that $r(y_{q+1}) = r_{q+1}$.

For large enough q the two sequences thus constructed have the following list of properties:

(P1) $\frac{1}{2\cdot 12^k}\varepsilon r_i^k \leq \mathcal{V}_{y_i}(r_i) \leq \varepsilon r_i^k$;
(P2) $r_i \leq \frac{\rho}{6}$;
(P3) $\mathcal{M}(y_i, 2r_i)$ and $\mathcal{M}(y_j, 2r_j)$ have no chamber in common when $i \neq j$;
(P4) $\mathrm{Vol}(\partial\mathfrak{h}(y_i, r_i)) \leq C_k \varepsilon^{\frac{1}{k}} \mathcal{V}_{y_i}(r_i)^{\frac{k-1}{k}}$, where $C_k = k(k+1)3^{k-1}$;
(P5) $\sum_{i=1}^q \mathcal{V}_{y_i}(r_i) \geq \frac{1}{2\cdot 12^k}\,\mathrm{card}\,\mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)$.

Properties (P1), (P2) and (P4) follow from Lemma 4.11, (L3), (L2) and (L4), respectively, while (P3) follows from the construction of the sequences (y_i) and (r_i). Property (P5) follows for q large enough because the process can continue until $\mathrm{Folded}_\varepsilon(\mathfrak{h},\rho)\backslash\bigcup_{i=1}^q \mathcal{M}(y_i, 6r_i)$ is empty, in which case

$$\mathrm{card}\,\mathrm{Folded}_\varepsilon(\mathfrak{h},\rho) \leq 2\sum_{i=1}^q \mathcal{V}_{y_i}(6r_i) \leq 2\cdot 12^k \sum_{i=1}^q \mathcal{V}_{y_i}(r_i).$$

The last inequality above uses Lemma 4.11, (L1).

Let $i \in \{1, 2, \ldots, q\}$. Assume $k = 2$. If $\mathfrak{h}$ is a sphere then $\mathfrak{h}(y_i, r_i)$ is a disk with other open disks removed from its interior. By filling each boundary circle with a quadratic area one obtains a sphere $\mathfrak{h}_i$ of area $\leq \mathcal{V}_{y_i}(r_i)(1+C_2^2\varepsilon)$, and the remainder $\mathfrak{r}$ is a disjoint union of spheres.

If we take $\mathfrak{h}$ to be a surface, then $\mathfrak{h}(y_i, r_i)$ is a surface with boundary. In that case, again fill each of the boundary circles with a quadratic area. This will yield a surface $\mathfrak{h}_i$ of area $\leq \mathcal{V}_{y_i}(r_i)(1 + C_2^2\varepsilon)$, and the remainder $\mathfrak{r}$ is a disjoint union of surfaces.

If $k = 3$ then $\mathfrak{h}(y_i, r_i)$ is a handlebody with other handlebodies removed from its interior. We fill each of the surfaces that compose $\partial\mathfrak{h}(y_i, r_i)$ with cubic volumes and obtain a 3-dimensional hypersurface $\mathfrak{h}_i$ of volume $\leq \mathcal{V}_{y_i}(r_i)(1 + C_3^{3/2}\varepsilon^{1/2})$; the remainder is a disjoint union of a 3-sphere with 3-dimensional hypersurfaces, and we apply Proposition 3.11 (1). A similar argument works in the case $k \geq 4$.

Note that for ε small enough we obtain in all cases that

$$\mathrm{Vol}(\mathfrak{h}_i) < 2\mathrm{Vol}(\mathfrak{h}(y_i, r_i)). \tag{11}$$

All the properties listed in Proposition 4.10 are satisfied. Indeed:

- property (1) follows from (11), (P1) and (P2);
- inequality (2) is implied by the fact that the diameter of $\mathfrak{h}(y_i, r_i)$ is at most $2r_i$ and by (P1), as well as by (P4) and Proposition 3.18;
- inequality (3) follows from the fact that $\mathrm{Vol}(\mathfrak{h}_i) \geq \mathrm{Vol}(\mathfrak{h}(y_i, r_i))$ and from (P5);
- inequality (4) is an immediate consequence of (11). □

An iteration of Proposition 4.10 allows to find a partition of an arbitrary sphere into unfolded spheres, hypersurfaces of volume and filling volume zero, and hypersurfaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$ with the properties (1), (2) and (4) as in Proposition 4.10 (with $\mathfrak{r}$ in (4) representing the union of the elements in the partition other than $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$).

This is described in detail below.

Proposition 4.12. (folded–unfolded decomposition) *Consider an integer $k \geq 2$. If $k \neq 3$ then assume that $\mathrm{Iso}_{k-1}(x) \leq Bx^k$ for some constant $B > 0$. If $k = 3$ then assume that for every handlebody V in $\mathbb{R}^3, \mathrm{Iso}_V(x) \leq Bx^3$, where $B > 0$ is independent of the handlebody.*

For every $\varepsilon \in (0, 1)$ small enough and for every $\delta > 0$ the following holds.

Assume that $k \geq 3$. Then every k-dimensional sphere $\mathfrak{h}$ admits a partition with contours $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$ and $\mathfrak{r}$, where $\mathfrak{h}_i$ are hypersurfaces and $\mathfrak{r}$ is composed of a disjoint union of k-dimensional spheres which are each adjoined with a hypersurface of volume and filling volume zero, let $\mathfrak{r}' : \mathcal{R}' \to X$ and $\mathfrak{r}'' : \mathcal{R}'' \to X$ denote the simplicial maps representing the union of k-dimensional spheres, respectively the union of adjoined hypersurfaces, such that:

(1) ($\mathfrak{r}'$ *is unfolded*) *every vertex v in $\mathcal{R}'_{\mathrm{Vol}}$ and every $r \in [1, 6\delta\mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}]$ have the property that*

$$\mathrm{Vol}(\mathfrak{r}'(v, r)) \geq \frac{1}{2 \cdot 12^k}\varepsilon r^k;$$

(2) (*volumes of hypersurfaces $\mathfrak{h}_i$ are proportionally small*) *for every i*, $0 < \mathrm{Vol}(\mathfrak{h}_i) \leq 2\delta^k\varepsilon\mathrm{Vol}(\mathfrak{h})$;

(3) ($\mathfrak{h}_i$ *are round*) $\mathrm{diam}(\mathfrak{h}_i) \leq \frac{\sigma}{\varepsilon^{\frac{1}{k}}}\mathrm{Vol}(\mathfrak{h}_i)^{\frac{1}{k}}$, *where* $\sigma = \sigma(A, B, k)$;

(4) (*sum of volumes is controlled by* $\mathrm{Vol}(\mathfrak{h})$)$\mathrm{Vol}(\mathfrak{r}) + \frac{1}{2}\sum_{i=1}^{q}\mathrm{Vol}(\mathfrak{h}_i) \leq \mathrm{Vol}(\mathfrak{h})$.

Assume that $k = 2$. Then all the above is true with $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$ spheres and $\mathfrak{r} = \mathfrak{r}'$ a disjoint union of spheres.

Moreover one can take $\mathfrak{h}$ to be an arbitrary surface, in which case the statement holds with $\mathfrak{h}_1, \ldots, \mathfrak{h}_q$ surfaces and $\mathfrak{r}$ a disjoint union of surfaces.

Proof. The proof is by recursive applications of Proposition 4.10. We apply Proposition 4.10 to the sphere $\mathfrak{h}$ and $\rho = 6\delta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$. We obtain a partition $\mathfrak{r}_1, \mathfrak{h}_1, \ldots, \mathfrak{h}_{q_1}$, where $\mathfrak{r}_1$ is a disjoint union of k-dimensional spheres each adjoined with a hypersurface of volume and filling volume zero. Let $\mathfrak{r}'_1$ be the union of disjoint k-dimensional spheres and $\mathfrak{r}''_1$ denote the union of hypersurfaces of volume and filling volume zero adjoined to each of the spheres.

At step i we apply Proposition 4.10 to the union of spheres $\mathfrak{r}'_{i-1}$ and to $\rho_{i-1} = 6\delta \mathrm{Vol}(\mathfrak{r}_{i-1})^{\frac{1}{k}}$ and we obtain a partition of $\mathfrak{r}_{i-1}$ with contours $\mathfrak{r}_i, \mathfrak{h}_{q_{i-1}+1}, \ldots, \mathfrak{h}_{q_i}$.

We have that

$$\mathrm{Vol}(\mathfrak{r}_i) + \frac{1}{2}\sum_{j=1}^{q_i} \mathrm{Vol}(\mathfrak{h}_j) \leq \mathrm{Vol}(\mathfrak{h}).$$

This in particular implies that the sum must be finite, hence the process must stabilize at some point. Hence at some step i we must find that $\mathrm{Folded}_\varepsilon(\mathfrak{r}'_i, \rho_i)$ is empty.

We take $\mathfrak{r}' = \mathfrak{r}'_i$ and we know that $\mathrm{Folded}_\varepsilon(\mathfrak{r}', 6\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}})$ is empty. □

Note that "unfolded" does not imply "round", because the property of being "unfolded" only takes into account vertices in $\mathcal{M}_{\mathrm{Vol}}$. For instance, in a hypersurface $\mathfrak{h} : \mathcal{M} \to X$ that is ε-unfolded at scale $6\delta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}$, one can take a maximal set S in $\mathcal{M}_{\mathrm{Vol}}$ whose image by $\mathfrak{h}$ is $[12\delta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}]$-separated, and its cardinality must be at most a number $N = N(\varepsilon, \delta)$. Still, the $[12\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}]$-neighborhood of $\mathfrak{h}(S)$ only covers $\mathfrak{h}(\mathcal{M}_{\mathrm{Vol}})$, not $\mathfrak{h}(\mathcal{M})$, therefore no bound on the diameter of the form $(N+2)[24\delta \mathrm{Vol}(\mathfrak{h})^{\frac{1}{k}}]$ can be obtained.

In the next proposition we strengthen the hypotheses: instead of Euclidean isoperimetric inequalities one dimension down, we require the existence of a bounded quasi-geodesic combing. Recall that the latter implies the former by combining Corollary 4.6 and Lemma 3.7.

Proposition 4.13. (decomposition of unfolded into unfolded and round) *Assume that the simplicial complex X has a bounded quasi-geodesic combing and let $k \geq 2$ be an integer.*

For every two numbers ε and δ in $(0,1)$ there exists $N = N(\varepsilon, \delta, k)$ such that the following holds.

If $k \geq 3$ then consider an arbitrary disjoint union of k-dimensional spheres $\mathfrak{r} : \mathcal{R} \to X$ such that $\mathrm{Vol}(\mathfrak{r}) \geq V_0$ for some large enough constant V_0.

Assume that $\mathfrak{r}$ is ε-unfolded at scale $6\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}$, in the sense that

($*$) *every vertex v in $\mathcal{R}_{\mathrm{Vol}}$ and every $r \in [0, 6\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}]$ have the property that*

$$\mathrm{Vol}(\mathfrak{r}(v, r)) \geq \frac{\varepsilon}{2 \cdot 12^k} r^k.$$

Then $\mathfrak{r}$ *has a partition into a union* $\mathfrak{r}_0$ *of hypersurfaces of volume zero and* m *round spheres of dimension* $k, \mathfrak{r}_i : \mathcal{R}^{(i)} \to X$ *with* $i \in \{1, 2, \ldots, m\}$ *and* $m \leq N$, *such that*

(1) *for every* $i \in \{1, \ldots, m\}$, *every vertex* v *in* $\mathcal{R}^{(i)}_{\mathrm{Vol}}$ *and every* $r \in [0, 6\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}]$ *have the property that*

$$\mathrm{Vol}(\mathfrak{r}_i(v, r)) \geq \frac{1}{2 \cdot 12^k} \varepsilon r^k;$$

(2) $\mathrm{diam}(\mathfrak{r}_i) \leq \frac{\kappa}{\varepsilon \delta^{k-1}} \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}$, *where* κ *depends only on the constants* L *and* C *of the combing*;

(3) $\frac{\varepsilon \delta^k}{2} \mathrm{Vol}(\mathfrak{r}) \leq \mathrm{Vol}(\mathfrak{r}_i) \leq \mathrm{Vol}(\mathfrak{r})$.

If $k = 2$ *then* $\mathfrak{r}_0$ *does not appear.*

Moreover, if $\mathfrak{r}$ *is taken to be a disjoint union of surfaces then the statement holds with* $\mathfrak{r}_i, i \in \{1, \ldots, m\}$, *surfaces, and* $\mathfrak{r}_0$ *again does not appear.*

Proof. Note that since X is a simplicial complex with a bounded quasi-geodesic combing, by Theorem 4.8(1), we have a Euclidean bound on its isoperimetric functions.

If $\mathrm{Vol}(\mathfrak{r}) = 0$ then take $\mathfrak{r}_0 = \mathfrak{r}$. We assume that $\mathrm{Vol}(\mathfrak{r}) > 0$.

Let q be the maximal number of connected components of $\mathfrak{r}$ with positive volume. Property $(*)$ implies that q has a uniform upper bound depending on ε. Thus, without loss of generality, we may assume that $\mathfrak{r}$ is one k-dimensional sphere.

Let v be an arbitrary vertex in $\mathcal{R}_{\mathrm{Vol}}$. Let W denote $24\delta \mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}$.

We choose the constant V_0 in the hypothesis large enough so that $W \geq 6$.

Suppose that for every $r \in [W, \frac{17}{\varepsilon \delta^k} W]$, $\mathcal{R}(v, r + W) \backslash \mathcal{R}(v, r)$ contains a chamber from $\mathcal{R}_{\mathrm{Vol}}$.

We divide $[W, \frac{17}{\varepsilon \delta^k} W]$ into consecutive intervals with disjoint interiors $[r_1 - 2W, r_1 + 2W], \ldots, [r_q - 2W, r_q + 2W]$. Each $\mathcal{R}(v, r_i + W) \backslash \mathcal{R}(v, r_i)$ contains a chamber from $\mathcal{R}_{\mathrm{Vol}}$, and for a vertex v_i of that chamber the chambers in $\mathcal{R}(v_i, W)$ are all contained in $\mathcal{R}(v, r_i + 2W) \backslash \mathcal{R}(v, r_i - 2W)$. Therefore the sets of chambers in $\mathcal{R}(v_i, W)$ are pairwise disjoint, and the cardinality of each set intersected with $\mathcal{R}_{\mathrm{Vol}}$ is at least $\frac{\varepsilon}{2 \cdot 12^k} W^k$ by property $(*)$.

It follows that $\mathrm{Vol}(\mathfrak{r})$ is at least $\frac{\frac{17}{\varepsilon \delta^k} W - W}{4W} \frac{\varepsilon}{2 \cdot 12^k} W^k \geq \frac{\frac{16}{\varepsilon \delta^k} W}{8W} \varepsilon \delta^k \mathrm{Vol}(\mathfrak{r}) \geq 2 \mathrm{Vol}(\mathfrak{r})$. This is a contradiction.

We conclude that there exists $r \in [W, \frac{17}{\varepsilon \delta^k} W]$ such that $\mathcal{R}(v, r + W) \backslash \mathcal{R}(v, r)$ does not contain a chamber from $\mathcal{R}_{\mathrm{Vol}}$. The boundary of $\mathcal{R}_1 = \mathcal{R}(v, r + W/2)$ has no volume. Assume for a contradiction that there exists a $(k-1)$-chamber H of the boundary which is sent by $\mathfrak{r}$ onto a $(k-1)$-dimensional simplex of X. Then there exists a non-collapsed chamber containing H, and hence contained in $\mathcal{R}(v, r + W)$, but not in $\mathcal{R}(v, r + W/2)$. This contradicts the hypothesis.

It follows that the boundary of $\mathcal{R}_1 = \mathcal{R}(v, r + W/2)$ has no volume. This boundary is composed of hypersurfaces contained in $X^{(k-2)}$, each can be filled with a domain inside $X^{(k-1)}$ using the combing (see Lemma 3.7). For $k \geq 3$, $\mathfrak{r}_1$ becomes a sphere $\mathfrak{r}_1'\colon \mathcal{R}_1' \to X$ with a hypersurface $\mathfrak{r}_1''$ of volume and filling volume zero adjoined to it, and $\mathfrak{r}\backslash\mathfrak{r}_1$ becomes $\overline{\mathfrak{r}}_1$, a disjoint union of spheres with hypersurfaces of volume and filling volume zero adjoined to them. We denote the disjoint union of spheres by $\overline{\mathfrak{r}}_1'$ and the union of hypersurfaces by $\overline{\mathfrak{r}}_1''$.

Assume $k=2$. Assume $\mathfrak{r}$ is a disjoint union of spheres. Then $\mathcal{R}_1$ is topologically a disc with open discs removed from its interior. The hypothesis that the boundary of $\mathcal{R}_1$ has no volume implies that each boundary circle is sent onto one vertex. Therefore, by replacing $\mathcal{R}$ with the quotient simplicial complex in which the above mentioned boundary circles become points, $\mathfrak{r}_1$ becomes a sphere, while $\mathfrak{r}$ restricted to $\mathcal{R}\backslash\mathcal{R}_1$ becomes a disjoint union of spheres $\overline{\mathfrak{r}}_1'$.

Assume now that $\mathfrak{r}$ is a disjoint union of surfaces. Then $\mathcal{R}_1$ is topologically a surface with boundary. An argument as above implies that by replacing $\mathcal{R}$ with the quotient simplicial complex in which each boundary circle becomes a point, $\mathfrak{r}_1$ becomes a closed surface $\mathfrak{r}_1'$, while $\mathfrak{r}$ restricted to $\mathcal{R}\backslash\mathcal{R}_1$ becomes a disjoint union of closed surfaces $\overline{\mathfrak{r}}_1'$.

We prove that, for all $k \geq 2$, both $\mathfrak{r}_1'$ and $\overline{\mathfrak{r}}_1'$ satisfy the property $(*)$. Indeed, the part added to $\mathfrak{r}_1$ to become $\mathfrak{r}_1'$, and to $\mathfrak{r}$ restricted to $\mathcal{R}\backslash\mathcal{R}_1$ to become the union of $\overline{\mathfrak{r}}_1'$ with $\overline{\mathfrak{r}}_1''$ does not contribute to the volume, it suffices therefore to check $(*)$ for vertices a in $\mathcal{R}_1^{\mathrm{Vol}}$, respectively in $(\mathcal{R}\backslash\mathcal{R}_1)^{\mathrm{Vol}}$.

Let $a \in (\mathcal{R}_1)_{\mathrm{Vol}}$. Then a is in a non-collapsed chamber in $\mathcal{R}(v,r)$, since $\mathcal{R}_1\backslash\mathcal{R}(v,r)$ does not contain non-collapsed chambers. It follows that $\mathcal{R}(a,\frac{W}{4}) \subseteq \mathcal{R}(v,r+\frac{W}{2})$. Thus, for every $t \in [0,\frac{W}{4}]$, $\mathfrak{r}_1(a,t) = \mathfrak{r}(a,t)$, hence $(*)$ is satisfied for $\mathfrak{r}_1$.

Likewise, let a be in $(\mathcal{R}\backslash\mathcal{R}_1)_{\mathrm{Vol}}$. Hence $a \in \mathcal{R}\backslash\mathcal{R}(v,r+W)$. If $\mathcal{R}(a,\frac{W}{4})$ would intersect $\mathcal{R}_1$ in a non-collapsed chamber then a would be in $\mathcal{R}(v,r+W)$, a contradiction. Thus $\mathcal{R}(a,\frac{W}{4})_{\mathrm{Vol}}$ is in $\mathcal{R}\backslash\mathcal{R}_1$ and as before we conclude that $\mathfrak{r}$ restricted to $\mathcal{R}\backslash\mathcal{R}_1$ satisfies $(*)$.

In particular the volume of $\mathfrak{r}_1'$ is at least $\frac{\varepsilon\delta^k}{2}\mathrm{Vol}(\mathfrak{r})$.

The diameter of $\mathfrak{r}_1'$ is at most $\kappa_1(r+W)$, where κ_1 depends on the constants L and C of the combing. An upper bound for $r+W$ is $(\frac{17}{\varepsilon\delta^k}+1)W \leq \frac{18}{\varepsilon\delta^k}24\delta\mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}} = \frac{\kappa_2}{\varepsilon\delta^{k-1}}\mathrm{Vol}(\mathfrak{r})^{\frac{1}{k}}$.

We thus obtain the estimate in (2) for the diameter of $\mathfrak{r}_1'$.

We repeat the argument for $\overline{\mathfrak{r}}_1'$, and find $\mathfrak{r}_2$ etc. The process must stop after finitely many steps because of inequality (3). □

We now state and deduce the main result of this section:

Theorem 4.14. *Let X be a simplicial complex with a bounded quasi-geodesic combing, and let η,ε and δ be small enough positive constants.*

(1) *Let $k \geq 2$ be an integer. If every k-dimensional sphere of volume at most Ax^k that is η-round and ε–unfolded at scale δx, in the sense of Proposition* 4.13(1), *has filling volume at most Bx^α with $\alpha \geq k$, then $\mathrm{Iso}_k(x) \leq Cx^\alpha$, where $C = C(\eta,\varepsilon,\delta)$.*

(2) *If every (closed) surface of volume at most Ax^2 that is η-round and ε-unfolded at scale δx has filling volume at most Bx^α with $\alpha \geq 2$ and B independent of the genus, then* $\mathrm{Iso}_V(x) \leq Cx^\alpha$, *for every handlebody V, where $C = C(\eta, \varepsilon, \delta)$.*

Proof. Proposition 4.13 implies that every disjoint union of k-dimensional spheres (respectively, surfaces), that is of volume at most Ax^k and ε-unfolded at scale δx, has filling volume at most BNx^α, where $N = N(\varepsilon, \delta, k)$.

We prove by induction on n that k-dimensional spheres (respectively surfaces) $\mathfrak{h}$ of volume at most 2^n satisfy

$$\mathrm{FillVol}(\mathfrak{h}) \leq A(\mathrm{Vol}(\mathfrak{h}))^{\frac{\alpha}{k}}$$

for A large enough.

Indeed, for $k = 2$ it suffices to use the decomposition in Proposition 4.12, and to apply the inductive hypothesis to each $\mathfrak{h}_i$. For $k \geq 3$, according to Proposition 3.11, each $\mathfrak{h}_i$ has a partition composed of a sphere $\mathfrak{h}_i'$ and a hypersurface $\mathfrak{h}_i''$ of volume and filling volume zero. We apply the inductive hypothesis to each $\mathfrak{h}_i'$. □

5. Divergence

We begin by recalling a few facts about the 0-*dimensional divergence*. There are several, essentially equivalent versions of 0-dimensional divergence. The first careful study of the notion was undertaken by S. Gersten [20, 21]. The main reference for the first part of this section is [16, §3.1].

Let X denote a geodesic metric space, quasi-isometric to a one-ended finitely generated group (this assumption can be replaced by a weaker technical hypothesis called $(\mathrm{Hyp}_{\kappa,\mathrm{L}})$, but we do not need that generality here). Also, we fix a constant $0 < \delta < 1$.

For an arbitrary triple of distinct points $a, b, c \in X$ we define $\mathrm{div}(a, b, c; \delta)$ to be the infimum of the lengths of paths connecting a, b and avoiding the ball centered at c and of radius $\delta \cdot \mathrm{dist}(c, \{a, b\}$. If no such path exists, define $\mathrm{div}(a, b, c; \delta) = \infty$.

The *divergence function* $\mathrm{Div}^X(n, \delta)$ of the space X is defined as the supremum of all finite numbers $\mathrm{div}(a, b, c; \delta)$ with $\mathrm{dist}(a, b) \leq n$. When the ambient space is clear we omit X from the notation.

It is proven in [16, Lemma 3.4] that, as long as δ is sufficiently small and n sufficiently large, $\mathrm{Div}^X(n, \delta)$ is always defined and, by construction, takes only finite values. In [16, Corollary 3.12], it is proven that, up to the equivalence relation $\asymp$ (which in this case means up to affine functions), the various standard notions of 0-dimensional divergence agree and that the $\asymp$ class of the divergence function is invariant under quasi-isometry. As our main results are only about $\asymp$ classes of functions, it is no loss of generality to assume that the value of n used in the above function is taken sufficiently large so that $\mathrm{Div}^X(n, \delta)$ is defined; hence we will make this assumption for n for the remainder of the paper.

An important feature of the 0-dimensional divergence function is its relationship to the topology of asymptotic cones as described in [16, Proposition 1.1].

We now proceed to discuss an extension to higher dimensions of the divergence function defined above. In an arbitrary dimension, the divergence may be seen as a filling function when moving towards infinity (e.g. when moving closer and closer to the boundary, if a boundary ∂_∞ can be defined).

The notion has been used mostly in the setting of non-positive curvature (see e.g., [10] for a version defined for Hadamard manifolds). Another homological notion of higher dimensional filling was provided by [1], generalizing the definitions in [10] and provided for a finer measurement, allowing, for instance, one to use the invariant to distinguish between various degrees of polynomial divergence. Among other things, in a Hadamard space the divergence can distinguish the rank. Indeed, for a symmetric space X of non-compact type, Div_k grows exponentially when $k = \mathrm{Rank}(X) - 1$ [10, 32], while when $k \geq \mathrm{Rank}(X)$ the divergence $\mathrm{Div}_k = O(x^{k+1})$ [29]. More generally, for a cocompact Hadamard space X and for a homological version of the divergence, defined in terms of integral currents, if $k = \mathrm{Rank}(X) - 1$ then $\mathrm{Div}_k \succeq x^{k+2}$, while if $k \geq \mathrm{Rank}(X)$ then $\mathrm{Div}_k \preceq x^{k+1}$ [40].

In what follows, we define a version of the higher dimensional divergence functions in the setting of simplicial complexes, in particular of groups of type $\mathcal{F}_n$. Therefore, we fix an n-connected simplicial complex X of dimension $n+1$ which is the universal cover of a compact simplicial complex K with fundamental group G. Recall that we assume edges in X to be of length one, and that we endow $X^{(1)}$ with the shortest path metric. We also fix a constant $0 < \delta < 1$. Given a vertex c in X, a k-dimensional hypersurface $\mathfrak{h} : \mathcal{M} \to X$ modelled on ∂V such that $k \leq n-1$, and a number $r > 0$ that is at most the distance from c to $\mathfrak{h}(\mathcal{M}^{(1)})$, *the divergence* of this quadruple, denoted $\mathrm{div}(\mathfrak{h}, c; r, \delta)$, is the infimum of all volumes of domains modelled on V filling $\mathfrak{h}$ and disjoint from $B(c, \delta r)$. If no such domain exists then we set $\mathrm{div}(\mathfrak{h}, c; r, \delta) = \infty$.

Definition 5.1. Let V be a manifold as described in Convention 3.2.

The *divergence function modelled on* V of the complex X, denoted $\mathrm{Div}_V(r, \delta)$, is the supremum of all finite values of $\mathrm{div}(\mathfrak{h}, c; r, \delta)$, where $\mathfrak{h}$ is a hypersurface modelled on ∂V with the distance from c to $\mathfrak{h}(\mathcal{M}^{(1)})$ at least r and $\mathrm{Vol}(\mathfrak{h})$ at most Ar^k.

When V is the $(k+1)$-dimensional unit ball, $\mathrm{Div}_V(r, \delta)$ is denoted $\mathrm{Div}^{(k)}(r, \delta)$, and it is called the *$k$-dimensional divergence function* (or the *kth divergence function*) of X.

In the above, as for the filling functions, we fix the constant $A > 0$ once and for all, and we do not mention it anymore. We note that when $k = 0$, the volume condition is vacuously satisfied and thus the above notion coincides with that of the 0-dimensional divergence given previously.

An immediate consequence of Proposition 3.18 is that when the isoperimetric function Iso_V is smaller than the Euclidean one, the divergence function Div_V coincides with the isoperimetric function. Although quasi-isometric invariance of the higher-divergence functions in not known in general, it is in this setting, since Iso_V is quasi-isometric invariant in general.

Proposition 5.2. *Let V be a manifold as described in Convention* 3.2, *and let ε and δ be small enough positive constants. Assume that* $\mathrm{Iso}_V(x) \leq \varepsilon x^{k+1}$. *Then* $\mathrm{Div}_V(x, \delta) = \mathrm{Iso}_V(x)$ *for every x large enough.*

Proof. Proposition 3.18, (3), implies that $\mathrm{Rad}_V(x) \leq 2L\varepsilon x$ for $x \geq x_\varepsilon$. It follows that for every k-dimensional hypersurface $\mathfrak{h} : \mathcal{M} \to X$ of area at most Ax^k, there exists a filling domain $\mathfrak{d} : \mathcal{D} \to X$ realizing $\mathrm{FillVol}(\mathfrak{h})$ and with the image $\mathfrak{d}(\mathcal{D}^{(1)})$ entirely contained in a tubular neighborhood of $\mathfrak{h}(\mathcal{M}^{(1)})$ of radius $2L\varepsilon x$. Therefore, if $\mathfrak{h}(\mathcal{M}^{(1)})$ is disjoint from a ball $B(c, x)$ then, for x large enough, $\mathfrak{d}(\mathcal{D}^{(1)})$ is disjoint from the (δx)-ball around c, provided that $2L\varepsilon + \delta < 1$. □

More importantly, the cutting arguments that we have described previously allow to reduce the problem of estimating the divergence to hypersurfaces that are round, in the particular case when a bounded combing exists.

Theorem 5.3. *Assume that X is a simplicial complex of dimension n endowed with a bounded quasi-geodesic combing. Let V be a $(k+1)$-dimensional connected compact sub-manifold of $\mathbb{R}^{k+1}$ with connected boundary, where $2 \leq k \leq n-1$.*

For every $\varepsilon > 0$ there exists $\eta > 0$ such that the following holds.

Consider the restricted divergence function $\mathrm{Div}^r_V(x, \delta)$, *obtained by taking the supremum only over hypersurfaces modelled on ∂V that are η-round, of volume at most $2Ax^k$ and situated outside balls of radius x.*

Assume that $\mathrm{Div}^r_V(x, \delta) \leq Br^\beta$ *for some $\beta \geq k+1$ and $B > 0$ universal constant. Then the general divergence function* $\mathrm{Div}_V(x, \delta(1-\varepsilon))$ *is at most $B' r^\beta$ for some $B' > 0$ depending on B, ε, η and X.*

Proof. Let $\mathfrak{h}$ be a hypersurface modelled on ∂V, of volume of most Ax^k and with image outside the ball $B(c, x)$.

According to Theorem 4.4 and Remark 4.5, for every ϵ there exists $\eta = \eta(\varepsilon)$ such that $\mathfrak{h}$ can be decomposed into η-round spheres $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ and hypersurfaces $\mathfrak{r}_1, \dots, \mathfrak{r}_m$ of volume and filling volume zero, all of them contained in $\mathcal{N}_{\varepsilon x}(\mathfrak{h})$.

All the spheres $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ have area at most $2Ax^k$.

We put aside the spheres $\mathfrak{h}_i$ that have volume at most εx^k. For all these, by Proposition 3.18, the filling radius is at most $L\varepsilon x$ for x large enough, where L is a universal constant, therefore they can be filled in the usual way outside $B(c, \delta x)$ if ε is small enough.

In the end we obtain finitely many round components $\mathfrak{h}_1, \dots, \mathfrak{h}_N$ of volume at least εx^k and at most $2Ax^k$, where $N = O(\frac{A}{\varepsilon})$, by Theorem 4.4, (1). Moreover $\mathfrak{h}_1, \dots, \mathfrak{h}_N$ have images outside $B(c, (1-\varepsilon)x)$ If $\mathfrak{h}_1, \dots, \mathfrak{h}_N$ can be filled with a domain of volume at most Bx^β outside $B(c, \delta(1-\varepsilon)x)$, then $\mathfrak{h}$ can be filled outside the same ball with a volume at most $NBx^\beta + A'x^{k+1}$, where the second term comes from the filling of the hypersurfaces with small volume that we had put aside, and from Theorem 4.4. □

Acknowledgments

The authors thank Bruce Kleiner, Enrico Leuzinger and Stefan Wenger for useful comments on an earlier version of this paper. J.B. thanks the Mathematics Departments of Barnard/Columbia and Exeter College/Oxford University for their hospitality during the writing of this paper. The authors thank the anonymous referees for a very careful job and for their helpful suggestions for improving the exposition in the paper.

The research of J.B. was supported as an Alfred P. Sloan Fellow, a Simons Fellow, and by NSF grant #DMS-1006219.

The research of C.D. was supported in part by the EPSRC grant "Geometric and analytic aspects of infinite groups", by the project ANR Blanc ANR-10-BLAN 0116, acronym GGAA, and by the LABEX CEMPI.

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