A threshold for relative hyperbolicity in random right-angled Coxeter groups

Jason Behrstock* R. Altar Çiçeksiz† Victor Falgas-Ravry†

July 14, 2024

Abstract

We consider the random right-angled Coxeter group W_Γ whose presentation graph $\Gamma \sim \mathcal{G}_{n,p}$ is an Erdős–Rényi random graph on n vertices with edge probability $p = p(n)$. We establish that $p = 1/\sqrt{n}$ is a threshold for relative hyperbolicity of the random group W_Γ. As a key step in the proof, we determine the minimal number of pairs of generators that must commute in a right-angled Coxeter group which is not relatively hyperbolic, a result which is of independent interest.

We also show that there is a interval of edge probabilities of width $\Omega(1/\sqrt{n})$ in which the random right-angled Coxeter group has precisely cubic divergence. This interval is between the thresholds for relative hyperbolicity (whence exponential divergence) and quadratic divergence. Moreover, any simple random walk on any Cayley graph of the random right-angled Coxeter group for p in this interval satisfies a central limit theorem.

1 Introduction

In his famous treatise [23], Gromov initiated the study of random groups with the statement that "almost all" groups are hyperbolic. This idea was developed further in work by Gromov and others, who established that for a number of random group models as one varies a model parameter one sees a threshold at which the typical behaviour changes from one regime into another: on one side of the threshold, random groups outputted by the model are typically infinite and hyperbolic, while on the other side of the threshold they are typically finite, see e.g. Gromov [24], Ollivier [28] and Ol'shanskii [29]. The world of these random group models thus splits into two distinct regimes. As we shall see, this stands in some contrast with the model studied in this paper.

The right-angled Coxeter group (or RACG) W_Γ with presentation graph $\Gamma = (V, E)$ is the group with generators V and relations $a^2 = \mathrm{id}$ and $ab = ba$ for all $a \in V$ and $ab \in E$. We will use the Erdős–Rényi random graph model, which is defined as follows: for $n \in \mathbb{N}$ and a sequence of probabilities $p = p(n) \in [0, 1]$, we define a random graph $\mathcal{G}_{n,p}$ on the vertex set $[n] := \{1, 2, \ldots n\}$ by including each of the $\binom{n}{2}$ possible edges with probability $p(n)$, independently at random. We write $\Gamma \sim \mathcal{G}_{n,p}$ to denote the fact that Γ is a random graph with the same distribution as $\mathcal{G}_{n,p}$. To

*Department of Mathematics, Lehman College and The Graduate Center, CUNY, New York, USA. Email: jason.behrstock@lehman.cuny.edu. Research supported by a Simons Fellowship.

†Institutionen för Matematik och Matematisk Statistik, Umeå Universitet, Sweden. Emails: altar.ciceksiz, victor.falgas-ravry@umu.se. Research supported by Swedish Research Council grant VR 2021-03687.

any $\Gamma \sim \mathcal{G}_{n,p}$ one can associate the RACG W_Γ thereby obtaining a model for a *random right-angled Coxeter group.*

The random right-angled Coxeter group was shown by Charney–Farber to be typically infinite and hyperbolic if $pn \to 0$, and finite if $p(1-n)^2 \to 0$ [10, Corollary 2]. However, unlike in the types of models introduced by Gromov, there is a universe of interesting behavior which occurs between these two extremes. In this paper we make progress towards understanding the algebra and geometry of such groups via combinatorial and probabilistic tools.

The past two decades have revealed that generalizations of hyperbolicity provide a powerful framework for studying finitely generated groups. One of the strongest types of such "non-positive curvature" is called *relative hyperbolicity.* This notion was introduced by Gromov [23], then developed by Farb [20] and eventually given many equivalent geometric, topological, and dynamical formulations, see Bowditch [9], Dahmani [12], Druţu and Sapir [17], Osin [30], Sisto [32, 33], Yaman [37], and others. Relative hyperbolicity is both general enough to include many important classes of groups including for example all fundamental groups of finite-volume hyperbolic manifolds and all finitely generated groups with infinitely many ends, while being sufficiently restrictive to yield powerful implications concerning geometric, algebraic, and algorithmic properties, see Arzhantseva, Minasyan and Osin [1], Druţu [16], Druţu and Sapir [19], Farb [20], and many others.

Roughly speaking, a group is *relatively hyperbolic* when all its 'non-hyperbolicity' is confined to certain subgroups that do not interact with each other in any substantial way. In this paper, rather than working directly with the definition of relative hyperbolicity, we will rely on two of the main results of Behrstock, Hagen and Sisto [7, Theorems I and II], which, in the context of RACGs, establishes a necessary and sufficient criterion for relative hyperbolicity in terms of combinatorial properties of the presentation graph.

The main result in this paper is the following which shows that a random RACG transitions from being asymptotically almost surely (a.a.s.) relatively hyperbolic to being a.a.s. not relatively hyperbolic around $p = 1/\sqrt{n}$.

Theorem 1.1. *Let $p = p(n)$ and $\Gamma \sim \mathcal{G}(n,p)$. Then the following hold:*

(i) if $p \leq \frac{1}{4\sqrt{n \log n}}$ then Γ is a.a.s. relatively hyperbolic;

(ii) if $p \geq \frac{\sqrt{\sqrt{6}-2}}{\sqrt{n}}$ then Γ is a.a.s. not relatively hyperbolic.

We note that this threshold provides a wide range of RACGs which are both relatively hyperbolic and one-ended, since the threshold for one-endedness occurs much lower, at $p = \frac{\log(n)}{n}$, see [7, Theorem 3.2].

A powerful geometric invariant for distinguishing finitely generated groups is their *divergence.* This notion was introduced by Gromov [24] and refined by Gersten [21, 22] and, roughly speaking, is a function of r providing a measure of the length of a shortest path needed to connect a pair of points at distance r in a geodesic space while avoiding a ball of radius linear in r around a third point. In Euclidean space the divergence is linear in r, while in hyperbolic and relatively hyperbolic spaces it is exponential in r.

Dani and Thomas [13] gave a construction of RACGs with each possible degree of polynomial divergence. On the other hand, combining work of Behrstock–Hagen–Sisto [7] and Levcovitz [26] it follows that anytime a RACG is not relatively hyperbolic, then its divergence is polynomial.

The degree of polynomial divergence of the Cayley graph of a group can be computed via a geometric invariant called *thickness*[1], which in turn can be characterized in terms of the combinatorics of the presentation graph by results of Behrstock, Hagen and Sisto [7] and Levcovitz [25]. This combinatorial characterization results in a notion of *graph theoretic thickness*, which is how we shall define and apply thickness in this paper (see Section 2.4 for the definition).

We investigate the order of thickness and divergence in the random RACG W_Γ, $\Gamma \sim \mathcal{G}_{n,p}$, in the p–regime when W_Γ is a.a.s. not relatively hyperbolic. The problem of determining the threshold for thickness and divergence of various orders was raised by Behrstock, Hagen and Sisto in [7, Question 1]. In prior work, Behrstock, Falgas–Ravry and Susse [6] established a sharp threshold for thickness of order 1 and quadratic divergence.

Adapting the ideas and arguments of [6], we show that thickness of order 2 and cubic divergence in random RACG occurs a.a.s. for $p = p(n)$ in some interval of values of width $\Omega\left(1/\sqrt{n}\right)$ below the sharp threshold for thickness of order 1 and quadratic divergence.

Theorem 1.2. *There exists an absolute constant $c > 0$ such that the following hold:*

(i) for $p = p(n)$ satisfying $\frac{\sqrt{\sqrt{6}-2-c}}{\sqrt{n}} \leq p(n) \leq 1 - \Omega(\frac{\log n}{n})$, a.a.s. the random graph $\Gamma \sim \mathcal{G}_{n,p}$ is thick of order at most 2;

(ii) for every fixed $\varepsilon > 0$ and $p = p(n) \in [\frac{\sqrt{\sqrt{6}-2-c}}{\sqrt{n}}, \frac{\sqrt{\sqrt{6}-2-\varepsilon}}{\sqrt{n}}]$, a.a.s. the random graph $\Gamma \sim \mathcal{G}_{n,p}$ is thick of order exactly 2, and the random group W_Γ has cubic divergence.

As an immediate consequence of Theorem 1.2 and a result of Chawla, Choi, He and Rafi [11, Proposition 1.1] (a result about RACGs which are thick of order at least 2) we have the following:

Corollary 1.3. *There exists an absolute constant $c > 0$ such that for any fixed ε with $0 < \varepsilon < c$ and $p = p(n) \in [\frac{\sqrt{\sqrt{6}-2-c}}{\sqrt{n}}, \frac{\sqrt{\sqrt{6}-2-\varepsilon}}{\sqrt{n}}]$, a.a.s. the random graph $\Gamma \sim \mathcal{G}_{n,p}$ has the property that the simple random walk on any Cayley graph of the right-angled Coxeter group W_Γ satisfies a central limit theorem and has a normal limit law.*

A key ingredient in the proof of Theorem 1.1 is the following extremal result on the minimum number of edges required for a graph to be thick (meaning thick of some finite order), and which is of independent interest.

Theorem 1.4. *Let Γ be an m–vertex graph which is thick. Then $|E(\Gamma)| \geq 2m - 4$.*

The lower bound in Theorem 1.4 is best possible, since, e.g., for all $m \geq 4$ the complete bipartite graph $K_{2,m-2}$ is thick of order 0 and has exactly $2m - 4$ edges. There are however many other extremal examples: consider the graph on $\{1, 2, \ldots, m\}$ obtained by joining i, j by an edge if and only if $|\lceil i/2\rceil - \lceil j/2\rceil| = 1$ (this graph can be visualized as a 'path of squares', see Figure 1 for an example). Again, it is easy to verify that this graph is thick of order 1 and has exactly $2m-4$ edges. Further, one can identify any pair of non-edges present in two extremal configurations on m_1 and m_2 vertices to obtain a new extremal configuration on $m_1 + m_2 - 2$ vertices — see Figure 2 for an example. In this way one can obtain a wide variety of extremal examples, presenting a particular

[1]In the cases of linear and quadratic divergence the characterization is in terms of *strong algebraic thickness*, for higher orders this notion does not characterize (see [25] for an obstruction), but a weaker invariant called *strong thickness* does characterize.

challenge in the proof of Theorem 1.4. A somewhat surprising feature of our result is that the order of thickness of Γ does not change the minimum number of edges required: in general, the least price to pay for thickness of order 0 is not lower than the least price to pay for thickness of higher orders.

Theorem 1.4, taken together with a result of Behrstock, Hagen and Sisto [7, Theorem I] showing that a RACG is relatively hyperbolic if and only it is not thick, immediately implies the following tight lower bound on the minimum number of commutative relations needed in a RACG to prohibit relative hyperbolicity.

Corollary 1.5. *Consider a graph Γ which we take to be the presentation graph of a right-angled Coxeter group W_Γ. If $|E(\Gamma)| < 2|V(\Gamma)| - 4$, then W_Γ is relatively hyperbolic.*

Questions and conjectures

We believe the following sharpening of Theorem 1.1 might hold. We expect that to prove such a result will require either an entirely new approach or a substantial quantitative strengthening of our techniques:

Conjecture 1.6. *Let $\Gamma \sim \mathcal{G}_{n,p}$. Then the following hold:*

(i) if $p = p(n) = o\left(\frac{1}{\sqrt{n}}\right)$, then a.a.s. the right-angled Coxeter group W_Γ is relatively hyperbolic;

(ii) if $p = p(n) = \Omega\left(\frac{1}{\sqrt{n}}\right)$, then a.a.s. the right-angled Coxeter group W_Γ is thick of order $O(1)$ and has polynomial divergence.

A key difficulty presented by this conjecture is the question of whether or not a random graph $\Gamma \sim \mathcal{G}_{n,p}$ with $p = p(n) = o\left(\frac{1}{\sqrt{n}}\right)$ may have thickness of order $t = t(n)$ finite but tending to infinity as $n \to \infty$.

In an even more challenging direction, we believe that Theorem 1.2 can be generalized, and that for any fixed $k \in \mathbb{Z}_{\geq 3}$ there exist intervals of width $\Omega\left(1/\sqrt{n}\right)$ of values of $p(n)$ for which W_Γ a.a.s. exhibits degree k polynomial divergence. Accordingly, we make the following bold conjecture, strengthening Conjecture 1.6 part (ii):

Conjecture 1.7. *There exists an infinite strictly decreasing sequence of strictly positive real numbers $(\lambda_k)_{k\in\mathbb{N}}$*

$$\lambda_1 = \sqrt{\sqrt{6} - 2} > \lambda_2 > \lambda_3 > \cdots > \lambda_k > \ldots,$$

such that for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ fixed, the following hold:

(i) for $p = p(n) \leq \frac{\lambda_k - \varepsilon}{\sqrt{n}}$, a.a.s. the random graph $\Gamma \sim \mathcal{G}_{n,p}$ is not thick of order at most k;

(ii) for $p = p(n)$ satisfying $\frac{\lambda_k + \varepsilon}{\sqrt{n}} \leq p(n) \leq 1 - \Omega(\frac{\log n}{n})$, a.a.s. the random graph $\Gamma \sim \mathcal{G}_{n,p}$ is thick of order at most k.

Showing the existence of such constants λ_k (let alone determining their values!) seems however an extremely difficult problem even in the case $k = 2$ (although Theorem 1.2 is evidence in this direction). Indeed, it is not even clear what the right rate of decay for the sequence λ_k should be:

does it tend to 0, and if so at what speed? As discussed in Remark 2.9 the property of being thick of order k is not monotonic, which is another difficulty in approaching this conjecture.

Finally, as already discussed by Behrstock, Falgas–Ravry and Susse in [6], it could be fruitful to investigate similar questions to the ones considered in this paper when the presentation graph Γ is taken from a different random graph distribution than that given by the Erdős–Rényi model. Random d-regular graphs or random graph models with clustering such as random intersection graphs or random geometric graphs could be interesting models to study in this way, as the typical geometric properties of the resulting RACG may differ in novel ways from those established in this paper.

Related results

The study of random Right-angled Coxeter groups and their geometric and cohomological properties was initiated in papers of Charney and Farber [10], Davis and Kahle [14] and Behrstock, Hagen and Sisto [7] amongst others. Theorem 1.1 part (i) of this paper represents a dramatic improvement on [7, Theorem III], where it was established that for $p(n) = o(n^{-5/6})$, a.a.s. the random RACG W_Γ, $\Gamma \sim \mathcal{G}_{n,p}$, is relatively hyperbolic: this represents significant progress towards the completion of a systematic picture of the geometric properties of random RACG proposed in [7, Figure 4].

Improving on the earlier work of [7], Behrstock, Falgas–Ravry, Hagen and Susse [5] established a threshold result for thickness of order 1 and quadratic divergence of the random RACG W_Γ, $\Gamma \sim \mathcal{G}_{n,p}$ around $p = 1/\sqrt{n}$. More precisely, they showed that for $p = p(n) \leq 1/\left(\log(n)\sqrt{n}\right)$, W_Γ exhibits a.a.s. at least cubic divergence, while for $p = p(n) \geq 5\sqrt{\log n}/\sqrt{n}$ it has a.a.s. at most quadratic divergence. This result was later sharpened by Behrstock, Falgas–Ravry and Susse, who determined in [6, Theorem 1.6] the precise location of the transition between cubic and quadratic divergence with the following result.

Theorem 1.8 (Behrstock, Falgas–Ravry and Susse). *Let $c > 0$ be fixed, and let $\Gamma \sim \mathcal{G}_{n,p}$. The following hold:*

(i) if $c < \sqrt{\sqrt{6} - 2}$ and $p = p(n) \leq c/\sqrt{n}$, then a.a.s. W_Γ is not algebraically thick of order 1 and has at least cubic divergence;

(ii) if $c > \sqrt{\sqrt{6} - 2}$ and $c/\sqrt{n} \leq p(n)$, then a.a.s. W_Γ is algebraically thick of order at most 1 and has at most quadratic divergence. Moreover if there is some constant $\varepsilon > 0$ such that $p(n) \leq 1-(1+\varepsilon)\frac{\log n}{n}$ then in fact W_Γ is algebraically thick of order exactly 1 and has quadratic divergence.

More recently, Susse [34] studied Morse subgroups and Morse boundaries of random RACG. In forthcoming work [2], the authors of the present paper establish a threshold for connectivity of the square graph of a random graph, a result that settles a conjecture of Susse and has geometric applications to the study of cubical coarse rigidity in random RACG.

Finally, it would be remiss of us to close this section without mentioning the closely related topic of *clique percolation*, which arises in the study of random graphs. Clique percolation first appeared as a simple model for community detection in a work of Derényi, Palla and Vicsek [15], and was then extensively featured in the network science literature [27, 31, 35, 36]. In (k, ℓ)–clique percolation, given a graph Γ one forms an auxiliary graph $K_{k,\ell}(\Gamma)$ whose vertices are the k–cliques of Γ and whose edges are those pairs of k–cliques having ℓ vertices in common. One of the main research questions in the area was to determine the threshold p for the emergence of a giant component in $K_{k,\ell}(\Gamma)$ when $\Gamma \sim \mathcal{G}_{n,p}$ is an Erdős–Rényi random graph. This was achieved in a landmark 2009 paper of Bollobás and Riordan [8], using highly sophisticated branching process techniques that in turn underpinned the arguments deployed by Behrstock, Falgas–Ravry and Susse to study square percolation and determine the threshold for quadratic divergence in random RACG in [6].

1.1 Organization of the paper

In Section 2 we summarize the graph theoretic and probabilistic notions we shall use, as well as discuss divergence and a combinatorial description of thickness that will play a key role in the proof of Theorem 1.4. We then prove our extremal results Theorem 1.4 and Corollary 1.5 in Section 3, while Section 4 is devoted to proofs of Theorems 1.1 and 1.2.

2 Preliminaries

2.1 Graph theoretic notions and notation

We recall here some standard graph theoretic notions and notation that we will use throughout the paper. We write $[n] := \{1, 2, \ldots, n\}$, and $x_1x_2 \ldots x_r$ to denote the r-set $\{x_1, x_2, \ldots, x_r\}$. Given a set S, we let $S^{(r)}$ denote the collection of all subsets of S of size r.

A graph is a pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ is a set of vertices and $E = E(\Gamma)$ is a subset of $V^{(2)}$. All graphs considered in this paper are thus simple graphs, with no loops or multiple edges. We use $v(\Gamma) := |V(\Gamma)|$ and $e(\Gamma) := |E(\Gamma)|$ to denote respectively the order and the size (i.e. the number of vertices and the number of edges) of a graph. Two subgraphs Γ and Γ' are isomorphic if there exists a bijection from $V(\Gamma)$ to $V(\Gamma')$ taking edges to edges and non-edges to non-edges; we denote this fact by $\Gamma \cong \Gamma'$.

A subgraph of Γ is a graph Γ' with $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. We say Γ' is the subgraph of Γ induced by a set of vertices S if $\Gamma' = (S, S^{(2)} \cap E(\Gamma))$, and denote this fact by writing $\Gamma' = \Gamma[S]$. The complement of Γ is the graph $\overline{\Gamma} := (V, V^{(2)} \setminus E)$. Given a vertex x in Γ, we denote the set of its neighbors by $N_\Gamma(x) := \{y \in V(\Gamma) : \ xy \in E(\Gamma)\}$.

A path of length $\ell \geq 0$ in Γ is a sequence of distinct vertices $v_0, v_1, \ldots, v_\ell$ with $v_iv_{i+1} \in E(\Gamma)$ for all $i \in [\ell - 1] \cup \{0\}$. The vertices v_0 and v_ℓ are called the endpoints of the path. Two vertices are said to be connected in Γ if they are the endpoints of some path of finite length. Being connected in Γ is an equivalence relation on the vertices of Γ, whose equivalence classes form the connected components of Γ. If there is a unique connected component, then the graph Γ is said to be connected. A minimally connected subgraph of a connected graph is called a spanning tree. Two useful facts about trees we shall use in our argument are (i) that every connected graph contains a spanning tree, and (ii) that the vertex set of a tree can be ordered starting from any vertex v as $v = v_1$, v_2, …, v_n in such a way that for every $j > 1$ the vertex v_j sends at least one edge into the set $\{v_i : \ 1 \leq i < j\}$.

Finally, we denote by $K_{1,2}$ the *cherry*, or induced path on three vertices, $K_{1,2} := ([3], \{12, 23\})$; we let C_4 denote the *square* (or 4-*cycle*), $C_4 := ([4], \{12, 23, 34, 14\})$; and we write K_m for the *clique* (or *complete graph*)of order m, $K_m := ([m], [m]^{(2)})$.

2.2 Probabilistic notation and tools

We write $\mathbb{P}$ and $\mathbb{E}$ and Var for probability, expectation and variance respectively. We say that a sequence of events $\mathcal{E} = \mathcal{E}(n)$, $n \in \mathbb{N}$ holds *a.a.s.* (asymptotically almost surely) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}(n)) = 1$. Throughout the paper we use standard Landau notation: for functions $f, g : \mathbb{N} \to \mathbb{R}$ we write $f = o(g)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, $f = O(G)$ if there exists a real constant $C > 0$ such that $\limsup_{n\to\infty} |f(n)/g(n)| \leq C$. We further write $f = \omega(g)$ if $g = o(f)$ and $f = \Omega(g)$ if $g = O(f)$.

We shall make repeated use of Markov's inequality: given a non-negative integer-valued random variables X, $\mathbb{P}(X > a) \leq \frac{1}{a+1}\mathbb{E}X$ for any integer $a \geq 0$.

2.3 Divergence

A useful object in geometry is the *divergence function* of a geodesic space. Roughly speaking, this function measures the length of a shortest path between pairs of points at distance r apart in the space when forced to avoid a ball of radius r around a third point. There are a number of different ways to formalize this and it is proven in [18, Proposition 3.5] that the various definitions in the literature carry the same information. For more background on divergence, we refer the reader to, e.g., Behrstock and Druţu [3] or Druţu, Mozes and Sapir [18].

A quasi-isometry preserves the divergence function up to affine functions. Hence the *growth rate of the divergence function* is a quasi-isometry invariant and thus is the same for all Cayley graphs of a given finitely generated group, independently of the choice of the generating set. Accordingly, we will abuse language slightly and just say the divergence function is linear, quadratic, polynomial, etc, when strictly speaking we are talking about the growth rate of the divergence function.

2.4 Thickness and the square graph

An important role in this paper is played by a graph theoretic notion of thickness. *Graph-theoretic thickness* was introduced by Behrstock, Hagen and Sisto in [7] as a combinatorial way of capturing a geometric/algebraic property of groups called *algebraic thickness*.

This latter notion of thickness first appeared in the context of geometric group theory in work of Behrstock, Druţu and Mosher [4]. Algebraic thickness is defined inductively. Roughly speaking, the base level of this property (algebraic thickness of order zero) holds in groups that geometrically look like the direct product of two infinite groups. Higher levels of this property hold for groups that are a union of subgroups, each of which look like direct products and which can be chained together through infinite diameter intersections. (When the chaining can be done in a nice group-theoretic way this leads to algebraic thickness, in this paper, we only require adjacent steps have infinite intersection, which corresponds to a more general invariant called *thickness*.)

As we explain below, the graph theoretic thickness of a presentation graph completely encodes the thickness of the associated right-angled Coxeter group. Accordingly, throughout this paper, without creating any ambiguity, we thus use the term 'thickness' to simultaneouly refer to both notions. We now formally define thickness for graphs.

Given graphs Γ_1, Γ_2, the *join* of Γ_1 and Γ_2 is the graph $\Gamma_1 * \Gamma_2$ obtained by taking the disjoint union of Γ_1 and Γ_2 and adding a complete bipartite graph between them. In other words, $V(\Gamma_1 * \Gamma_2) := V(\Gamma_1) \sqcup V(\Gamma_2)$ and $E(\Gamma_1 * \Gamma_2) := E(\Gamma_1) \cup E(\Gamma_2) \cup \{\{v_1, v_2\} : v_1 \in V(\Gamma_1), v_2 \in V(\Gamma_2)\}$. Note that the RACG of a join of two graphs is the direct product of the RACGs associated to the two factors of the join. So for example the square C_4, which is the join of two non-edges, corresponds to the direct product of two free groups of rank two on generators of order 2.

With this in hand, we now give the base case of our inductive definition of thickness of order zero in graphs. By [7, Proposition 2.11] this graph theoretic notion corresponds to the associated RACG being algebraically thick of order 0.

Definition 2.1 (Thickness of order 0)**.** *A graph* Γ *is* thick of order 0 *if there exists a partition* $V(\Gamma) = A \sqcup B$ *of its vertex set for which neither* $\Gamma[A]$ *nor* $\Gamma[B]$ *is a clique and such that* $\Gamma = \Gamma[A] * \Gamma[B]$.

Identifying vertex sets in Γ *with the corresponding induced subgraphs, we say that a subset* $S \subseteq V(\Gamma)$ *is* thick of order 0 *if the induced subgraph* $\Gamma[S]$ *is thick of order* 0*. Further, we say that* S *is a* maximal *thick of order* 0 *subset if for every* S' *with* $S \subsetneq S' \subseteq V(\Gamma)$ *we have that* S' *is not thick of order* 0*.*

Thus a subset $S \subset V(\Gamma)$ is thick of order 0 if and only if there exists a bipartition $S = A \sqcup B$ such that the associated sets $S_A = A^{(2)} \cap E(\overline{\Gamma})$ and $S_B = B^{(2)} \cap E(\overline{\Gamma})$ are both non-empty (ensuring $\Gamma[A]$, $\Gamma[B]$ are non-cliques) and $\Gamma[A \cup B] = \Gamma[A] * \Gamma[B]$. Further, S is a maximal thick of order 0 subset if in addition for every such partition $A \sqcup B$ and every $v \in V(\Gamma) \setminus S$, there is some $a \in A$ and $b \in B$ where neither av and bv are edges of Γ.

To inductively define higher levels of thickness, we shall glue together subsets which are thick of lower orders. Accordingly, we define the following, which will form the base level for the inductive definition.

Definition 2.2 (T_0: the level 0 subsets)**.** *The* level 0 subsets of Γ, $T_0(\Gamma)$*, will denote the collection of sets* $S \subseteq V(\Gamma)$ *such that* S *is a maximal thick of order* 0 *subset. We refer to elements of* $T_0(\Gamma)$ *as* level 0 components *of* Γ*.*

Given $S \in T_0(\Gamma)$ it will often be convenient to consider the associated set of non-edges $S^{(2)} \cap E(\overline{\Gamma})$. Note that $T_0(\Gamma)$ is just a collection of sets and not a graph. Below we shall construct a sequence of auxiliary graphs $T_k(\Gamma)$, $k \in \mathbb{Z}_{\geq 1}$. Each of these graphs will have as its vertex set the collection $E(\overline{\Gamma})$ of non-edges of Γ, with connected components merging as we increase the value of k according to specific rules, which we specify below.

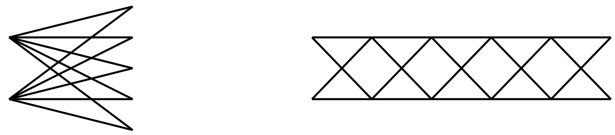
The first of these auxiliary graphs is known as the *square graph*, as it encodes the induced squares of Γ and their pairwise interactions.

Definition 2.3 (T_1: the square graph)**.** *Given a graph* Γ*, the* square graph of Γ*, denoted by* $T_1(\Gamma)$*, is defined as follows. The vertex set of* $T_1(\Gamma)$ *is the collection* $E(\overline{\Gamma})$ *of non-edges of* Γ*. The edges of* $T_1(\Gamma)$ *consist of those pairs of non-edges* $f, f' \in E(\overline{\Gamma})$ *such that the* 4*–set of vertices* $f \cup f'$ *induces a copy of the* square C_4 *in* Γ*.*

We refer to connected components in $T_1(\Gamma)$ *as square components, or level* 1 *components. Given such a level* 1 *component* C*, we define its support to be* $\mathrm{supp}_1(C) := \bigcup_{f \in C} f$*, the collection of vertices in* $V(\Gamma)$ *belonging to some* $f \in C$*. Further, we define the latch-set of* C *to be the collection of pairs from* $\mathrm{supp}_1(C)$ *that are non-edges in* Γ, $\mathrm{latch}_1(C) := E(\overline{\Gamma}) \cap (\mathrm{supp}_1(C))^{(2)}$.

Remark 2.4. *An equivalent definition of the square graph of* Γ *is to let* $T_1(\Gamma)$ *be the graph on* $E(\overline{\Gamma})$ *obtained by replacing each level* 0 *component* S *by a clique on* $S^{(2)} \cap E(\overline{\Gamma})$ *(and replacing multi-edges by single edges), whence the connection to* $T_0(\Gamma)$*.*

Remark 2.5. *For certain applications in geometric group theory, it is natural to work with another auxiliary graph closely related to (but distinct from) the square graph* $T_1(\Gamma)$*, namely the line graph*



$K_{2,5}$ A 'path of squares'

Figure 1: examples of a thick of order 0 graph (left) and a thick of order 1 graph (right).

of $T_1(\Gamma)$, denoted by $S(\Gamma)$, see [5, 13]. This other graph (which in [5, 13] is referred to as the square graph) has the induced squares of Γ as its vertices, and as its edges those pairs of induced squares having a diagonal in common.

The square graph given in Definition 2.3 was introduced by Behrstock, Falgas–Ravry and Susse in [6] as a more natural object to study from a combinatorial viewpoint. As noted in [6, Remark 1.2] the two non-equivalent definitions of a square graph carry essentially the same information, so working with one rather the other is primarily a question of which definition is convenient for the application one has in mind.

The study of the properties of $S(\Gamma)$ and $T_1(\Gamma)$ when Γ is a random graph is known as *square percolation*, by analogy with the well-studied *clique percolation* model from network science mentioned in the introduction.

Having defined the level 0 subsets and the level 1 graph, we will define the level k graphs inductively after first giving a preliminary definition.

Definition 2.6 (Suspension). *Given $f = \{u_1, u_2\} \in E(\overline{\Gamma})$, we let $\operatorname{susp}(f) := \{v \in V(\Gamma) : v \in N_\Gamma(u_1) \cap N_\Gamma(u_2)\}$ denote the collection of common neighbors of the endpoints of f, or, equivalently, the collection of all vertices v such that the 3–set of vertices $\{v\} \cup f$ induces a copy of the cherry $K_{1,2}$ in Γ. We refer to $\operatorname{susp}(f)$ as the suspension based at f.*

Note that if a (sub)graph Γ is a suspension then there exists a partition $V(\Gamma) = A \sqcup B$ of its vertex set for which $|A| = 2$, $\Gamma[A]$ is a non-edge, and $\Gamma = \Gamma[A] * \Gamma[B]$. If additionally $\Gamma[B]$ is a clique, then we call Γ a *strip* (sub)graph.

The level k graph will be built by merging lower level components if they have a non-edge of Γ in common; more formally:

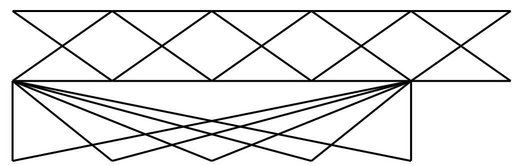
Definition 2.7 (T_k, $k \geq 2$: the level k graph). *For $k \geq 1$, we define $T_{k+1}(\Gamma)$ as a graph on the vertex set $E(\overline{\Gamma})$ (the non-edges of Γ) by joining $f_1, f_2 \in E(\overline{\Gamma})$ by an edge if for $i \in \{1, 2\}$ there exist level k components C_1, C_2 such that $f_i \in \operatorname{latch}_k(C_i)$ and $\operatorname{latch}_k(C_1) \cap \operatorname{latch}_k(C_2) \neq \emptyset$.*

Given a connected component C in T_{k+1}, we define its support to be

$$\operatorname{supp}_{k+1}(C) := \bigcup_{f \in C} f \cup \operatorname{susp}(f),$$

i.e. the collection of vertices in $V(\Gamma)$ that either belong to a member f of C or to a suspension based at some $f \in C$. Further, we define the latch-set of C to be $\operatorname{latch}_{k+1}(C) := E(\overline{\Gamma}) \cap \left(\operatorname{supp}_{k+1}(C)\right)^{(2)}$.

Definition 2.8 (Thickness). *Given a collection of non-edges $C \subseteq E(\overline{\Gamma})$ that forms a level k connected component for some $k \geq 0$, we say that C is a level k_0 component if k_0 is the least integer*



A thick union of $K_{2,5}$ and a path of squares

Figure 2: an example of a thick of order 2 graph

$k \geq 0$ such that C forms a connected component in T_k. We say that such a component has full support if $\mathrm{supp}_{k_0}(C) = V(\Gamma)$ *or, equivalently, if* $\mathrm{latch}_{k_0}(C) = E(\overline{\Gamma})$*. If this occurs, we say that the graph Γ is* thick of order k_0*. More generally, we say that Γ is* thick *if it is thick of some finite order.*

Remark 2.9 (Non-monotonicity under the addition/removal of edges)**.** *The thickness properties of the graphs $T_k(\Gamma)$ are* not *monotone under the addition of edges to Γ. Indeed, adding an edge to Γ could potentially create some new induced squares in Γ (and thus new edges in $T_1(\Gamma)$) as well as destroy existing ones. As an example, consider the complete bipartite graph $\Gamma = K_{2,n-2}$ with one part of size 2 and another of size $n-2$. This graph is clearly thick of order 0. However if we delete any edge from Γ, or if we add an edge inside the part of size 2, then Γ ceases to be thick of any order.*

The non-monotonicity of thickness under the addition or deletion of edges make its study a delicate matter, and explains in particular why many results on the geometric properties of random RACG feature both a lower and an upper bound on the edge probability p required for a certain property to hold a.a.s..

To compute the order of thickness of RACGs, the following definition of hypergraph index was introduced by Levcovitz in [25]. Levcovitz proved in [26, Theorem A] that a graph Γ has finite hypergraph index if and only if the corresponding RACG W_Γ is thick of order k; moreover, this occurs if and only if the RACG W_Γ has divergence which is polynomial of degree $k+1$. This strengthens earlier partial results from [4, 13, 7, 5].

Definition 2.10 (Hypergraph index)**.** *Let Γ be a graph. Let $\Psi(\Gamma)$ denote the collection of subsets of $V(\Gamma)$ that induce strip subgraphs in Γ. Let Λ_0 be the (non-uniform) hypergraph whose vertex set is $V(\Gamma)$ and whose set of hyperedges is $T_0(\Gamma) \cup \Psi(\Gamma)$.*

For all integers $n \geq 0$, inductively define a hypergraph Λ_{n+1} as follows. Introduce an equivalence relation $\cong_n$ on pairs of hyperedges of Λ_n by setting $E \equiv_n E'$ when there exists a finite sequence of hyperedges $E = E_1, E_2, \ldots, E_k, E_{k+1} = E'$ such that for every $i \in [k]$ the subgraph of Γ induced by $E_i \cap E_{i+1}$ contains a non-edge. The hypergraph Λ_{n+1} then has $V(\Gamma)$ as its vertex set, and a hyperedge $\bigcup_{E \in C} E$ for each $\equiv_n$-equivalence class C.

If $T_0(\Gamma) \neq \emptyset$ we define the hypergraph index *of Γ to be the smallest non-negative integer n for which Λ_n contains $V(\Gamma)$ as a hyperedge. When no such n exists or when $T_0(\Gamma) = \emptyset$ we define the* hypergraph index *of Γ to be ∞.*

Our definition of higher order thickness for graphs is defined so that it encodes this notion. Indeed, it is not hard to see that the hypergraph index of Levcovitz coincides precisely with the order of thickness as defined in Definition 2.8. Thus Levcovitz's results imply the order of thickness of a

RACG W_Γ coincides precisely with the order of graph theoretic thickness of its presentation graph Γ, and the two notions can be conflated in the context of RACGs.

3 An extremal result for thickness in graphs

We shall prove the following strengthening of Theorem 1.4 using an inductive strategy. Recall that the union $\Gamma_1 \cup \Gamma_2$ of the graphs Γ_1 and Γ_2 is the graph Γ with $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.

Theorem 3.1. *Let $k \in \mathbb{Z}_{\geq 0}$. Let $\Gamma_1 = (V_1, E_1)$ be a graph, and let $\Gamma_2 = (V_2, E_2)$ be either a copy of the cherry $K_{1,2}$ or a thick of order at most k graph. Set $I := V_1 \cap V_2$, and suppose that $\Gamma_2[I]$ is not a clique of order at most 2. Then*

$$e(\Gamma_1 \cup \Gamma_2) \geq e(\Gamma_1) + 2|V_2 \setminus V_1|. \tag{3.1}$$

In particular if Γ is thick of order at most $k+1$, then $e(\Gamma) \geq 2v(\Gamma) - 4$.

Remark 3.2. *If $\Gamma_2 \cong K_{1,2}$, then (3.1) is easily seen to hold: if $\Gamma_2[I]$ is a non-clique, then it must either be the entirety of Γ_2, or it must consist of the unique non-edge in Γ_2, and in both cases the claimed upper bound holds. Thus the content of Theorem 3.1 lies in the case where Γ_2 is thick of order at most k; the formulation including the cherry as a special case is nonetheless useful as it will make the formulation of our inductive argument easier.*

Our proof of Theorem 3.1 follows an inductive strategy that relies on the following characterization of thick of order $k+1$ graphs.

Proposition 3.3. *Let Γ be a thick of order $k+1$ graph, for some $k \geq 0$. Then for some $T > 0$, there exists a collection of induced subgraphs of Γ, $\Gamma_i = (V_i, E_i)$ for $i \in [T]$, and a tree $\mathcal{T}$ on the vertex set $[T]$ such that:*

(a) for every $i \in [T]$, Γ_i is a copy of the cherry $K_{1,2}$ or is thick of order at most k;

(b) for every $ij \in E(\mathcal{T})$, the induced subgraph $\Gamma[V_i \cap V_j]$ is not complete;

(c) $\bigcup_{i=1}^{T} V_i = V$.

Proof. Follows immediately from the combinatorial characterization of thickness in right-angled Coxeter groups given in [7, Theorem II] which implies that any thick graph gives a connected graph whose vertices are associated to full induced subgraphs which are thick of lower order and which are connected by an edge if the associated graphs overlap in a non-clique. The result then follows from the fact that every connected graph contains a spanning tree as a subgraph. □

Proposition 3.4. *Let $k \in \mathbb{Z}_{\geq 0}$. Suppose that every thick of order at most k graph Γ satisfies $e(\Gamma) \geq 2v(\Gamma) - 4$ and that in addition (3.1) holds for all graphs Γ_1 and all thick of order at most k graphs Γ_2. Then every graph Γ' that is thick of order at most $k+1$ satisfies $e(\Gamma') \geq 2v(\Gamma') - 4$.*

Proof. Let Γ' be a thick of order at most $k+1$ graph, and let $\Gamma'_i = (V_i, E_i)$, $i \in [T]$ be the collection of induced thick of order at most k subgraphs of Γ' whose existence is guaranteed by Proposition 3.3. Since $\mathcal{T}$ is a tree, we can relabel the indices of the Γ'_i so that for every $j > 1$ there exists $i \in [j-1]$ with $ij \in E(\mathcal{T})$.

By our assumption on graphs which are thick of order at most k, we have $e(\Gamma'_1) \geq 2|V_1| - 4$. Now for each $j > 1$, there exists $i \in [j-1]$ such that $ij \in E(\mathcal{T})$ and hence (by Proposition 3.3.(b)) $\Gamma'[V_i \cap V_j]$ is non-complete. Since Γ'_j is thick of order at most k or a copy of the cherry $K_{1,2}$, our assumption allows us to apply (3.1), which yields:

$$e\left(\left(\bigcup_{j'<j} \Gamma'_{j'}\right) \cup \Gamma'_j\right) \geq e\left(\bigcup_{j'<j} \Gamma'_{j'}\right) + 2\left|V_j \setminus \left(\bigcup_{j'<j} V_{j'}\right)\right|.$$

Iterating $T-1$ times, we get the desired bound on $e(\Gamma')$:

$$e(\Gamma') \geq e\left(\bigcup_{j=1}^{T} \Gamma'_i\right) \geq 2|V_1| - 4 + \sum_{j>1} 2\left|V_j \setminus \left(\bigcup_{j'<j} V_{j'}\right)\right| = 2\left|\bigcup_{j=1}^{T} V_i\right| - 4 = 2v(\Gamma') - 4,$$

where the last equality follows from Proposition 3.3.(c). $\square$

Thus armed, we begin our proof of Theorem 3.1 by proving the base case $k = 0$.

Proposition 3.5. *Let Γ be thick of order 0, and let $V(\Gamma) = A \sqcup B$ be a partition of its vertex-set such that neither of $\Gamma[A]$ nor $\Gamma[B]$ is a clique and $\Gamma = \Gamma[A] * \Gamma[B]$. Setting $a := |A|$ and $m := v(\Gamma)$, we have*

$$e(\Gamma) \geq a(m-a) \geq 2(m-2) = 2m - 4.$$

Proof. As neither of $\Gamma[A]$ and $\Gamma[B]$ is a clique, we have $2 \leq a \leq m-2$. Now clearly $e(\Gamma) \geq |A| \cdot |B| = a(m-a)$, which for $a \in [2, m-2]$ is at least $2(m-2)$. $\square$

Proposition 3.6. *The statement of Theorem 3.1 holds for $k = 0$.*

Proof. We have already established in Proposition 3.5 that a thick of order 0 graph on m vertices must have at least $2m-4$ edges. By Proposition 3.4, it is thus enough to establish (3.1) for $k = 0$. Further, by Remark 3.2 we need not consider the case where Γ_2 is a cherry.

Let $\Gamma_1 = (V_1, E_1)$ be a graph, and let $\Gamma_2 = (V_2, E_2)$ be a thick of order 0 graph. Set $I := V_1 \cap V_2$, and suppose this is a non-empty subset of V_2. If $V_2 \subseteq V_1$, then (3.1) holds trivially. We may thus assume $V_2 \setminus V_1$ is non-empty. Since Γ_2 is thick of order zero it admits a partition $V(\Gamma_2) = A \sqcup B$ where $\Gamma_2[A]$ and $\Gamma_2[B]$ are both non-complete and $\Gamma_2 = \Gamma_2[A] * \Gamma_2[B]$. Denote $a := |A|$, $b := |B|$, $x := |A \cap I|$ and $y := |B \cap I|$. Assume without loss of generality that $x \leq y$.

Then we have the following key inequality:

$$\begin{aligned} e(\Gamma_1 \cup \Gamma_2) - e(\Gamma_1) - 2|V_2 \setminus V_1| &\geq |B \setminus I|\,(|A| - 2) + |A \setminus I|\,(|B \cap I| - 2) \\ &= (b-y)(a-2) + (a-x)(y-2). \end{aligned} \tag{3.2}$$

To see why this inequality holds note: every vertex in A sends edges to every vertex in $B \setminus I$, giving us a first set of $|B \setminus I| \cdot |A|$ edges; that every vertex in $B \cap I$ sends edges to every vertex in $A \setminus I$, giving us a second set of $|A \setminus I| \cdot |B \cap I|$ edges disjoint from the first; and, that none of the edges in these two sets belong to Γ_1 since all of them are incident to a vertex of $(A \sqcup B) \setminus I = V(\Gamma_2) \setminus V(\Gamma_1)$.

To conclude the proof, it suffices to show that either $\Gamma_2[I] \cong K_\epsilon$ for some $\epsilon \in \{1, 2\}$ or that the expression on the right hand side of (3.2) is non-negative. By thickness of order 0, we know that

$a = |A|$ and $b = |B|$ are both at least 2, and we also have $a \geq x$ and $b \geq y$. In particular the first term in (3.2) above is always non-negative.

Consider now the second term. We have $2y \geq x + y = |I| > 0$, whence $y > 0$. If $y = 1$ and $x = 0$, then $\Gamma_2[I] \cong K_1$ and we are done. Similarly if $y = 1$ and $x = 1$, then $\Gamma_2[I] \cong K_2$ (since Γ_2 contains as a subgraph the complete bipartite graph on the bipartition $A \sqcup B$) and we are done. Finally if $y \geq 2$, then $(b-y)(a-2) + (a-x)(y-2) \geq 0$, as desired. The proposition follows. $\square$

Proof of Theorem 3.1. We perform induction on k. We established the base case $k = 0$ in Proposition 3.6. For the inductive step, assume we have proved the theorem for all $k \leq K$, for some $K \geq 0$. By Proposition 3.4, it suffices then to show (3.1) holds for $k = K + 1$.

Let $\Gamma_1 = (V_1, E_1)$ be an arbitrary graph, and let $\Gamma_2 = (V_2, E_2)$ be a thick of order $K+1$ graph. Set $I = V_1 \cap V_2$, and suppose this is a non-empty subset of V_2 with $\Gamma_2[I] \ncong K_1, K_2$. Note first of all that if $|V_2 \setminus V_1| = 0$, then (3.1) holds trivially. We may thus assume that $V_2 \setminus V_1$ is non-empty.

Applying Proposition 3.3 to Γ_2, provides a collection of induced subgraphs $\Gamma_{2,i} = (V_{2,i}, E_{2,i})$, $i \in [T]$, of our graph Γ_2, each of which is either a $K_{1,2}$ or thick of order at most K, together with a tree $\mathcal{T}$ on $[T]$ satisfying properties (a)—(c) from the statement of Proposition 3.3. Here we must consider two cases.

Case 1: suppose first of all that there is some i_0 such that

$$e(\Gamma_1 \cup \Gamma_{2,i_0}) \geq e(\Gamma_1) + 2|V_{2,i_0} \setminus V_1|. \tag{3.3}$$

Reordering the indices of the $\Gamma_{2,i}$ as necessary, we may assume that $i_0 = 1$ and that for every $j > 1$ there exists $i \in [j-1]$ with $ij \in E(\mathcal{T})$ (the existence of such an ordering is implied by the fact that $\mathcal{T}$ is a tree), which in turn implies $\Gamma_2[V_{2,i} \cap V_{2,j}]$ is non-complete (by property (b)). In particular, we must have that $\Gamma_2\left[\left(V_1 \cup \bigcup_{i<j} V_{2,i}\right) \cap V_{2,j}\right]$ is a graph on at least two vertices not isomorphic to K_1 or K_2. Applying our inductive hypothesis $T-1$ times and appealing to (3.3), we conclude that

$$\begin{aligned} e(\Gamma_1 \cup \Gamma_2) \geq e\left(\Gamma_1 \cup \left(\bigcup_j \Gamma_{2,j}\right)\right) &\geq e(\Gamma_1 \cup \Gamma_{2,1}) + \sum_{j>1} 2\left|V_{2,j} \setminus \left(V_1 \cup \left(\bigcup_{j'<j} V_{2,j'}\right)\right)\right| \\ &\geq e(\Gamma_1) + 2\left|\bigcup_{j \geq 1} V_{2,j} \setminus V_1\right| = e(\Gamma_1) + 2|V_2 \setminus V_1|, \end{aligned}$$

and (3.1) holds as required.

Case 2: suppose that (3.3) does not hold for any $i_0 \in [T]$. By our inductive hypothesis this implies the following:

$(\star)$ for every $i \in [T]$, $\Gamma_2[V_1 \cap V_{2,i}]$ is a clique on at most two vertices

By property (c), there exists some i_0 such that $V_{2,i_0} \cap V_1$ is a non-empty subset of $I = V_1 \cap V_2$. Reordering the indices of the $\Gamma_{2,i}$ as necessary, we may assume that $i_0 = 1$, that $|V_1 \cap V_{2,1}| \geq |V_1 \cap V_{2,i}|$ for all $i \in [T]$, and that for every $j > 1$ there exists $i \in [j-1]$ with $ij \in E(\mathcal{T})$.

Now $\Gamma_2[V_{2,1} \cap V_1] = \Gamma_{2,1}[V_{2,1} \cap V_1]$, which by $(\star)$ above is a clique on at least one and at most two vertices. Since $\Gamma_2[V_2 \cap V_1]$ is not a clique on at most two vertices, it follows by property (c) again that there is some $j > 1$ such that $(V_{2,j} \cap V_1) \setminus (V_{2,1} \cap V_1)$ is non-empty. Let j_0 be the least such j.

Consider now the graph $\Gamma_{2,[j_0]} = \bigcup_{i=1}^{j_0} \Gamma_{2,i}$, and write $V_{2,[j_0]} := \bigcup_{i=1}^{j_0} V_{2,i}$ for its vertex set. Since for every $j > 1$ there exists $i \in [j-1]$ with $ij \in E(\mathcal{T})$, it follows from property (b) and the definition of thickness that $\Gamma_{2,[j_0]}$ is thick of order at most $K+1$. By the 'in particular' part of our inductive hypothesis, we have

$$E(\Gamma_{2,[j_0]}) \geq 2\left|V_{2,[j_0]}\right| - 4. \tag{3.4}$$

Set $I_0 := V_1 \cap V_{2,1}$ and $I_1 := V_1 \cap V_{2,j_0}$, so that $V_1 \cap V_{2,[j_0]} = I_0 \cup I_1$ by the minimality of j_0. Further we have $I_0 \setminus I_1$ and $I_1 \setminus I_0$ both non-empty — indeed, the latter follows by definition of j_0, and the former from the maximality of $|V_1 \cap V_{2,i}|$.

Since the elements of $I_1 \setminus I_0$ do not appear in any $V_{2,i}$ with $i \in [j_0 - 1]$, and since the elements of $I_0 \setminus I_1$ do not belong to V_{2,j_0} it follows that there is no edge from $I_1 \setminus I_0$ to $I_0 \setminus I_1$ in $\Gamma_{2,[j_0]}$. By the inclusion-exclusion principle and inequality (3.4), we have

$$\begin{aligned} e(\Gamma_1 \cup \Gamma_{2,[j_0]}) &\geq e(\Gamma_1) + E(\Gamma_{2,[j_0]}) - e\left(\Gamma_{2,[j_0]}[I_0 \cup I_1]\right) \\ &\geq e(\Gamma_1) + 2\left|V_{2,[j_0]}\right| - 4 - e\left(\Gamma_{2,[j_0]}[I_0 \cup I_1]\right) \\ &\geq e(\Gamma_1) + 2\left|V_{2,[j_0]}\right| - 4 - \binom{|I_0 \cup I_1|}{2} + |I_0 \setminus I_1| \cdot |I_1 \setminus I_0|. \end{aligned} \tag{3.5}$$

Claim 3.7. $2|V_{2,[j_0]}| - 4 - \binom{|I_0 \cup I_1|}{2} + |I_0 \setminus I_1| \cdot |I_1 \setminus I_0| \geq 2|V_{2,[j_0]} \setminus V_1|$.

Proof. Set $t := |I_0 \cup I_1|$ and $N := |V_{2,[j_0]}|$, so that $|V_{2,[j_0]} \setminus V_1| = N - t$.

Since I_0 and I_1 have size at least 1 and at most 2 (by $(\star)$), and since $I_0 \Delta I_1 \neq \emptyset$ we have $2 \leq t \leq 4$. Furthermore, $t = 4$ is possible if and only I_0 and I_1 are disjoint sets of size two.

If $2 \leq t \leq 3$, then we have

$$2|V_{2,[j_0]}| - 4 - \binom{|I_0 \cup I_1|}{2} + |I_0 \setminus I_1| \cdot |I_1 \setminus I_0| \geq 2N - 4 - \binom{t}{2} + 1 = 2(N-t) - \frac{(t-2)(t-3)}{2},$$

which is equal to $2(N-t)$ as desired. On the other hand if $t = 4$, then we have

$$2|V_{2,[j_0]}| - 4 - \binom{|I_0 \cup I_1|}{2} + |I_0 \setminus I_1| \cdot |I_1 \setminus I_0| = 2N - 4 - \binom{4}{2} + 4 = 2(N-4) + 2 > 2(N-4),$$

as required. The claim follows. □

Combining Inequality (3.5) and Claim 3.7, we deduce that

$$e(\Gamma_1 \cup \Gamma_{2,[j_0]}) \geq e(\Gamma_1) + 2|V_{2,[j_0]} \setminus V_1|. \tag{3.6}$$

We can now conclude the proof of this case much as we did in Case 1: for every $j > j_0$ there exists $i \in [j-1]$ with $ij \in E(\mathcal{T})$, which in turn implies $\Gamma_2[V_{2,i} \cap V_{2,j}]$ is non-complete. In particular, we must have that $\Gamma_2\left[\left(V_1 \cup \bigcup_{i<j} V_{2,i}\right) \cap V_{2,j}\right]$ is a graph on at least two vertices not isomorphic to K_1 or K_2. Applying our inductive hypothesis $T - j_0$ times and appealing to (3.6), we conclude that

$$\begin{aligned} e\left(\Gamma_1 \cup \Gamma_2\right) \geq e\left(\Gamma_1 \cup \left(\bigcup_j \Gamma_{2,j}\right)\right) &\geq e\left(\Gamma_1 \cup \Gamma_{2,[j_0]}\right) + \sum_{j>j_0} 2\left|V_{2,j} \setminus \left(V_1 \cup \left(\bigcup_{j'<j} V_{2,j'}\right)\right)\right| \\ &\geq e(\Gamma_1) + 2\left|\bigcup_{j\geq 1} V_{2,j} \setminus V_1\right| = e(\Gamma_1) + 2|V_2 \setminus V_1|, \end{aligned}$$

and (3.1) holds as required.

□

4 Thresholds in random right-angled Coxeter groups

4.1 Relative hyperbolicity: proof of Theorem 1.1

The key novelty in this paper is in Theorem 1.1 part (i) and its proof, which we shall now give. By results of Behrstock Hagen and Sisto [7] and Levcovitz [25] discussed in the introduction, a RACG W_Γ is relatively hyperbolic if and only if its presentation graph Γ fails to be thick. Thus part (i) of Theorem 1.1 is implied by the following stronger theorem giving upper bounds on the order of thick components in $\Gamma \sim \mathcal{G}_{n,p}$.

Theorem 4.1. *Let $p = p(n) \leq \frac{1}{4\sqrt{n \log n}}$ and $\Gamma \sim \mathcal{G}(n,p)$. Then a.a.s. for every $k \in \mathbb{N}$, every component in $T_k(\Gamma)$ has support of size at most $\log n$. In particular, Γ is a.a.s. not thick of order k.*

We prove Theorem 4.1 in an inductive fashion by combining our extremal result, Theorem 1.4, with the following simple result about random graphs.

Proposition 4.2. *Let $p = p(n) \leq \frac{1}{4\sqrt{n \log n}}$ and $\Gamma \sim \mathcal{G}_{n,p}$. Then a.a.s. for every $m \in [\log n, 2\log n]$, every m–vertex subset of Γ supports at most $2m-5$ edges.*

Proof. Let X_m denote the number of m–vertex subsets in Γ supporting at least $2m-4$ edges. Set $X := \sum_{m \in [\log n, 2\log n]} X_m$. Applying the inequalities $\binom{N}{r} \leq \left(\frac{eN}{r}\right)^r$ and $2m - 4 \geq \frac{2e}{3}m$ for $m \geq 7$, we have that for all $n \geq e^7$,

$$\mathbb{E}(X) = \sum_{m=\lceil \log n \rceil}^{\lfloor 2\log n \rfloor} X_m \leq \sum_{m=\lceil \log n \rceil}^{\lfloor 2\log n \rfloor} \binom{n}{m} \binom{\binom{m}{2}}{2m-4} p^{2m-4} \leq \sum_{m=\lceil \log n \rceil}^{\lfloor 2\log n \rfloor} \left(\frac{ne}{m}\right)^m \left(\frac{3mp}{4}\right)^{2m-4}$$
$$\leq \sum_{m=\lceil \log n \rceil}^{\lfloor 2\log n \rfloor} \left(\frac{9em}{256 \log n}\right)^m \left(\frac{16 n \log n}{m^2}\right)^2 = O\left(\frac{1}{\log n}\right).$$

where in the last inequality we used the fact that the function

$$x \mapsto x \log\left(\frac{256}{9ex}\right) = x\left(8\log(2) - 2\log(3) - 1 - \log x\right)$$

is strictly greater than 2 in the interval $x \in [1,2]$. It follows from Markov's inequality that a.a.s. $X = 0$ and thus Γ contains no m–vertex subset supporting more than $2m-5$ edges for any $m \in [\log n, 2\log n]$, as claimed. □

Proof of Theorem 4.1. Let $\mathcal{E}$ denote the event that for every $m \in [\log n, 2\log n]$, every m–set of vertices in Γ supports at most $2m-5$ edges. By Proposition 4.2, $\mathcal{E}$ occurs a.a.s., so it is enough to show the conclusions of the theorem hold conditional on $\mathcal{E}$.

Suppose therefore that $\mathcal{E}$ occurs. We shall then prove by induction on $k \geq 0$ that every component of $T_k(\Gamma)$ has support of size at most $\log n$ (which implies both parts of Theorem 4.1).

For the base case $k = 0$, suppose that Γ contains an induced thick of order 0 subgraph on more than $\log n$ vertices. Let $A \sqcup B$ be any partition of this induced subgraph such that $\Gamma[A]$ and $\Gamma[B]$

are both non-complete and $\Gamma[A \sqcup B] = \Gamma[A] * \Gamma[B]$. Then, provided n is sufficiently large, there exists $A' \subseteq A$ and $B' \subseteq B$ such that $\Gamma[A'], \Gamma[B']$ are both non-complete, $\Gamma[A' \sqcup B'] = \Gamma[A'] * \Gamma[B']$ and $\log n \leq |A'| + |B'| \leq 2 \log n$. In other words, $\Gamma[A' \sqcup B']$ is an induced thick of order 0 subgraph of Γ on m vertices, for some m with $\log n \leq m \leq 2 \log n$. By Proposition 3.5, we have $e(\Gamma[A' \sqcup B']) \geq 2m - 4$, contradicting our assumption that $\mathcal{E}$ holds. Thus $T_0(\Gamma)$ contains no level 0 component supported on more than $\log n$ vertices, as required.

For the inductive step, suppose a component C in $T_{k+1}(\Gamma)$ has support of size at least $\log n$. Observe that C is obtained by successively glueing together components or cherries $C'_1, C'_2, \ldots$ from $T_k(\Gamma)$, each of which has support of size at most $\log n$, and that the sequence of glueing can be done in such a way that for every i, the subgraph of Γ induced by the union $U_i := \bigcup_{j=1}^{i} \mathrm{supp}(C'_j)$ is thick of order at most $k+1$.

In particular, we must have $|U_{i+1}| < |U_i| + \log n$ for every i, and hence there must be a least i_0 such that $|U_{i_0}| > \log n$ satisfying in addition $m := |U_{i_0}| < 2 \log n$. Now the m–vertex induced subgraph $\Gamma[U_{i_0}]$ is thick of order at most $k+1$, and hence by Theorem 1.4 must support at least $2m - 4$ edges. Since $m \in [\log n, 2 \log n]$, this again contradicts our assumption that $\mathcal{E}$ holds.

It follows by induction that, conditional on the a.a.s. event $\mathcal{E}$, for all integers $k \geq 1$, $T_0(\Gamma)$ contains no component with support of size greater than or equal to $\log n$. In particular Γ is a.a.s. not thick. □

Part (ii) of Theorem 1.1 follows directly from Theorem 1.2 part (i), which is proven below. (In fact, for $p \geq (\sqrt{\sqrt{6} - 2} + \varepsilon)/\sqrt{n}$, the statement of part (ii) already follows from Theorem 1.8 part (ii), so we only appeal to Theorem 1.2 to remove the ε.)

4.2 Thickness of order two: proof of Theorem 1.2

To prove Theorem 1.2, we will employ a twist on the argument used in [6] to prove Theorem 1.8. The crux of the proof of Theorem 1.8 lay in the analysis of an exploration process for the square graph $T_1(\Gamma)$ and its comparison with a supercritical Bienaymé–Galton–Watson branching process. We describe that exploration process below and explain how a slight modification of it allows us to explore thick of order 2 rather than thick of order 1 components, and to keep the associated branching process supercritical for p a little below the threshold for thickness of order 1.

The exploration process introduced in [6, Section 6.1] is as follows. Starting at time $t = 0$ from an induced square of $\Gamma \sim \mathcal{G}_{n,p}$ on $\{v_1, v_2, v_3, v_4\}$ with non-edges $v_1 v_3$ and $v_2 v_4$ one defines a set of discovered vertices $D_0 = \{v_1, v_2, v_3, v_4\}$, an (ordered) set of active pairs $A_0 = \{v_1 v_3, v_2 v_4\}$ and a set of reached pairs $R_0 = \emptyset$. At every time $t \geq 0$, D_t is a subsets of $V(\Gamma)$, while A_t and R_t are disjoint subsets of $E(\overline{\Gamma}) \cap (D_t)^{(2)}$, i.e. A_t and R_t are disjoint subsets of non-edges in the subgraph of Γ induced by the set of discovered vertices D_t.

At each time step $t \geq 0$ of our exploration process, we proceed as follows:

1. **If $|R_t| + |A_t|$ is large**, meaning $|R_t| + |A_t| > (\log n)^4$, then we terminate the process and output LARGE STOP.

2. **If there are no active pairs left** (i.e. if $A_t = \emptyset$), then we terminate the process and output EXTINCTION STOP.

3. **Otherwise**, we select the first active pair $x_1 y_1 \in A_t$, which by construction induces a C_4 in Γ with some pair $F_t \in A_t \cup R_t$. For every undiscovered vertex $z \in V(\Gamma) \setminus D_t$, we test whether or

not z sends an edge of Γ to both of x_1 and y_1; in this way we form a set $Z_t := \{z \in V(\Gamma) : \{x_1 z, y_1 z\} \subset E(\Gamma)\}$.

Finally we update our triple (D_t, A_t, R_t) by setting $D_{t+1} = D_t \cup Z_t$, $A_{t+1} = (A_t \setminus \{x_1, y_1\}) \cup \big((F_t \cup Z_t)^{(2)} \setminus E(\Gamma)\big)$ and $R_{t+1} = R_t \cup \{x_1 y_1\}$.

Note that our update rule under 3. above ensures $A_t \cup R_t$ is a collection of non-edges of Γ which lie in the same component of the square graph $T_1(\Gamma)$ and in particular preserves the property that every pair in A_t induces a square with some pair $F_t \in D_t^{(2)}$.

The key result in [6, Section 6.2] is that for any $\varepsilon > 0$ fixed and any $\lambda = \lambda(n)$ satisfying $\sqrt{\sqrt{6}-2} + \varepsilon \leq \lambda \leq 5\sqrt{\log n}$, for $p = \lambda/\sqrt{n}$ the above exploration process is supercritical, and that a.a.s. a strictly positive proportion of induced squares in Γ are part of large square components, in the sense that their non-edges are part of components of size at least $(\log n)^4$ in $T_1(\Gamma)$; this is the content of [6, Lemma 6.2].

Supercriticality for the process follows from the fact that as long as the number of discovered vertices D_t is small, the number $|A_{t+1} \setminus A_t|$ of active pairs discovered at each time step of the process (i.e. the offspring distribution) stochastically dominates a random variable $X' \sim \binom{Z'+2}{2} - 1$, where $Z' \sim \mathrm{Binom}(n - o(n), p^2 - o(p^2))$. For $p = \lambda/\sqrt{n}$ and $\lambda = \theta(1)$, we have $\mathbb{E}X' = \frac{\lambda^4}{2} + 2\lambda^2 + o(1)$, which for fixed $\lambda > \sqrt{\sqrt{6}-2}$ is at least $1 + \eta$ for some fixed $\eta > 0$. For such λ, we can thus apply standard results from branching process theory to conclude that the extinction probability is at most $1 - \delta$, for some fixed $\delta > 0$, i.e. that with probability bounded away from zero our exploration process ends with a LARGE STOP.

Having shown that a.a.s. a strictly positive proportion of squares lie in large square components, it easily followed that a.a.s. a positive proportion of non-edges of Γ belong to large square-components of $T_1(\Gamma)$ (see [6, Corollary 6.3]). With this fact in hand, a.a.s. thickness of order 1 for Γ was proved in [6, Section 6.3-6.5] via a somewhat elaborate vertex sprinkling argument relying on Janson's inequality and partition arguments. That part of the proof, however, only required $p(n) = \Omega(1/\sqrt{n})$ together with the aforementioned fact that a.a.s. $\Omega(n^2)$ non-edges of Γ belong to square components of order at least $(\log n)^4$ in $T_1(\Gamma)$.

To prove Theorem 1.2, it thus suffices to show that there exists some absolute constant $c > 0$ such that for $\sqrt{\sqrt{6}-2} - c \leq \lambda \leq 5\sqrt{\log n}$, a.a.s. a strictly positive proportion of non-edges of Γ lie in thick of order 2 components of size at least $(\log n)^4$. As we observe below, this can be done with a modification of the exploration algorithm that increases the expected number of offspring and thus allows the exploration process to remain supercritical a little below $p = \sqrt{\sqrt{6}-2}/\sqrt{n}$. As the technical modification of the analysis from [6] is fairly straightforward, and yields an upper bound on the threshold for thickness of order 2 which we do not believe is optimal, we only sketch the argument and leave the details to the reader.

Given an active pair $x_1 y_1 \in A_t$, observe that if there is a quadruple of vertices $\{x_1, x_3, y_2, y_3\} \subseteq D_t$ such that setting $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ we have that the subgraph of Γ induced by $X \sqcup Y$ is the complete bipartite graph on $X \sqcup Y$ with the edge $x_1 y_1$ removed, then the pairs $x_i x_j$ and $y_i y_j$ all lie in the same component of $T_2(\Gamma)$ as $x_1 y_1$ (since $\Gamma[X \sqcup Y]$ is thick of order 1 and has $x_1 y_1$ as a member of its latch-set but not as the diagonal of an induced square), and may thus be added to A_{t+1} if we are looking to explore $T_2(\Gamma)$ rather than $T_1(\Gamma)$. This leads us to make the following modification of Step 3:

3′ **Otherwise**, we select the first active pair $x_1 y_1 \in A_t$, which by construction induces a C_4 in Γ

with some pair $F_t \in A_t \cup R_t$. For every undiscovered vertex $z \in V(\Gamma) \setminus D_t$, we test whether or not z sends an edge of Γ to both x_1 and y_1; in this way we form a set $Z_t := \{z \in V(\Gamma) : \{x_1 z, y_1 z\} \subset E(\Gamma)\}$.

Next, for every pair of pairs $\{x_2x_3, y_2y_3\}$ drawn from $V(\Gamma) \setminus (D_t \cup Z_t)$, we test wether we have $x_iy_j \in E(\Gamma)$ for all $(i,j) \neq (1,1)$ and $x_2x_3, y_2y_3' \notin E(\Gamma)$ both holding; if this is the case, we say x_2x_3 and y_2y_3 are bridge pairs, and we denote by B'_t the collection of all such bridge pairs.

Finally we update our triple (D_t, A_t, R_t) by setting $D_{t+1} = D_t \cup Z_t$, $A_{t+1} = (A_t \setminus \{x_1, y_1\}) \cup \big((F_t \cup Z_t)^{(2)} \setminus E(\Gamma)\big) \cup B'_t$ and $R_{t+1} = R_t \cup \{x_1y_1\}$.

The arguments of [6] are readily adapted to show that $\mathbb{E}B'_t \geq \frac{1}{2}\left(\binom{n-o(n)}{2}\right)^2 p^8(1-p)^6$ and that as a result the expected number of offspring in our modified exploration process when $p = \lambda/\sqrt{n}$ and $\lambda = \theta(1)$ is $\frac{\lambda^4}{2} + 2\lambda^2 + \frac{\lambda^8}{8} + o(1)$. In particular, there exist constants $c, \eta > 0$ such that for $\sqrt{\sqrt{6}-2} - c \leq \lambda \leq 2\sqrt{\sqrt{6}-2}$ the expected number of offspring in our modified process is at least $1+\eta$, whereupon the rest of the machinery from [6] can be deployed essentially without any further alterations to ensure the a.a.s. existence of a strictly positive proportion of non-edges in large components of $T_2(\Gamma)$, and, from there, a.a.s. thickness of order 2. Theorem 1.2 follows.

Acknowledgements

The initial research that led to this paper was carried out when the first author visited the second and third author in Umeå in January 2023 at the occasion of the third Midwinter Meeting in Discrete Probability. The hospitality of the Umeå mathematics department as well as the financial support of the Simons Foundation which made this visit possible are gratefully acknowledged. In addition the research of the first author was supported by the Simons Foundation as a Simons Fellow while the research of the second and third authors is supported by the Swedish Research Council grant VR 2021-03687.

References

[1] G. Arzhantseva and A. Minasyan. The SQ-universality and residual properties of relatively hyperbolic groups. *J. Algebra*, 315(1):165–177, 2007.

[2] J. Behrstock, R. A. Çiçeksiz, V. Falgas-Ravry Connectivity for square percolation and unique cubical coarse medians in random right-angled Coxeter groups. In preparation.

[3] J. Behrstock and C. Druţu. Divergence, thick groups, and short conjugators. *Illinois J. Math.*, 58(4):939–980, 2014.

[4] J. Behrstock, C. Druţu, and L. Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Mathematische Annalen*, 344(3):543–595, 2009.

[5] J. Behrstock, V. Falgas-Ravry, M.F. Hagen, and T. Susse. Global structural properties of random graphs. *International Mathematics Research Notices*, 2018(5):1411–1441, 2018.

[6] J. Behrstock, V. Falgas-Ravry, and T. Susse. Square percolation and the threshold for quadratic divergence in random right-angled Coxeter groups. *Random Structures & Algorithms*, 60(4):594–630, 2022.

[7] J. Behrstock, M.F. Hagen, and A. Sisto. Thickness, relative hyperbolicity, and randomness in Coxeter groups. *Algebr. Geom. Topol.*, 17(2):705–740, 2017.

[8] B. Bollobás and O. Riordan. Clique percolation. *Random Structures Algorithms*, 35(3):294–322, 2009.
[9] B.H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.
[10] R. Charney and M. Farber. Random groups arising as graph products. *Alg. Geom. Topol.*, (12):979–996, 2012.
[11] K. Chawla, I. Choi, V. He, and K. Rafi. Random walks on groups and superlinear divergent geodesics. arXiv:2310.18506.
[12] F. Dahmani. *Les groupes relativement hyperboliques et leurs bords.* PhD thesis, Université de Strasbourg, 2003. Available at `https://www-fourier.ujf-grenoble.fr/~dahmani/`.
[13] P. Dani and A. Thomas. Divergence in right-angled Coxeter groups. *Trans. Amer. Math. Soc.*, 367(5):3549–3577, 2015.
[14] M.W. Davis and M. Kahle. Random graph products of finite groups are rational duality groups. *J. Topol.*, 7(2):589–606, 2014.
[15] I. Derényi, G. Palla, and T. Vicsek. Clique percolation in random networks. *Phys. Rev. Lett.*, 94:160202, Apr 2005.
[16] C. Druţu. Relatively hyperbolic groups: geometry and quasi-isometric invariance. *Comment. Math. Helv.*, 84(3):503–546, 2009.
[17] C. Druţu and M.V Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.
[18] C. Druţu, S. Mozes, and M.C. Sapir. Divergence in lattices in semisimple Lie groups and graphs of groups. *Trans. Amer. Math. Soc.*, 362(5):2451–2505, 2010.
[19] C. Druţu and M.V. Sapir. Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups. *Adv. Math.*, 217(3):1313–1367, 2008.
[20] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.
[21] S.M. Gersten. Quadratic divergence of geodesics in CAT(0)-spaces. *Geom. Funct. Anal.*, 1(4):37–51, 1994.
[22] S.M. Gersten. Divergence in 3-manifold groups. *Geom. Funct. Anal.*, 4(6):633–647, 1994.
[23] M. Gromov. Hyperbolic groups. In S. Gersten, editor, *Essays in group theory*, volume 8 of *MSRI Publications*. Springer, 1987.
[24] M. Gromov. Asymptotic invariants of infinite groups, Geometric Group Theory, vol. 2 (Niblo, Roller, eds.). *LMS Lecture Notes Ser*, 182, 1993.
[25] I. Levcovitz. A quasi-isometry invariant and thickness bounds for right-angled Coxeter groups. *Groups Geom. Dyn.*, 13(1):349–378, 2019.
[26] I. Levcovitz. Characterizing divergence and thickness in right-angled Coxeter groups. *J. Topol.*, 15(4):2143–2173, 2022.
[27] M. Li, Y. Deng, and B.-H. Wang. Clique percolation in random graphs. *Phys. Rev. E (3)*, 92(4):042116–042122, 2015.
[28] Y. Ollivier. Sharp phase transition theorems for hyperbolicity of random groups. *Geom. Funct. Anal.*, 3(14):595–679, 2004.
[29] A. Ol'shanskii. Almost every group is hyperbolic. *Internat. J. Algebra Comput.*, 1(2):1–17, 1992.
[30] D.V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.
[31] G. Palla, I. Derényi, Illés Farkas, and T. Vicsek. Uncovering the overlapping community

structure of complex networks in nature and society. *Nature*, 435:814–818, June 2005.
[32] A. Sisto. On metric relative hyperbolicity. Preprint, arXiv:math/1210.8081, 2012.
[33] A. Sisto. Projections and relative hyperbolicity. *Enseign. Math.*, 59:165–181, 2013.
[34] T. Susse. Morse subgroups and boundaries of random right-angled Coxeter groups. *Geometriae Dedicata*, 217(2):15, 2023.
[35] B. Tóth, T. Vicsek, and G. Palla. Overlapping modularity at the critical point of k-clique percolation. *J. Stat. Phys.*, 151(3-4):689–706, 2013.
[36] B. Wang, L. Cao, H. Suzuki, and K. Aihara. Impacts of clustering on interacting epidemics. *J. Theoret. Biol.*, 304:121–130, 2012.
[37] A. Yaman. A topological characterization of relatively hyperbolic groups. *J. Reine Angew. Math. (Crelle's Journal)*, 566:41–89, 2004.