STRUCTURE INVARIANT PROPERTIES OF THE HIERARCHICALLY HYPERBOLIC BOUNDARY

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Abstract. We prove several topological and dynamical properties of the boundary of a hierarchically hyperbolic group are independent of the specific hierarchically hyperbolic structure. This is accomplished by proving that the boundary is invariant under a "maximization" procedure introduced by the first two authors and Durham.

1. Introduction

A geodesic metric space $\mathcal{X}$ has a *hierarchically hyperbolic structure* if there exist an index set $\mathfrak{S}$ parameterizing a collection of hyperbolic spaces and projections from $\mathcal{X}$ to each of these hyperbolic spaces satisfying some conditions that encode the presence/absence of certain quasi-isometrically embedded products in $\mathcal{X}$; see Definition 3.1. A *hierarchically hyperbolic group* (HHG) is a finitely generated group where the word metric on the group has a hierarchically hyperbolic structure that is compatible with the group action.

The notion of hierarchical hyperbolicity was introduced by Behrstock, Hagen, and Sisto [BHS17b, BHS19] and includes a number of important examples in geometric group theory including mapping class groups, most three-manifold groups [BHS19], right-angled Coxeter groups, large classes of Artin groups [BHS17b, HMS], hyperbolic groups, and various combinations of these examples [BHS19, BR20a, BR20b].

A hierarchically hyperbolic group is typically studied by fixing a particular hierarchically hyperbolic structure and deducing results about the group using the geometric and combinatorial properties associated to that structure. On the other hand, once a group admits one hierarchically hyperbolic structure it automatically admits many. For example, a hyperbolic group has a hierarchically hyperbolic structure where the index set consists of a single element and the associated hyperbolic space is the group itself. However, one can also take a more complicated index set consisting of a collection of quasi-convex subgroups of the hyperbolic group together with an electrification of the original group collapsing those subgroups and their cosets. (Note that a hyperbolic group is not hyperbolic relative to such a collection if they are not almost malnormal, but one does get hierarchical hyperbolicity without such an assumption.) This point of view provides an upside to having multiple structures by yielding new techniques for studying hyperbolic groups and their boundaries; see e.g., [Spr18].

In a few cases, the possible structures for an HHG are understood; for instance a hierarchically hyperbolic group is virtually abelian if and only if the associated hyperbolic spaces are either bounded or quasi-lines [PS20]. However, at this point, it remains out of reach to understand all the possible structures on a given group in general. A natural question in this direction is whether or not a hierarchically hyperbolic group possesses a most "natural" or "simplest" structure. In a hierarchically hyperbolic structure, there is a partial order, called *nesting*, on the set of hyperbolic spaces and a unique nest-maximal element. Understanding the geometry of the nest-maximal hyperbolic space is one way to make precise the notion of a simplest structure. Some progress on this has been made in [ABD21], where the authors gave a construction, which we call *maximization*, that modifies any given structure to produce one where the nest-maximal hyperbolic space is canonical. The canonicality can be seen in a number of ways, including being unique up to quasi-isometry, as well as encoding all the Morse elements of the group. This paper begins with the work of [ABD21] as a starting point in order to study the effect of maximization on the boundary of a hierarchically hyperbolic group.

Durham, Hagen, and Sisto introduced a boundary that provides a compactification for a hierarchically hyperbolic group and coincides with the Gromov boundary when the group is hyperbolic [DHS17]. Their construction depends *a priori* on the choice of hierarchically hyperbolic structure $\mathfrak{S}$ for the group G, a pair which we denote $(G, \mathfrak{S})$; accordingly, we denote this boundary $\partial(G, \mathfrak{S})$. Question 1 in Durham, Hagen, and Sisto's paper is: given two different structures on a hierarchically hyperbolic group, does the identity map from the group to itself extend to a homeomorphism between the boundaries of the two different structures?

In this paper we resolve Durham–Hagen–Sisto's question for any structure and its maximized version.

Theorem 4.1 *If $(G, \mathfrak{S})$ is an HHG and $\mathfrak{T}$ is the structure obtained by maximizing $\mathfrak{S}$, then the identity map on G extends continuously to a G–equivariant map $\partial(G, \mathfrak{S}) \to \partial(G, \mathfrak{T})$ that is both a simplicial isomorphism and a homeomorphism.*

As part of our proof of Theorem 4.1, we also prove that two other important notions in hierarchical hyperbolicity—hierarchical quasiconvexity and hierarchy paths—are also invariant under the maximization procedure; see Section 4.2 for the precise statements.

Theorem 4.1 allows one to convert questions about the HHG boundary to questions about a maximized structure. In particular, this allows us to obtain a number of results about HHG boundaries which are independent of the choice of HHG structure used to build the boundary.

One consequence is that some topological properties of the boundary of the maximized hyperbolic space can be shown to hold in every HHG boundary, for instance:

Corollary 5.6 *Let G be an HHG. If the hyperbolic space associated to the nest-maximal element in some (and hence any) maximized hierarchically hyperbolic structure is one-ended, then for any HHG structure $\mathfrak{S}$ for G, the HHS boundary $\partial(G, \mathfrak{S})$ is connected.*

The converse of Corollary 5.6 is an interesting open question.

Theorem 4.1 and the fact that the maximized hyperbolic space encodes the Morse elements of the group implies that the Morse elements are precisely the set of elements that act with north-south dynamics in *any* HHG boundary.

Corollary 5.9 *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group that is not virtually cyclic. An element $g \in G$ acts with north-south dynamics on $\partial(G, \mathfrak{S})$ if and only if g is a Morse element of G. In particular, the set of elements of G that act with north-south dynamics does not depend on the HHG structure $\mathfrak{S}$.*

We can also show that the set of attracting fixed points of the Morse elements is dense in the boundary, regardless of the choice of HHG structure.

Corollary 5.10 *Let $(G, \mathfrak{S})$ be an HHG that is not virtually cyclic. Either G is quasi-isometric to a product of two unbounded spaces or the set of attracting fixed points of Morse elements in $\partial(G, \mathfrak{S})$ is dense in $\partial(G, \mathfrak{S})$. In the latter case, the Morse boundary is a dense subset of the HHS boundary.*

Note that for the above corollary, as well as the following one, the hypothesis that G is quasi-isometric to a product of two unbounded spaces could be replaced by the equivalent statement that G does contain a Morse element. This equivalence is obtained by first applying the rank rigidity theorem [DHS17, Theorem 9.13] to know that a group is either a product or contains a rank-one element, then applying maximization to ensure that a rank-one element is irreducible axial, and finally appealing to [DHS17, Theorem 6.15] (or alternatively [ABD21, Theorem 4.4]), which implies that irreducible axials are Morse.

Finally, we use the density of Morse elements to show the limit set of a normal subgroup is the entire HHS boundary. Examples of such normal subgroups include the kernel of the Birman exact sequence [Bir69], Bestvina–Brady subgroups of RAAGs [BB97], the normal closure of sufficiently high powers of Dehn twists, as studied in [Dah18], and the infinitely generated RAAG subgroups of mapping class groups considered in [CMM21].

Corollary 5.12 *Let $(G, \mathfrak{S})$ be an HHG that is not quasi-isometric to a product of two unbounded spaces and is not virtually cyclic. If N is an infinite normal subgroup of G, then the limit set of N in $\partial(G, \mathfrak{S})$ is all of $\partial(G, \mathfrak{S})$.*

For hyperbolic groups, if the normal subgroup N is also hyperbolic, then a remarkable theorem of Mj says there is a continuous surjection of the Gromov boundary of N onto the Gromov boundary of the ambient group G induced by the (highly distorted) inclusion of N into G [Mit98]. These maps are often called *Cannon–Thurston maps* in honor of the fact that they were first discovered by Cannon and Thurston in the case where G is the fundamental group of a fibered hyperbolic 3–manifold [CT07]. Corollary 5.12 therefore inspires the following question.

Question 1.1. For which HHGs do Cannon–Thurston maps exist? That is, if $(G, \mathfrak{S})$ is an HHG and N is a normal subgroup of G that has an HHG structure $\mathfrak{T}$, when does the inclusion $N \to G$ induce a continuous surjection $\partial(N, \mathfrak{T}) \to \partial(G, \mathfrak{S})$?

Natural test cases of Question 1.1 are the kernel of the Birman exact sequence and cases when a Bestvina–Brady group is itself a RAAG. The answer is "no" when G is the direct product of two hyperbolic groups, but no other obstructions are currently known.

A more general formulation of Question 1.1 is to remove the normal hypothesis and ask for which hierarchically hyperbolic subgroups there is a continuous extension from the HHS boundary of the subgroup to its limit set. This version holds for quasiconvex subgroups in hyperbolic groups. Moreover, the naive obstruction noted above provided by the product of two hyperbolic groups is not an obstruction to this version of the question. On the other hand, Mousley showed that for a family of RAAG subgroups of mapping class groups, there are obstructions to extending, while for a family of free groups she characterizes exactly when they do extend [Mou18].

Organization of the paper. In Section 2, we set notation and collect preliminary facts on hyperbolic spaces and their boundaries. In Section 3, we define hierarchically hyperbolic spaces and their boundaries. We also describe several tools from the theory of hierarchical hyperbolicity that we will use. We begin Section 4 by describing the maximization procedure in detail (Section 4.1), and then devote the remainder of the section to the proof of Theorem 4.1. Section 5 contains the applications of Theorem 4.1 including Corollaries 5.6, 5.9, 5.10, and 5.12. Section 4 contains the bulk of the technical work of the paper. Section 5 is an essentially self-contained collection of applications where the only reference to the rest of the paper is the statement of Theorem 4.1.

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2. Preliminaries on hyperbolic spaces

In this section, we establish notation and recall some basic notions about hyperbolic spaces. We also deduce a few general results about hyperbolic spaces, which we will use later in the paper.

2.1. **Coarse Geometry.** We begin by gathering several facts about metric spaces and coarse geometry. We refer the reader to [BH99] for further details.

Let (X, d_X) be a metric space. If $Y \subseteq X$ is a subspace, then for any constant $C \geqslant 0$, we denote the closed C–neighborhood of Y in X by

$$\mathcal{N}_C(Y) = \{x \in X \mid d_X(x, Y) \leqslant C\}.$$

We say two subsets $Y, Z \subseteq X$ are *C–coarsely equal*, for some $C \geqslant 0$, if $Y \subseteq \mathcal{N}_C(Z)$ and $Z \subseteq \mathcal{N}_C(Y)$. When Y and Z are C–coarsely equal, we write $Y \asymp_C Z$.

A map of metric spaces $f\colon (X, d_X) \to (Y, d_Y)$ is a *(λ, c)–quasi-isometric embedding* if for all $x, y \in X$

$$\frac{1}{\lambda} d_X(x,y) - c \leqslant d_Y(f(x), f(y)) \leqslant \lambda d_X(x,y) + c.$$

A *(λ, c)–quasi-geodesic* is a (λ, c)–quasi-isometric embedding of a closed interval $I \subseteq \mathbb{R}$ into X, and a *geodesic* is an isometric embedding of I into X. We let $[x, y]$ denote a geodesic in X from x to y. In the case of quasi-geodesics, we allow f to be a *coarse map*, that is, a map which sends points in I to uniformly bounded diameter sets in X. Accordingly, we can assume that the domain of a quasi-geodesic is an interval in $\mathbb{Z}$ instead of $\mathbb{R}$ when convenient. A (coarse) map $f\colon [0, T] \to X$ is an *unparametrized (λ, c)–quasi-geodesic* if there exists a non-decreasing function $g\colon [0, T'] \to [0, T]$ such that the following hold:

- $g(0) = 0$,
- $g(T') = T$,
- $f \circ g\colon [0, T'] \to X$ is a (λ, c)–quasi-geodesic.
- for each $j \in [0, T'] \cap \mathbb{N}$, we have the diameter of $f(g(j)) \cup f(g(j+1))$ is at most c.

For $\delta \geqslant 0$, a geodesic metric space X is *δ–hyperbolic* if we have $[x, y] \subseteq \mathcal{N}_\delta([x, z] \cup [z, y])$. If the particular choice of δ is not important, we simply say that X is *hyperbolic*.

Quasi-geodesics in a hyperbolic metric space satisfy the following Morse property, which roughly states that quasi-geodesics with the same endpoints remain in a uniform neighborhood of each other. This is also known as *quasi-geodesic stability*.

Lemma 2.1 (Morse Lemma)**.** *Let X be a δ–hyperbolic metric space, and fix $\lambda \geqslant 1$ and $c \geqslant 0$. There exists a constant σ depending only on δ, λ, and c such that if γ_1 and γ_2 are (λ, c)–quasi-geodesics in X with the same endpoints, then $\gamma_1 \asymp_\sigma \gamma_2$.*

We say σ is the *Morse constant* associated to (λ, c)–quasi-geodesics in a δ–hyperbolic space.

2.2. **The Gromov product and the Gromov boundary.** Let X be a δ–hyperbolic metric space. For any $x, y, z \in X$, the *Gromov product of x and y with respect to z* is

$$(x \mid y)_z = \frac{1}{2}\left(d_X(x,z) + d_X(y,z) - d_X(x,y)\right).$$

The Gromov product $(x \mid y)_z$ is uniformly close to the distance from z to a geodesic connected x and y:

Lemma 2.2 ([BH99, p. 410])**.** *For any δ–hyperbolic space X and $x, y, z \in X$, we have*

$$|(x \mid y)_z - d_X(z, [x, y])| \leqslant \delta.$$

Given a fixed basepoint x_0 of X, a sequence of points (x_n) in X *converges to infinity* if

$$(x_n \mid x_k)_{x_0} \to \infty$$

as $n, k \to \infty$. Two sequences (x_n) and (y_n) are *asymptotic* if $(x_n \mid y_n)_{x_0} \to \infty$ as $n \to \infty$. Note, this is equivalent to requiring that $(x_n \mid y_k)_{x_0} \to \infty$ as $n, k \to \infty$. The *Gromov boundary* ∂X of X is the set of sequences in X that converge to infinity modulo the equivalence relation of being asymptotic.

The Gromov product extends to $x, y \in X \cup \partial X$ and $z \in X$ by taking the supremum of

$$\liminf_{n,k} (x_n \mid y_k)_z$$

over all sequences (x_n) and (y_k) that are either asymptotic to x or y when they are boundary points or converge to x or y when they are points in X. We can then topologize $X \cup \partial X$ by declaring a sequence (x_n) in $X \cup \partial X$ to converge to $x \in \partial X$ if and only if

$$\lim_{n\to\infty} (x_n \mid x)_{x_0} = \infty.$$

Definition 2.3. For each $p \in \partial X$, the sets

$$M(r;p) = \{x \in X \cup \partial X : (p \mid x)_{x_0} > r\}$$

where $r > 0$ form a neighborhood basis for p in $X \cup \partial X$. Note that if $r \leqslant r'$, then $M(r';p) \subseteq M(r;p)$.

Despite the presence of the basepoint in the above definitions, convergence to infinity, being asymptotic, the Gromov boundary, and the topology of $X \cup \partial X$ are all independent of the choice of basepoint.

If $\gamma\colon [0,\infty) \to X$ is a quasi-geodesic ray, then there exists a unique $p \in \partial X$ so that $\gamma(t_n) \to p$ for every increasing sequence (t_n) that approaches infinity. In this case, we say that γ *represents* $p \in \partial X$. Every point in ∂X can be represented by a $(1, 20\delta)$–quasi-geodesic ray based at any point in X; see e.g., [KB02].

The next two lemmas allow geodesics to assist in calculating Gromov products. The first is straight-forward, and its proof is left to the reader.

Lemma 2.4. *For every $\lambda \geqslant 1$, $c \geqslant 0$, and $\delta \geqslant 0$, there exist $B \geqslant 0$ and $t_0 \geqslant 0$ so that the following holds.*

Let X be a δ–hyperbolic space with basepoint x_0, and let $p \in \partial X$ and $x \in X$. Let $\gamma\colon [0,\infty) \to X$ be a (λ, c)–quasi-geodesics starting at x_0 and representing p. For all $t \geqslant t_0$, we have

$$|(\gamma(t) \mid x)_{x_0} - (p \mid x)_{x_0}| \leqslant B.$$

Lemma 2.5. *Let X be a δ–hyperbolic space and $x, y, z \in X$. If q is a point on $[z, x]$ so that $q \in \mathcal{N}_C([x, y])$, then*

$$|(x \mid y)_z - (q \mid y)_z| \leqslant C.$$

Proof. Since $q \in [z, x]$, we have

$$d_X(q, z) + d_X(y, z) - d_X(q, y) = d_X(x, z) - d_X(q, x) + d_X(y, z) - d_X(q, y).$$

Let p be point on $[x, y]$ with $d_X(p, q) \leqslant C$. Thus,

$$d_X(q, y) \leqslant d_X(p, y) + C \text{ and } d_X(q, x) \leqslant d_X(p, x) + C.$$

Hence,

$$\begin{aligned} d_X(x, z) - d_X(q, x) + d_X(y, z) - d_X(q, y) &\geqslant d_X(x, z) - d_X(p, x) + d_X(y, z) - d_X(p, y) - 2C \\ &= d_X(x, z) + d_X(y, z) - d_X(x, y) - 2C. \end{aligned}$$

For the other inequality, we apply

$$d_X(q, y) \geqslant d_X(p, y) - C \text{ and } d_X(q, x) \geqslant d_X(p, x) - C$$

to conclude

$$\begin{aligned} d_X(x, z) - d_X(q, x) + d_X(y, z) - d_X(q, y) &\leqslant d_X(x, z) - d_X(p, x) + d_X(y, z) - d_X(p, y) + 2C \\ &= d_X(x, z) + d_X(y, z) - d_X(x, y) + 2C. \end{aligned}$$

Dividing everything by 2 produces $|(x \mid y)_z - (p \mid y)_z| \leqslant C$. □

2.3. **Quasiconvex subsets.** A subset Y of a δ–hyperbolic space X is *μ–quasiconvex* if every geodesic in X between points in Y is contained in the closed μ–neighborhood of Y. We recall a few basic facts about quasiconvex subset of hyperbolic spaces and verify a simple lemma. We direct the reader to [DK18, §11.7] for full details.

When Y is a μ–quasiconvex subset of a δ–hyperbolic space X and $x \in X$, the set of points $\{y \in Y : d_X(x,y) \leqslant d_X(x,Y) + 1\}$ is uniformly bounded in terms of μ and δ. Hence, there is a well defined coarse map $\mathfrak{p}_Y : X \to Y$ so that

$$\mathfrak{p}_Y(x) = \{y \in Y : d_X(x,y) \leqslant d_X(x,Y) + 1\}.$$

We call the map $\mathfrak{p}_Y$ the *closest point projection onto* Y.

For a quasiconvex subset $Y \subseteq X$, we let ∂Y denote the set of points in ∂X that are represented by sequences of points in ∂Y. The following lemma shows that quasi-geodesics in X that represent points in ∂Y can be modified to be eventually contained in Y.

Lemma 2.6. *Let Y be a μ–quasiconvex subset of a δ–hyperbolic space X. Let $\gamma\colon [0,\infty) \to X$ be a $(1, 20\delta)$–quasi-geodesic ray from a point $x \in X$ to a point $p \in \partial Y$. There then exists a constant $A \geqslant 1$, depending only on δ and μ, such that the following holds. There is a $(1, 20\delta + 2A)$–quasi-geodesic ray $\gamma'\colon (0,\infty) \cap \mathbb{Z} \to X$ from x to p and a constant $T \in [0,\infty) \cap \mathbb{Z}$ such that $\gamma'(t) \in Y$ for all $t \geqslant T$ and γ' is uniformly close to γ.*

Proof. Since $p \in \partial Y$ and Y is μ–quasiconvex, there is a constant A, depending only on δ and μ, and some $t_0 \in (0,\infty)$ such that $\gamma|_{[t_0,\infty)} \subseteq \mathcal{N}_A(Y)$. We may assume without loss of generality that $t_0 \in \mathbb{Z}_{\geqslant 0}$. Let $t_i = t_0 + i$, and for each i choose a point $y_i \in Y$ such that $d_X(\gamma(t_i), y_i) \leqslant A$.

Define $\gamma'\colon [0,\infty) \cap \mathbb{Z} \to X$ by

$$\gamma'(t) = \begin{cases} \gamma(t) & \text{if } t \in [0,t_0] \cap \mathbb{Z} \\ y_{t-t_0} & \text{if } t \in (t_0,\infty) \cap \mathbb{Z}. \end{cases}$$

We will show that γ' is a $(1, 20\delta + 2A)$–quasi-geodesic. Let $t, s \in [0,\infty) \cap \mathbb{Z}$. If $t, s \in [0, t_0]$, then the result is clear. Suppose $t, s \in (t_0, \infty) \cap \mathbb{Z}$. Then $d_X(\gamma'(t), \gamma'(s)) = d_X(y_{t-t_0}, y_{s-t_0})$, and

$$d_X(\gamma(t-t_0), \gamma(s-t_0)) - 2A \leqslant d_X(y_{t-t_0}, y_{s-t_0}) \leqslant d_X(\gamma(t-t_0), \gamma(s-t_0)) + 2A$$
$$|(t-t_0) - (s-t_0)| - 20\delta - 2A \leqslant d_X(y_{t-t_0}, y_{s-t_0}) \leqslant |(t-t_0) - (s-t_0)| + 20\delta + 2A.$$

Since $|(t-t_0) - (s-t_0)| = |t-s|$, we conclude that $\gamma'|_{(t_0,\infty)}$ is a $(1, 20\delta + 2A)$–quasi-geodesic.

Finally, suppose $t \in [0, t_0]$ and $s \in (t_0, \infty)$. Then $\gamma'(s) = y_{s-t_0}$ and $t_{s-t_0} = t_0 + (s-t_0) = s$, and so $d_X(y_{s-t_0}, \gamma(s)) = d_X(y_{s-t_0}, \gamma(t_{s-t_0})) \leqslant A$. Thus we have

$$d_X(\gamma(t), \gamma(s) - A \leqslant d_X(\gamma'(t), \gamma'(s)) = d_X(\gamma(t), y_{s-t_0}) \leqslant d_X(\gamma(t), \gamma(s)) + A.$$

Therefore γ' is a $(1, 20\delta + 2A)$–quasi-geodesic which, by construction, is from x to p and is uniformly close to γ, completing the proof. □

3. Preliminaries on hierarchically hyperbolic spaces

In this section, we recall some of the tools we will use to work with hierarchically hyperbolic spaces and groups and define the HHS boundary. We begin with the definition of an HHS.

Definition 3.1 (HHS)**.** Let $E > 0$ and $\mathcal{X}$ be an (E,E)–quasi-geodesic space. A *hierarchically hyperbolic space structure with constant* E for $\mathcal{X}$ is an index set $\mathfrak{S}$ and a set $\{\mathcal{C}W : W \in \mathfrak{S}\}$ of E–hyperbolic spaces $(\mathcal{C}W, d_W)$ such that the following axioms are satisfied.

(1) **(Projections.)** For each $W \in \mathfrak{S}$, there exists a *projection* $\pi_W : \mathcal{X} \to 2^{\mathcal{C}W}$ such that for all $x \in \mathcal{X}$, $\pi_W(x) \neq \emptyset$ and $\operatorname{diam}(\pi_W(x)) \leqslant E$. Moreover, each π_W is (E,E)–coarsely Lipschitz and $\mathcal{C}W \subseteq \mathcal{N}_E(\pi_W(\mathcal{X}))$ for all $W \in \mathfrak{S}$.

(2) **(Nesting.)** If $\mathfrak{S} \neq \emptyset$, then $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$ and contains a unique $\sqsubseteq$–maximal element. When $V \sqsubseteq W$, we say V is *nested* in W. For each $W \in \mathfrak{S}$, we denote by $\mathfrak{S}_W$ the set of all $V \in \mathfrak{S}$ with $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with

$V \sqsubsetneq W$ there is a specified non-empty subset $\rho_W^V \subseteq \mathcal{C}W$ with $\operatorname{diam}(\rho_W^V) \leqslant E$, and a map $\rho_V^W : \mathcal{C}W - \mathcal{N}_E(\rho_W^V) \to 2^{\mathcal{C}V}$.

(3) **(Orthogonality.)** $\mathfrak{S}$ has a symmetric relation called *orthogonality*. If V and W are orthogonal, we write $V \perp W$ and require that V and W are not $\sqsubseteq$–comparable. Further, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. We denote by $\mathfrak{S}_W^\perp$ the set of all $V \in \mathfrak{S}$ with $V \perp W$.

(4) **(Transversality.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \pitchfork W$ there are non-empty sets $\rho_W^V \subseteq \mathcal{C}W$ and $\rho_V^W \subseteq \mathcal{C}V$ each of diameter at most E.

(5) **(Finite complexity.)** Any set of pairwise $\sqsubseteq$–comparable elements has cardinality at most E.

(6) **(Containers.)** For each $W \in \mathfrak{S}$ and $U \in \mathfrak{S}_W$ with $\mathfrak{S}_W \cap \mathfrak{S}_U^\perp \neq \emptyset$, there exists $Q \in \mathfrak{S}_W$ such that $V \sqsubseteq Q$ whenever $V \in \mathfrak{S}_W \cap \mathfrak{S}_U^\perp$. We call Q the *container of U in W*.

(7) **(Uniqueness.)** There exists a function $\theta\colon [0,\infty) \to [0,\infty)$ so that for all $r \geqslant 0$, if $x, y \in \mathcal{X}$ and $d_{\mathcal{X}}(x,y) \geqslant \theta(r)$, then there exists $W \in \mathfrak{S}$ such that $d_W(\pi_W(x), \pi_W(y)) \geqslant r$.

(8) **(Bounded geodesic image.)** For all $V, W \in \mathfrak{S}$ with $V \sqsubsetneq W$, if a $\mathcal{C}W$ geodesic γ does not intersect $\mathcal{N}_E(\rho_W^V)$, then $\operatorname{diam}_{\mathcal{C}V}(\rho_V^W(\gamma)) \leqslant E$.

(9) **(Large links.)** For all $W \in \mathfrak{S}$ and $x, y \in \mathcal{X}$, there exists $\{V_1, \ldots, V_m\} \subseteq \mathfrak{S}_W - \{W\}$ such that m is at most $Ed_W(\pi_W(x), \pi_W(y)) + E$, and for all $U \in \mathfrak{S}_W - \{W\}$, either $U \in \mathfrak{S}_{V_i}$ for some i, or $d_U(\pi_U(x), \pi_U(y)) \leqslant E$.

(10) **(Consistency.)** For all $x \in \mathcal{X}$ and $V, W, U \in \mathfrak{S}$:
- if $V \pitchfork W$, then $\min\left\{d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W)\right\} \leqslant E$,
- if $V \sqsubsetneq W$, then $\min\left\{d_W(\pi_W(x), \rho_V^W), \operatorname{diam}\left(\pi_V(x) \cup \rho_W^V(\pi_V(x))\right)\right\} \leqslant E$,
- if $U \sqsubseteq V$ and either $V \sqsubsetneq W$ or $V \pitchfork W$ and $W \not\perp U$, then $d_W(\rho_W^U, \rho_W^V) \leqslant E$.

(11) **(Partial realization.)** If $\{V_i\}$ is a finite collection of pairwise orthogonal elements of $\mathfrak{S}$ and $p_i \in \mathcal{C}V_i$ for each i, then there exists $x \in \mathcal{X}$ so that:
- $d_{V_i}(\pi_{V_i}(x), p_i) \leqslant E$ for all i;
- for each i and each $W \in \mathfrak{S}$, if $V_i \sqsubsetneq W$ or $W \pitchfork V_i$, we have $d_W(\pi_W(x), \rho_W^{V_i}) \leqslant E$.

We use $\mathfrak{S}$ to denote the hierarchically hyperbolic space structure, including the index set $\mathfrak{S}$, spaces $\{\mathcal{C}W : W \in \mathfrak{S}\}$, projections $\{\pi_W : W \in \mathfrak{S}\}$, and relations $\sqsubseteq$, $\perp$, $\pitchfork$. We call the elements of $\mathfrak{S}$ the *domains* of $\mathfrak{S}$ and call the ρ_W^V the *relative projection* from V to W. The number E is called the *hierarchy constant* for $\mathfrak{S}$.

We call a quasi-geodesic space $\mathcal{X}$ a *hierarchically hyperbolic space with constant E* if there exists a hierarchically hyperbolic structure on $\mathcal{X}$ with constant E. We use the pair $(\mathcal{X}, \mathfrak{S})$ to denote a hierarchically hyperbolic space equipped with the specific HHS structure $\mathfrak{S}$.

When writing the distances in the hyperbolic spaces $\mathcal{C}W$ between images of points under π_W, we will frequently suppress the π_W notation. That is, we will use $d_W(x,y)$ to denote $d_W(\pi_W(x), \pi_W(y))$ for $x, y \in \mathcal{X}$.

For a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, we are often most concerned with the domains $W \in \mathfrak{S}$ whose associated hyperbolic spaces $\mathcal{C}W$ have infinite diameter. We often also restrict to HHSs with the following regularity condition.

Definition 3.2 (Bounded domain dichotomy)**.** Given an HHS $(\mathcal{X}, \mathfrak{S})$, we let $\mathfrak{S}^\infty$ denote the set $\{W \in \mathfrak{S} : \operatorname{diam}(\mathcal{C}W) = \infty\}$. We say that $(\mathcal{X}, \mathfrak{S})$ has the *bounded domain dichotomy* if there is some $D \geqslant 0$ so that for all $W \in \mathfrak{S} - \mathfrak{S}^\infty$ we have $\operatorname{diam}(\mathcal{C}W) \leqslant D$.

The bounded domain dichotomy is a natural condition as it is satisfied by all *hierarchically hyperbolic groups (HHG)* which is a condition requiring equivariance of the HHS structure. In this paper we work with a class of finitely generated groups which is slightly more general than being an

HHG (see Remark 3.4); these are groups which have an HHS structure compatible with the action of the group in the following way.

Definition 3.3 (G–HHS)**.** Let G be a finitely generated group. A hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ with constant E that has the bounded domain dichotomy is a *G–HHS* if the following hold.

(1) $\mathcal{X}$ is a proper metric space with a proper and cocompact action of G by isometries.
(2) G acts on $\mathfrak{S}$ by $\sqsubseteq$–, $\perp$–, and $\pitchfork$–preserving bijection, and $\mathfrak{S}^\infty$ has finitely many G–orbits.
(3) For each $W \in \mathfrak{S}$ and $g \in G$, there exists an isometry $g_W : \mathcal{C}W \to \mathcal{C}gW$ satisfying the following for all $V, W \in \mathfrak{S}$ and $g, h \in G$.
   - The map $(gh)_W : \mathcal{C}W \to \mathcal{C}ghW$ is equal to the map $g_{hW} \circ h_W : \mathcal{C}W \to \mathcal{C}ghW$.
   - For each $x \in \mathcal{X}$, $g_W(\pi_W(x)) \asymp_E \pi_{gW}(g \cdot x)$.
   - If $V \pitchfork W$ or $V \subsetneq W$, then $g_W(\rho^V_W) \asymp_E \rho^{gV}_{gW}$.

We can and will assume that $\mathcal{X}$ is G equipped with a finitely generated word metric. We say that $\mathfrak{S}$ is a *G–HHS structure* for the group G and use the pair $(G, \mathfrak{S})$ the group G equipped with the specific G–HHS structure $\mathfrak{S}$.

Remark 3.4 (HHG vs G–HHS)**.** The only difference between the above definition of a G–HHS and a hierarchically hyperbolic group is that a hierarchically hyperbolic group is required to have finitely many orbits in $\mathfrak{S}$ and not just $\mathfrak{S}^\infty$. (Both HHGs and G–HHSs satisfy the bounded domain dichotomy, but for HHGs this is a theorem and for G–HHSs it is by definition.)

The primary reason we choose to work with the slightly broader definition of a G–HHS is that we are ultimately interested in the boundary defined by an HHS structure. Since the definition of the boundary does not involve uniformly bounded diameter domains, the natural class of structures to think about are those that only have restrictions on the set of infinite diameter domains.

We note that [ABD21, Corollary 3.8] states that applying the maximization procedure that we will introduce in Section 4 to an HHG results in an HHG. However, the argument in [ABD21] does not explicitly address the co-finiteness of the action on the added finite diameter domains ("dummy domains"). Thus that result only shows that the result is a G–HHS. Since that argument addresses the infinite diameter domains, it follows that applying the maximization procedure to a G–HHS again results in a G–HHS.

A hallmark of hierarchically hyperbolic spaces is that every pair of points can be joined by a special family of quasi-geodesics called *hierarchy paths*, each of which projects to a quasi-geodesic in each of the spaces $\mathcal{C}W$.

Definition 3.5. A *λ–hierarchy path* γ in an HHS $(\mathcal{X}, \mathfrak{S})$ is a (λ, λ)–quasi-geodesic with the property that $\pi_W \circ \gamma$ is an unparametrized (λ, λ)–quasi-geodesic for each $W \in \mathfrak{S}$.

Theorem 3.6 ([BHS19, Theorem 4.4])**.** *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with constant E. There exist $\lambda \geqslant 1$ depending only on E so that every pair of point points in $\mathcal{X}$ is joined by a λ–hierarchy path.*

In general, a quasi-geodesic (or even geodesic) in an HHS can be arbitrarily far from being a hierarchy path. Moreover, a given space might have different HHS structures, and the set of hierarchy paths with respect to each structure might be different.

3.1. **Hierarchical quasiconvexity and standard product regions.** The analogue of quasi-convex subsets of a hyperbolic space in the setting of hierarchical hyperbolicity are the following *hierarchically quasiconvex* subsets. We refer the reader to [BHS19, §5] for details on any of the background material in this subsection.

Definition 3.7. Let $k : [0, \infty) \to [0, \infty)$. A subset $\mathcal{Y}$ of an HHS $(\mathcal{X}, \mathfrak{S})$ is *k–hierarchically quasi-convex* if

(1) for each $W \in \mathfrak{S}$, $\pi_W(\mathcal{Y})$ is a $k(0)$–quasiconvex subset of $\mathcal{C}W$;

(2) if $x \in \mathcal{X}$ so that $d_W(x,\mathcal{Y}) \leqslant r$ for each $W \in \mathfrak{S}$, then $d_{\mathcal{X}}(x,\mathcal{Y}) \leqslant k(r)$.

As with hierarchy paths, whether or not a subset is hierarchically quasiconvex can depend on which HHS structure is put on the space, hence $\mathcal{Y}$ is a hierarchically quasiconvex subset of $(\mathcal{X},\mathfrak{S})$ and not just $\mathcal{X}$.

Each hierarchically quasiconvex subset $\mathcal{Y}$ comes equipped with a *gate map* denoted $\mathfrak{g}_{\mathcal{Y}}\colon \mathcal{X} \to \mathcal{Y}$. While this map might not be the coarse closest point projection, it has a number of nice properties that we summarize below.

Lemma 3.8 ([BHS19, Lemma 5.5] plus [BHS21, Lemma 1.20])**.** *Let $(\mathcal{X},\mathfrak{S})$ be an HHS with constant E. Suppose $\mathcal{Y} \subseteq \mathcal{X}$ is k–hierarchically quasiconvex. There is a coarse map $\mathfrak{g}_{\mathcal{Y}}\colon \mathcal{X} \to \mathcal{Y}$ and a constant $\kappa \geqslant 1$ depending only on k and E, so that the following hold.*

- *$\mathfrak{g}_{\mathcal{Y}}$ is coarsely the identity on $\mathcal{Y}$.*
- *$\mathfrak{g}_{\mathcal{Y}}$ is (κ,κ)–coarsely Lipschitz.*
- *For each $x \in \mathcal{X}$ and $W \in \mathfrak{S}$ we have*
$$\pi_W(\mathfrak{g}_{\mathcal{Y}}(x)) \asymp_\kappa \mathfrak{p}_{\pi_W(\mathcal{Y})}(\pi_W(x)).$$
- *For each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there is a κ–hierarchy path γ from x to y with the property that $\gamma \cap \mathcal{N}_\kappa\big(\mathfrak{g}_{\mathcal{Y}}(x)\big) \neq \emptyset$.*

Associated to each domain $W \in \mathfrak{S}$ of an HHS $(\mathcal{X},\mathfrak{S})$ is a pair of hierarchically quasiconvex subspaces $\mathbf{F}_W$ and $\mathbf{E}_W$. Since we will not work directly with the definition of these subsets, we will just state the salient properties that we will need. For details, we direct the interested reader to [BHS19, §5B] for the definition and proofs of their basic properties. In the sequel, we shall work primarily with the $\mathbf{F}_W$, but we include the companion facts about $\mathbf{E}_W$ for context.

Proposition 3.9. *Let $(\mathcal{X},\mathfrak{S})$ be an HHS with constant E. For each $W \in \mathfrak{S}$, the subsets $\mathbf{F}_W, \mathbf{E}_W \subseteq \mathcal{X}$ have the following properties.*

(1) There exists $k\colon [0,\infty) \to [0,\infty)$ depending only on E so that $\mathbf{F}_W$ and $\mathbf{E}_W$ are k–hierarchically quasiconvex.

(2) There exists $\kappa \geqslant 0$ depending only on E so that for each $V \in \mathfrak{S}$:
- *$\pi_V(\mathbf{F}_W) \asymp_\kappa \rho^V_W$ and $\pi_V(\mathbf{E}_W) \asymp_\kappa \rho^W_V$ when $W \subsetneq V$ or $W \pitchfork V$;*
- *$\mathcal{C}V = \mathcal{N}_\kappa(\pi_V(\mathbf{F}_W))$ and $\operatorname{diam}(\pi_V(\mathbf{E}_W)) \leqslant \kappa$ when $V \sqsubseteq W$; and*
- *$\operatorname{diam}(\pi_V(\mathbf{F}_W)) \leqslant \kappa$ and $\mathcal{C}V = \mathcal{N}_\kappa(\pi_V(\mathbf{E}_W))$ when $V \perp W$.*

Moreover, if $V \sqsubseteq W$, then $\pi_W(x) \asymp_\kappa \pi_W(\mathfrak{g}_{\mathbf{F}_W}(x))$ for each $x \in \mathcal{X}$.

(3) If $(\mathcal{X},\mathfrak{S})$ has the bounded domain dichotomy, then $\operatorname{diam}(\mathbf{F}_W) = \infty$ if and only if $\mathfrak{S}_W \cap \mathfrak{S}^\infty \neq \emptyset$. Similarly, $\operatorname{diam}(\mathbf{E}_W) = \infty$ if and only if $\mathfrak{S}^\perp_V \cap \mathfrak{S}^\infty \neq \emptyset$ in this case.

Remark 3.10. The construction of $\mathbf{F}_W$ and $\mathbf{E}_W$ in [BHS19] involves some choices, but all choices will produce coarsely equal subsets that satisfy the above properties.

While we will not use this structure directly, the $\mathbf{F}_W$ and $\mathbf{E}_W$ form natural product regions in $\mathcal{X}$ as follows: equipping each $\mathbf{F}_W$ and $\mathbf{E}_W$ with the metric induced from $\mathcal{X}$, there is a quasi-isometric embedding $\mathbf{F}_W \times \mathbf{E}_W \to \mathcal{X}$ that sends $\mathbf{F}_W \times \{e\}$ and $\{f\} \times \mathbf{E}_W$ onto $\mathbf{F}_W$ and $\mathbf{E}_W$ for some $e \in \mathbf{E}_W$ and $f \in \mathbf{F}_W$. The image of this quasi-isometric embedding is often called the *standard product region* for W and denoted $\mathbf{P}_W$.

3.2. **The boundary of a hierarchically hyperbolic space.** Durham, Hagen, and Sisto defined a boundary for an HHS $(\mathcal{X},\mathfrak{S})$ that is built from the boundaries of the hyperbolic spaces in $\mathfrak{S}$; [DHS17] is the reference for this subsection.

The points in the HHS boundary are organized in a simplicial complex that we denote $\partial_\Delta(\mathcal{X},\mathfrak{S})$. The vertex set of $\partial_\Delta(\mathcal{X},\mathfrak{S})$ is the set of all boundary points of all the hyperbolic spaces $\mathcal{C}U$ for $U \in \mathfrak{S}^\infty$. That is, the set of vertices is the set of points $\bigcup_{U \in \mathfrak{S}^\infty} \partial \mathcal{C}U$. The vertices $p_1,\dots,p_n$ of $\partial_\Delta(\mathcal{X},\mathfrak{S})$ will form an n–simplex if each $p_i \in \partial\mathcal{C}U_i$ and $U_i \perp U_j$ for each $i \neq j$. This means the set of points making up the HHS boundary can equivalently be described as the set of all linear combinations $\sum_{U\in\mathfrak{U}} a_U p_U$ where

- $\mathfrak{U}$ is a pairwise orthogonal subset of $\mathfrak{S}^\infty$,
- $p_U \in \partial\mathcal{C}U$ for each $U \in \mathfrak{U}$, and
- $\sum_{U\in\mathfrak{U}} a_U = 1$ and each $a_U > 0$.

Definition 3.11. For each $p \in \partial_\Delta(\mathcal{X},\mathfrak{S})$, we define $\operatorname{supp}(p)$, the *support of* p, to be the pairwise orthogonal set $\mathfrak{U} \subseteq \mathfrak{S}$ so that $p = \sum_{U\in\mathfrak{U}} a_U p_U$. Equivalently, the support of p is the pairwise orthogonal set $\mathfrak{U} \subseteq \mathfrak{S}$ so that the smallest dimensional simplex of $\partial_\Delta(\mathcal{X},\mathfrak{S})$ that contains p has exactly one vertex from $\partial\mathcal{C}U$ for each $U \in \mathfrak{U}$

For clarity, we will often decorate the coefficients a_U with the boundary point they are the coefficient for, that is, we write $p = \sum_{U\in\operatorname{supp}(p)} a_U^p p_U$ to emphasize a_U^p are the coefficients for the point p.

Durham, Hagen, and Sisto equip the HHS boundary with a topology beyond that coming from the simplicial complex described above. The definition of this topology is quite involved, combining the standard topology on the boundaries of the hyperbolic spaces $\mathcal{C}U$ with projections of boundary points onto certain domains of the HHS structure. We will define these boundary projections in Section 3.2.1 and use these boundary projections to define the topology in Section 3.2.2. After defining the boundary, we will describe when certain maps that respect the HHS structure extend to maps on the boundary in Section 3.2.3. When our HHS is in fact a G–HHS, this will produce a natural action of the group on the boundary by homeomorphisms and simplicial automorphisms.

We use $\partial(\mathcal{X},\mathfrak{S})$ to denote the HHS boundary equipped with the topology coming from the boundary projections while $\partial_\Delta(\mathcal{X},\mathfrak{S})$ will denote the simplicial complex that is the underlying set of boundary points.

3.2.1. *Boundary projections.* The following is a slight modification of the definition of a boundary projection from [DHS17].

Definition 3.12. Fix a point $q = \sum_{W\in\operatorname{supp}(q)} a_W q_W \in \partial(\mathcal{X},\mathfrak{S})$. For each $U \in \mathfrak{S}$ such that there exists $W \in \operatorname{supp}(q)$ with $U \not\perp W$, we define the *boundary projection* $\partial\pi_U(q)$ of q into $\mathcal{C}U$ as follows.

- If $W = U$, then we define $\partial\pi_U(q) := q_U = q_W$.
- If $W \subsetneq U$ or $W \pitchfork U$, let $\mathcal{V} = \{V \in \operatorname{supp}(q) \mid V \pitchfork U \text{ or } V \subsetneq U\}$, then define
$$\partial\pi_U(q) := \bigcup_{V\in\mathcal{V}} \rho_U^V.$$
- If $W \supsetneq U$, then $U \perp V$ for each $V \in \operatorname{supp}(q) - \{W\}$. In this case, let $Z \subseteq \mathcal{C}W$ be the set of all points on all $(1,20E)$–quasi-geodesics from a point in $\rho_W^U \in \mathcal{C}W$ to $q_W \in \partial\mathcal{C}W$ that are at distance at least $E+\sigma$ from ρ_W^U, where σ is the Morse constant from Lemma 2.1 for a $(1,20E)$–quasi-geodesics in a E–hyperbolic metric space. We then define $\partial\pi_U(q) := \rho_U^W(Z)$.

The difference between this definition and the original from [DHS17] is that [DHS17] only defines the boundary projection of q to a certain set of domains related to the support set of another point in the boundary. We define the boundary projection of q to any domain for which the definition makes sense. Because of this, our notation is different from what is used in [DHS17]: they use $(\partial\pi_{\overline{S}}(q))_U$, where $\overline{S}$ is the support set of some point in the boundary, while we use $\partial\pi_U(q)$.

3.2.2. *Topology on* $\mathcal{X} \cup \partial(\mathcal{X},\mathfrak{S})$. Before defining the topology, we define the notion of a remote point. This is a slight modification of [DHS17, Definition 2.5], where they define a point being remote to a support set.

Definition 3.13. Let $(\mathcal{X},\mathfrak{S})$ be a hierarchically hyperbolic space, and let $p \in \partial(\mathcal{X},\mathfrak{S})$. A point $q \in \partial(\mathcal{X},\mathfrak{S})$ is *remote to* p if:

(1) $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \varnothing$; and
(2) for all $Q \in \mathrm{supp}(q)$, there exists $P \in \mathrm{supp}(p)$ so that P and Q are *not* orthogonal.

Denote the set of points remote to p by $\partial_p^{rem}(\mathcal{X}, \mathfrak{S})$.

We are now ready to define the topology on $\partial(\mathcal{X}, \mathfrak{S})$. Fix a basepoint $x_0 \in \mathcal{X}$, and, for each $W \in \mathfrak{S}^\infty$, pick the basepoint for $\partial \mathcal{C}W$ to be a point in $\pi_W(x_0)$. Fix a point $p = \sum_{W \in \mathrm{supp}(p)} a_W^p p_W \in \partial(\mathcal{X}, \mathfrak{S})$. For each $r \geqslant 0$, each $\varepsilon > 0$, and each $W \in \mathrm{supp}(p)$, let $M(r; p_W)$ be a neighborhood of p_W in $\mathcal{C}W \cup \partial \mathcal{C}W$ as in Definition 2.3. We first define three sets depending on r and ε: the *remote part*, the *non-remote part*, and the *interior part*. In what follows, if U is *not* in the support of a boundary point q, then $a_U^q = 0$; if U is in $\mathrm{supp}(q)$ then a_U^q is the coefficient of q in the domain U so that $q = \sum_{U \in \mathrm{supp}(q)} a_U^q q_U$.

Definition 3.14. Given any $q \in \partial(\mathcal{X}, \mathfrak{S})$, let $\mathcal{S}_q$ be the union of $\mathrm{supp}(p)$ and the set of domains $T \in \mathrm{supp}(p)^\perp$ such that there exists some $W_T \in \mathrm{supp}(q)$ with $T \not\perp W_T$. The *remote part* $\mathcal{B}_{r,\varepsilon}^{rem}(p)$ is the set of all points $q \in \partial_p^{rem}(\mathcal{X}, \mathfrak{S})$ satisfying the following three conditions:

(R1) $\forall W \in \mathrm{supp}(p), \partial\pi_W(q) \subseteq M(r; p_W)$,
(R2) $\forall W \in \mathcal{S}_q, V \in \mathrm{supp}(p), \left| \dfrac{d_W(x_0, \partial\pi_W(q))}{d_V(x_0, \partial\pi_V(q))} - \dfrac{a_W^p}{a_V^p} \right| < \varepsilon$, and
(R3) $\sum_{T \in \mathrm{supp}(p)^\perp} a_T^q < \varepsilon$.

Definition 3.15. The *non-remote part* $\mathcal{B}_{r,\varepsilon}^{non}(p)$ is the set of all points

$$q = \sum_{T \in \mathrm{supp}(q)} a_T^q q_T \in \partial\mathcal{X} - \partial_p^{rem}(\mathcal{X}, \mathfrak{S})$$

satisfying the following three conditions, where $\mathcal{A} = \mathrm{supp}(p) \cap \mathrm{supp}(q)$.

(N1) $\forall T \in \mathcal{A}, q_T \in M(r; p_T)$,
(N2) $\sum_{V \in \mathrm{supp}(q) - \mathcal{A}} a_V^q < \varepsilon$,
(N3) $\forall T \in \mathcal{A}, \left| a_T^q - a_T^p \right| < \varepsilon$.

Definition 3.16. Finally, the *interior part* $\mathcal{B}_{r,\varepsilon}^{int}(p)$ is the set of all points $x \in \mathcal{X}$ satisfying the following three conditions:

(I1) $\forall W \in \mathrm{supp}(p)$, $\pi_W(x) \subseteq M(r; p_W)$,
(I2) $\forall W, V \in \mathrm{supp}(p)$, $\left| \dfrac{a_W^p}{a_V^p} - \dfrac{d_W(x_0, x)}{d_V(x_0, x)} \right| < \varepsilon$, and
(I3) $\forall W \in \mathrm{supp}(T)$, $T \in \mathrm{supp}(p)^\perp$, $\dfrac{d_T(x_0, x)}{d_W(x_0, x)} < \varepsilon$.

The disjoint union of these sets forms a basic set in the topology.

Definition 3.17. For each $\varepsilon > 0$ and each $r \geqslant 0$, a basic set in the topology on $\mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S})$ is the set $\mathcal{B}_{r,\varepsilon}(p)$ defined as follows:

$$\mathcal{B}_{r,\varepsilon}(p) := \mathcal{B}_{r,\varepsilon}^{rem}(p) \sqcup \mathcal{B}_{r,\varepsilon}^{non}(p) \sqcup \mathcal{B}_{r,\varepsilon}^{int}(p).$$

In [DHS17], the basic sets are defined slightly differently. We briefly describe their definition here; Lemma 3.18 shows that the two collections generate the same topology. Given $p \in \partial(\mathcal{X}, \mathfrak{S})$ and $W \in \mathrm{supp}(p)$, let K_W be *any* neighborhood of p_W in $\mathcal{C}W \cup \partial\mathcal{C}W$. Let $\mathcal{N}_{\{K_W\},\varepsilon}^{rem}(p)$, $\mathcal{N}_{\{K_W\},\varepsilon}^{non}(p)$, and $\mathcal{N}_{\{K_W\},\varepsilon}^{int}(p)$ be the sets of points satisfying (R1)–(R3), (N1)–(N3), and (I1)–(I3), respectively, using the neighborhoods K_W in place of $M(r; p_W)$. Then define $\mathcal{N}_{\{K_W\},\varepsilon}(p)$ to be the disjoint union of these three sets.

Lemma 3.18. *The collections of sets* $\mathcal{B} = \{\mathcal{B}_{r,\varepsilon}(p) \mid p \in \partial(\mathcal{X}, \mathfrak{S}), r \geqslant 0, \varepsilon \geqslant 0\}$ *and* $\mathcal{N} = \{\mathcal{N}_{\{K_W\},\varepsilon}(p) \mid p \in \partial(\mathcal{X}, \mathfrak{S}), \varepsilon \geqslant 0,$ *and* K_W *is a neighborhood of* p_W *when* $W \in \mathrm{supp}(p)\}$ *generate the same topology on* $\mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S})$.

Proof. Since $\mathcal{B} \subseteq \mathcal{N}$, we need only show that for every $p \in \partial(\mathcal{X}, \mathfrak{S})$ and every collection $\{K_W\}_{W \in \mathrm{supp}(p)}$ of neighborhoods of p_W, there is an $r \geqslant 0$ such that $\mathcal{B}_{r,\varepsilon}(p) \subseteq \mathcal{N}_{\{K_W\},\varepsilon}(p)$. For each $W \in \mathrm{supp}(p)$, there is some r_W such that $M(r_W; p_W) \subseteq K_W$. Since $\mathrm{supp}(p)$ consists of finitely many domains, the result follows by setting $r = \max\{r_W \mid W \in \mathrm{supp}(p)\}$. □

The following technical lemma will be useful in verifying when points in the boundary lie in a particular basic set.

Lemma 3.19. *Let A_n and B_n be sequences of positive numbers so that $A_n \to \infty$, $B_n \to \infty$ and $\lim_{n\to\infty} \frac{A_n}{B_n} = L$. If there exist $E \geqslant 0$ so that $|A_n - C_n| \leqslant E$ and $|B_n - D_n| \leqslant E$, then $\lim\limits_{n\to\infty} \frac{C_n}{D_n} = L$.*

Proof. Fix $\varepsilon > 0$. There exists $s > 0$ sufficiently large and $r > 0$ sufficiently small so that

$$L \cdot r + \frac{1}{s} + \frac{r}{s} < \varepsilon.$$

Since A_n and B_n tend to ∞,

$$\lim_{n\to\infty} \frac{1 + \frac{E}{A_n}}{1 + \frac{E}{B_n}} = 1.$$

Therefore, for all sufficiently large n we have

$$\frac{C_n}{D_n} \leqslant \frac{A_n + E}{B_n - E} \leqslant \frac{A_n}{B_n}(1 + r)$$

where r is the number fixed above. Since $\frac{A_n}{B_n} \to L$ as $n \to \infty$, for every large enough n, we have

$$\frac{A_n}{B_n}(1 + r) \leqslant \left(L + \frac{1}{s}\right)(1 + r) \leqslant L + L \cdot r + \frac{1}{s} + \frac{r}{s} \leqslant L + \varepsilon.$$

Hence, we have $\frac{C_n}{D_n} \leqslant L + \varepsilon$.

A completely analogous argument beginning with the inequality

$$\frac{C_n}{D_n} \geqslant \frac{A_n - E}{B_n + E}$$

gives the lower bound, completing the proof. □

The next lemma says that sequences that stay within uniformly bounded distance of each other in $\mathcal{X}$ converge to the same point in the boundary $\partial(\mathcal{X}, \mathfrak{S})$.

Lemma 3.20. *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS. Let (x_n) be a sequence of points in $\mathcal{X}$ that converges to $p \in \partial(\mathcal{X}, \mathfrak{S})$. If (y_n) is a sequence in $\mathcal{X}$ with $d_{\mathcal{X}}(x_n, y_n)$ uniformly bounded for all $n \in \mathbb{N}$, then y_n also converges to p.*

Proof. Let R be the uniform bound on $d_{\mathcal{X}}(x_n, y_n)$ and x_0 be the basepoint of the HHS boundary $(\mathcal{X}, \mathfrak{S})$. The sequence (y_n) will converge to $p = \sum a_W p_W \in \partial(\mathcal{X}, \mathfrak{S})$ if for each $r \geqslant 1$ and $\varepsilon \geqslant 0$, we have $y_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$ for all but finitely many n. Since x_n converges to p, for each $r \geqslant 1$ and $\varepsilon \geqslant 0$, there exists $n_0 = n_0(r, \varepsilon)$ so that for all $n \geqslant n_0$, $x_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$. Thus,

- for each $W \in \mathrm{supp}(p)$, $\pi_W(x_n) \in M(r; p_W)$;
- for each $W, V \in \mathrm{supp}(p)$, $\lim\limits_{n\to\infty} \frac{d_W(x_0,x_n)}{d_V(x_0,x_n)} = \frac{a_W}{a_V}$; and
- for each $W \in \mathrm{supp}(p)$ and $T \in \mathrm{supp}(p)^{\perp}$, $\lim\limits_{n\to\infty} \frac{d_T(x_0,x_n)}{d_W(x_0,x_n)} = 0$.

Let $W \in \mathrm{supp}(p)$. Since $d_W(x_n, y_n) \leqslant ER + E$ and $d_W(x_0, x_n) \to \infty$ as $n \to \infty$, there must exist $n_1 \in \mathbb{N}$ so that $y_n \in M(r; p_W)$ for all $n \geqslant n_1$. Since $d_V(x_n, y_n) \leqslant ER + E$ for every $V \in \mathfrak{S}$, we have $|d_V(x_0, x_n) - d_V(x_0, y_n)| \leqslant ER + E$ for each $V \in \mathfrak{S}$. Thus, Lemma 3.19 implies that

- for each $W, V \in \mathrm{supp}(p)$, $\lim\limits_{n\to\infty} \frac{d_W(x_0,y_n)}{d_V(x_0,y_n)} = \frac{a_W}{a_V}$; and

- for each $W \in \operatorname{supp}(p)$ and $T \in \operatorname{supp}(p)^{\perp}$, $\lim\limits_{n\to\infty} \frac{d_T(x_0,y_n)}{d_W(x_0,y_n)} = 0$.

Hence $y_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$ for all sufficiently large n. □

3.2.3. *Boundary maps induced by hieromorphisms and the group action on the boundary.* Given two hierarchically hyperbolic spaces, it is natural to wonder when maps between the spaces extend to maps between their respective HHS boundaries. A natural class of maps to consider for this question are the following *hieromorphisms*.

Definition 3.21. Let $(\mathcal{X},\mathfrak{S})$ and $(\mathcal{Y},\mathfrak{T})$ be HHSs and $\lambda \geqslant 1$. A λ–*hieromorphism* from $(\mathcal{X},\mathfrak{S})$ to $(\mathcal{Y},\mathfrak{T})$ consists of

- a map $f\colon \mathcal{X} \to \mathcal{Y}$;
- an injective map $f^{\mathfrak{s}}\colon \mathfrak{S} \to \mathfrak{T}$ that preserves nesting, transversality, and orthogonality; and
- a (λ,λ)–quasi-isometric embedding $f_V\colon \mathcal{C}_{\mathfrak{S}}V \to \mathcal{C}_{\mathfrak{T}}f^{\mathfrak{s}}(V)$ for each $V \in \mathfrak{S}$

satisfying the following properties:

- $f_V(\pi_V(x)) \asymp_\lambda \pi_{f^{\mathfrak{s}}(V)}(f(x))$ for each $x \in \mathcal{X}$ and $V \in \mathfrak{S}$;
- $f_V(\rho^W_V) \asymp_\lambda \rho^{f^{\mathfrak{s}}(W)}_{f^{\mathfrak{s}}(V)}$ whenever $W \pitchfork V$ or $W \subsetneq V$; and
- whenever $W \subsetneq V$, $f_W(\rho^V_W(z)) \asymp_\lambda \rho^{f^{\mathfrak{s}}(V)}_{f^{\mathfrak{s}}(W)}(f_W(z))$ for each $z \in \mathcal{C}V - \mathcal{N}_E(\rho^W_V)$.

Since f, $f^{\mathfrak{s}}$, and f_V all have different domains, is it often clear from context which is the relevant map. In these cases, we will abuse notation and call all maps f; we denote the hieromorphism by $f\colon (\mathcal{X},\mathfrak{S}) \to (\mathcal{Y},\mathfrak{T})$.

Give a hieromorphism $f\colon (\mathcal{X},\mathfrak{S}) \to (\mathcal{Y},\mathfrak{T})$, each quasi-isometric embedding $f_V\colon \mathcal{C}V \to \mathcal{C}f(V)$ induces a continuous inclusion $\partial f_V\colon \partial\mathcal{C}V \to \partial\mathcal{C}f(V)$. Since f respects the orthogonality relation on the index sets, there is an induced injective simplicial map $\partial f\colon \partial_\Delta(\mathcal{X},\mathfrak{S}) \to \partial_\Delta(\mathcal{X},\mathfrak{T})$ defined by

$$\partial f(p) = \partial f\left(\sum_{V\in\operatorname{supp}(p)} a_V p_V\right) = \sum_{V\in\operatorname{supp}(p)} a_V \partial f_V(p_V).$$

Work of Durham, Hagen, and Sisto implies the following sufficient conditions for this map ∂f to be continuous with respect to the topology on the HHS boundary.

Theorem 3.22 (A special case of [DHS17, Theorem 5.6])**.** *Let $f\colon (\mathcal{X},\mathfrak{S}) \to (\mathcal{Y},\mathfrak{T})$ be a hieromorphism. If for each $V \in \mathfrak{S}$, f_V is a $(1,\lambda)$–quasi-isometric embedding, then the map $\partial f\colon \partial_\Delta(\mathcal{X},\mathfrak{S}) \to \partial_\Delta(\mathcal{Y},\mathfrak{T})$ defines a continuous map from $\partial(\mathcal{X},\mathfrak{S})$ to $\partial(\mathcal{Y},\mathfrak{T})$.*

We will use Theorem 3.22 only in the following special case.

Corollary 3.23. *Let $(\mathcal{X},\mathfrak{S})$ be an HHS. If $\mathfrak{T}$ is another HHS structure for $\mathcal{X}$ and there is a hieromorphism $f\colon (\mathcal{X},\mathfrak{S}) \to (\mathcal{X},\mathfrak{T})$ so that $f\colon \mathfrak{S} \to \mathfrak{T}$ is a bijection and f_V is a $(1,\lambda)$–quasi-isometry for each $V \in \mathfrak{S}$, then $\partial f\colon \partial(\mathcal{X},\mathfrak{S}) \to \partial(\mathcal{X},\mathfrak{T})$ is a homeomorphism.*

The definition of a G–HHS ensures that the action of an element $g \in G$ on G by left multiplication gives a hieromorphism $g\colon (G,\mathfrak{S}) \to (G,\mathfrak{S})$ where for each $V \in \mathfrak{S}$, the map $g_V\colon \mathcal{C}V \to \mathcal{C}gV$ is an isometry. Thus, we can use Corollary 3.23 to extend the action of G on itself to an action of G on $\partial(G,\mathfrak{S})$ that is both a homeomorphism and a simplicial automorphism.

3.3. **Hyperbolic HHSs.** A hyperbolic space can itself have many different HHS structures [Spr18], but being hyperbolic puts a number of restriction on all of these HHS structures. The following result summarize the facts about hyperbolic HHSs that we will need.

Theorem 3.24. *Let $(\mathcal{X},\mathfrak{S})$ be an HHS with constant E, and suppose $\mathcal{X}$ is also an E–hyperbolic space.*

(1) For all $W \in \mathfrak{S}^\infty$, we have $\mathfrak{S}_W^\perp \cap \mathfrak{S}^\infty = \varnothing$ [DHS17, Lemma 4.1]*. In particular, the simplicial HHS boundary $\partial_\Delta(\mathcal{X}, \mathfrak{S})$ is a collection of 0–simplices.*
(2) For each $\ell \geqslant 1$ and $c \geqslant 0$, there exists $\lambda \geqslant 1$ (depending on ℓ, c, and E) so that every (ℓ, c)–quasi-geodesic in $\mathcal{X}$ is a λ–hierarchy path in $(\mathcal{X}, \mathfrak{S})$ [Spr18, Proposition 3.5]*.*
(3) The identity map $\mathcal{X} \to \mathcal{X}$ continuously extends to a homeomorphism from the Gromov boundary $\partial\mathcal{X}$ to the HHS boundary $\partial(\mathcal{X}, \mathfrak{S})$ [DHS17, Theorem 4.3]*.*

The next lemma describes how basis neighborhoods in $\mathcal{C}W \cup \partial\mathcal{C}W$ are related to basis neighborhoods in $\mathcal{X} \cup \partial\mathcal{X}$ when $(\mathcal{X}, \mathfrak{S})$ is a hyperbolic HHS. In the statement, we use $M(r, p)$ to denote basis neighborhoods in $\mathcal{X} \cup \partial\mathcal{X}$ and $M_W(r, p)$ to denote basis neighborhoods in $\mathcal{C}W \cup \partial\mathcal{C}W$.

Lemma 3.25. *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with constant E, and suppose $\mathcal{X}$ is also a E–hyperbolic space. There exists $r_0 \geqslant 0$ so that for each $r \geqslant r_0$ and each $W \in \mathfrak{S}^\infty$, there exists $r' \geqslant 0$ so that the following hold.*

- *r' is an increasing linear function of r with constant of linearity determined by E;*
- *for each $p \in \partial\mathcal{C}W \subseteq \partial\mathcal{X}$ and each $x \in \mathcal{X}$, if $\pi_W(x) \subseteq M_W(r, p)$, then $x \in M(r', p)$; and*
- *for each $p \in \partial\mathcal{C}W \subseteq \partial\mathcal{X}$ and each $q \in \partial\mathcal{C}W$, if $q \in M_W(r, p)$, then $q \in M(r'; p)$.*

Proof. Fix $W \in \mathfrak{S}^\infty$ and a basepoint $x_0 \in \mathcal{X}$. For the reader's convenience we will use $(\cdot \mid \cdot)_{x_0}$ to denote the Gromov product in $\mathcal{X}$ and $\langle \cdot \mid \cdot \rangle_{x_0}$ to denote the Gromov product in $\mathcal{C}W$. We first prove the second bullet point; the proof will determine the value of r_0 and calculate r' in terms of r, which will establish the first bullet point.

Since $\mathcal{X}$ is hyperbolic, Theorem 3.24 (2) provides $\lambda \geqslant 1$ so that each $(1, 20E)$–quasi-geodesic in $\mathcal{X}$ is a λ–hierarchy path in $(\mathcal{X}, \mathfrak{S})$. Fix $p \in \partial\mathcal{C}W$, and let α be a $(1, 20E)$–quasi-geodesic ray from x_0 to p in $\mathcal{X}$ (and hence a hierarchy path). Since the projections of hierarchy paths are unparametrized quasi-geodesics, the projection $\pi_W \circ \alpha$ is an unparametrized (λ, λ)–quasi-geodesic in $\mathcal{C}W$. Thus Lemma 2.4 provides a constant $B \geqslant 0$ that depends only on E and a point $y \in \alpha$ so that

$$|(x \mid p)_{x_0} - (x \mid y)_{x_0}| \leqslant B \text{ and } |\langle x \mid p \rangle_{x_0} - \langle x' \mid y' \rangle_{x_0}| \leqslant B,$$

where x' and y' are any point in $\pi_W(x)$ and $\pi_W(y)$ respectively. Let β be a $(1, 20E)$–quasi-geodesic from y to x in $\mathcal{X}$, and consider the unparametrized (λ, λ)–quasi-geodesic $\beta_W = \pi_W \circ \beta$ in $\mathcal{C}W$. By the Morse Lemma (Lemma 2.1), β (resp. β_W) and any $\mathcal{X}$–geodesic (resp. $\mathcal{C}W$–geodesic) from x to y (resp. x' to y') are each contained in the σ–neighborhood of each other for some σ determined by E. Combining this with Lemma 2.2 yields

$$|d_{\mathcal{X}}(x_0, \beta) - (x \mid y)_{x_0}| \leqslant E + \sigma \text{ and } |d_W(x_0, \beta_W) - \langle x' \mid y' \rangle_{x_0}| \leqslant E + \sigma.$$

Since $d_W(x_0, \beta_W) \leqslant E d_{\mathcal{X}}(x_0, \beta) + E$, we now have

$$\begin{aligned} r \leqslant \langle x' \mid p \rangle_{x_0} &\leqslant \langle x' \mid y' \rangle_{x_0} + B \\ &\leqslant d_W(x_0, \beta_W) + B + \sigma + E \\ &\leqslant E d_{\mathcal{X}}(x_0, \beta) + B + \sigma + 2E \\ &\leqslant E(x \mid y)_{x_0} + E(E + \sigma) + B + \sigma + 2E \\ &\leqslant E(x \mid p)_{x_0} + EB + E(E + \sigma) + B + \sigma + 2E. \end{aligned}$$

Hence if $r' = \frac{1}{E}(r - B - \sigma) - B - \sigma - E - 2$ and $r > (2B + 2\sigma + E + 2)E$, then $x \in M(r'; p)$.

Now we establish the third bullet. Let $q \in \partial\mathcal{C}W \cap M_W(r; p)$. By Theorem 3.24, the inclusion map continuously extends to a homeomorphism between $\partial\mathcal{X}$ and $\partial(\mathcal{X}, \mathfrak{S})$. Hence, we can consider $q \in \partial\mathcal{X}$ and find a sequence $(q_n) \subseteq \mathcal{X}$ that converges to q in both $\partial\mathcal{X}$ and $\partial(\mathcal{X}, \mathfrak{S})$. The definition of the topology on $\partial(\mathcal{X}, \mathfrak{S})$ ensures that $q_n \to q$ in $\mathcal{X}$ implies that for any choice $q_n' \in \pi_W(q_n)$, $q_n' \to q$ in $\mathcal{C}W$. Hence $\pi_W(q_n) \subseteq M_W(r; p)$ for all but finitely many n. The second bullet then says $q_n \in M(r'; p)$ for all sufficiently large n. Hence $q \in M(r'; p)$ as well. ∎

4. Maximization and the boundary

The goal of this section is to prove that the boundary of a proper hierarchically hyperbolic space with the bounded domain dichotomy is invariant under changing the structure by a procedure that we call *maximization.* Given a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ with the bounded domain dichotomy, maximization has two steps. First we replace $\mathfrak{S}$ with the set of *essential domains* (see Section 4.1 for the definition), denoted $\mathfrak{S}_{ess}$. Second, we apply the work of [ABD21] to obtain a new hierarchical structure on $\mathcal{X}$, denoted $(\mathcal{X}, \mathfrak{T})$, that satisfies several nice properties. If we want to emphasize the initial structure $\mathfrak{S}$, we call this procedure *maximizing the structure* $\mathfrak{S}$. The resulting structure $\mathfrak{T}$ is called the *maximized structure on* $\mathcal{X}$ *obtained from* $\mathfrak{S}$, or simply *a maximized structure on* $\mathcal{X}$, if the structure $\mathfrak{S}$ is implicit or irrelevant.

Our main result is that the HHS boundary of a G–HHS is invariant under maximization.

Theorem 4.1. *If* $(G, \mathfrak{S})$ *is a* G*–HHS and* $\mathfrak{T}$ *is the maximized structure on* $\mathcal{X}$ *obtained from* $\mathfrak{S}$*, then the identity map on* G *extends to a* G*–equivariant map* $\partial(G, \mathfrak{S}) \to \partial(G, \mathfrak{T})$ *that is both a simplicial isomorphism and a homeomorphism.*

While the case of G–HHSs is likely of primary interest, our proof will not use a group action in any way and will apply to any hierarchically hyperbolic space that is proper and has the bounded domain dichotomy; see Theorem 4.21 for this more general statement.

4.1. **The maximization procedure.** In this section, we will provide a detailed description of the two steps of maximization and show that the first step does not change the boundary of a hierarchically hyperbolic space. The proof that the boundary is invariant under the second step is more involved, and we will prove that in Section 4.6, after first developing some technical preliminaries in Sections 4.2–4.5.

Fix a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ with the bounded domain dichotomy. As every quasi-geodesic space is quasi-isometric to a geodesic space, we will assume for convenience that $\mathcal{X}$ is a geodesic metric space. In the context of G–HHSs, the space $\mathcal{X}$ can be taken to be a Cayley graph of the group with respect to a finite generating set.

Step 1: Essential domains. Let $\mathfrak{S}_{ess} \subseteq \mathfrak{S}$ be the set of domains $U \in \mathfrak{S}$ such that there exists some $V \sqsubseteq U$ so that $\mathcal{C}V$ has infinite diameter, that is, $V \in \mathfrak{S}^{\infty}$. We call elements of $\mathfrak{S}_{ess}$ *essential domains.* The first step of maximization is to replace $\mathfrak{S}$ with the set of essential domains $\mathfrak{S}_{ess}$.

Lemma 4.2. *Let* $(\mathcal{X}, \mathfrak{S})$ *be a hierarchically hyperbolic space with the bounded domain dichotomy. Then* $(\mathcal{X}, \mathfrak{S}_{ess})$ *is a hierarchically hyperbolic space and the identity* $\mathcal{X} \to \mathcal{X}$ *extends to map* $\partial(\mathcal{X}, \mathfrak{S}_{ess}) \to \partial(\mathcal{X}, \mathfrak{S})$ *that is both a simplicial automorphism and a homeomorphism.*

Proof. The set $\mathfrak{S} - \mathfrak{S}_{ess}$ is the set of domains $U \in \mathfrak{S}$ such that $\mathcal{C}V$ is uniformly bounded for every $V \sqsubseteq U$. Since this set is clearly closed under nesting, it follows from [BHS17a, Proposition 2.4] and the distance formula in hierarchically hyperbolic spaces [BHS19, Theorem 4.5] that $(\mathcal{X}, \mathfrak{S}_{ess})$ is a hierarchically hyperbolic space where all the relations, hyperbolic spaces, and projections are the same as in $(\mathcal{X}, \mathfrak{S})$. This yields a hieromorphism $f \colon (\mathcal{X}, \mathfrak{S}_{ess}) \to (\mathcal{X}, \mathfrak{S})$ where $f \colon \mathcal{X} \to \mathcal{X}$ is the identity, $f \colon \mathfrak{S}_{ess} \to \mathfrak{S}$ is the inclusion, and f_V is an isometry for all $V \in \mathfrak{S}_{ess}$. Therefore by Theorem 3.22, there is an injective simplicial map $\partial f \colon \partial_{\Delta}(\mathcal{X}, \mathfrak{S}_{ess}) \to \partial_{\Delta}(\mathcal{X}, \mathfrak{S})$ that is also a continuous map $\partial f \colon \partial(\mathcal{X}, \mathfrak{S}_{ess}) \to \partial(\mathcal{X}, \mathfrak{S})$. Moreover, since no domain in $\mathfrak{S} - \mathfrak{S}_{ess}$ contributes to $\partial(\mathcal{X}, \mathfrak{S})$, this map is a bijection and the basis neighborhoods (given by Definition 3.17) with respect to $\mathfrak{S}$ and $\mathfrak{S}_{ess}$ will be identical. Hence, the map ∂f is a homeomorphism from $\partial(\mathcal{X}, \mathfrak{S}_{ess}) \to \partial(\mathcal{X}, \mathfrak{S})$. □

We note that Lemma 4.2 implies that if a group G has two different G–HHS structures $\mathfrak{S}$ and $\mathfrak{S}'$ such that $\mathfrak{S}_{ess} = \mathfrak{S}'_{ess}$, then $\partial(G, \mathfrak{S})$ is homeomorphic to $\partial(G, \mathfrak{S}')$. More generally, the two boundaries associated to $\mathfrak{S}, \mathfrak{S}'$ are homeomorphic if there exists a hieromorphism $(G, \mathfrak{S}_{ess}) \to (G, \mathfrak{S}'_{ess})$ that satisfies the condition of Corollary 3.23.

Step 2: The new hierarchical structure. We describe the second and more involved step in the process of maximizing an HHS structure $(\mathcal{X}, \mathfrak{S})$. We refer the reader to [ABD21] for the proof that this process in fact gives an HHS structure on $\mathcal{X}$. We assume that we have already performed Step 1 so that $\mathfrak{S} = \mathfrak{S}_{ess}$.

Given $(\mathcal{X}, \mathfrak{S})$ an HHS with constant E satisfying the bounded domain dichotomy, define $\mathfrak{T}$ to be the subset of $\mathfrak{S}$ containing the $\sqsubseteq$–maximal element $S \in \mathfrak{S}$ as well as all domains $W \in \mathfrak{S}$ where $\mathbf{F}_W$ and $\mathbf{E}_W$ are both unbounded. Because $\mathfrak{S} = \mathfrak{S}_{ess}$ and $\mathfrak{S}$ has the bounded domain dichotomy, Proposition 3.9 says $\mathbf{F}_W$ and $\mathbf{E}_W$ will both be unbounded if and only if $\mathfrak{S}_W^{\perp} \neq \emptyset$. In particular, $W, V \in \mathfrak{S}$ are orthogonal if and only if $W, V \in \mathfrak{T} - \{S\}$ and are orthogonal in $\mathfrak{T}$.

The maximal structure on $\mathcal{X}$ obtained from $\mathfrak{S}$ has index set $\mathfrak{T}$. Before we describe the full hierarchy structure associated to the set of domains $\mathfrak{T}$, we fix some notation to differentiate in which structure a domain is being considered.

Notation 4.3. To distinguish which structure we are working in ($\mathfrak{S}$ vs $\mathfrak{T}$), we use the following convention. If nothing is appended to the notation, it occurs in $(\mathcal{X}, \mathfrak{S})$; for example, $\pi_W : \mathcal{X} \to \mathcal{C}W$ is the projection map in the structure $(\mathcal{X}, \mathfrak{S})$. For the hyperbolic spaces associated to the structure $(\mathcal{X}, \mathfrak{T})$, we use the notation $\mathcal{C}_{\mathfrak{T}}W$ for each $W \in \mathfrak{T}$. For most other notation $*$ that occurs in $(\mathcal{X}, \mathfrak{S})$, we will typically use $\overline{*}$ to denote the corresponding object in $(\mathcal{X}, \mathfrak{T})$. For example, $\overline{\pi}_W : \mathcal{X} \to \mathcal{C}_{\mathfrak{T}}W$ is a projection map in the structure $(\mathcal{X}, \mathfrak{T})$. Similarly, a point in $\partial(\mathcal{X}, \mathfrak{S})$ is simply denoted p, while a point in $\partial(\mathcal{X}, \mathfrak{T})$ is denoted $\overline{p}$. Given a point $\overline{p} \in \partial(\mathcal{X}, \mathfrak{T})$, we denote its support in $\mathfrak{T}$ by $\operatorname{supp}_{\mathfrak{T}}(\overline{p})$.

The relations between domains in $\mathfrak{T}$ are inherited from the relations in $\mathfrak{S}$, i.e., the relation between $W, V \in \mathfrak{T}$ is the same as the relation in $\mathfrak{S}$. If $W \in \mathfrak{T} - \{S\}$, then $\mathcal{C}W = \mathcal{C}_{\mathfrak{T}}W$ and the projection maps and relative projection maps are defined as in the original structure for any $W \in \mathfrak{T} - \{S\}$.

Thus the only associated hyperbolic space in the structure $(\mathcal{X}, \mathfrak{T})$ that is different is $\mathcal{C}_{\mathfrak{T}}S$. The hyperbolic space $\mathcal{C}_{\mathfrak{T}}S$ is defined as follows.

Definition 4.4. Let $\mathcal{C}_{\mathfrak{T}}S$ be the space obtained from $\mathcal{X}$ by adding an edge of length 1 between every pair of points x, y for which there is a $W \in \mathfrak{T} - \{S\}$ so that $x, y \in \mathbf{F}_W$.

For the $\sqsubseteq$–maximal domain $S \in \mathfrak{T}$ and any $W \in \mathfrak{T} - \{S\}$, we define $\overline{\pi}_S$ to be the inclusion map $\mathcal{X} \to \mathcal{C}_{\mathfrak{T}}S$, we define $\overline{\rho}_W^S$ be the map $\overline{\pi}_W \circ \overline{\pi}_S^{-1}$, and we define $\overline{\rho}_S^W$ to be the subset $\overline{\pi}_S(\mathbf{F}_W)$ in $\mathcal{C}_{\mathfrak{T}}S$.

Remark 4.5. Technically, $(\mathcal{X}, \mathfrak{T})$ as described is not a hierarchically hyperbolic space, because it may not satisfy the containers axiom (Definition 3.1(6)). In order to fix this problem, we actually define $\mathfrak{T}$ to be the union of the set described above along with a collection of *dummy domains*, whose associated hyperbolic spaces are points. These dummy domains essentially take the place of any containers that we may have removed when initially forming $\mathfrak{T}$ from $\mathfrak{S}$. Since to each dummy domain the associated hyperbolic space is defined to be a point, these domains do not contribute in any way to the HHS boundary, and hence we can ignore them in this paper. We refer the reader to [ABD21] for a detailed description of how the dummy domains are incorporated into the full hierarchy structure on $(\mathcal{X}, \mathfrak{T})$.

The fact that $\mathcal{C}_{\mathfrak{T}}S$ is a hyperbolic space is a consequence of the *factored space* construction in an HHS introduced in [BHS17a, §2]. In addition to hyperbolicity, this construction yields that $\mathcal{C}_{\mathfrak{T}}S$ inherits an HHS structure as described in the next result.

Proposition 4.6 ([BHS17a, Proposition 2.4] plus [BHS21, Corollary 2.16])**.** *Given an HHS $(\mathcal{X}, \mathfrak{S})$, there exists $E' \geqslant 0$, depending only on the HHS constant E of $(\mathcal{X}, \mathfrak{S})$, so that the space $\mathcal{C}_{\mathfrak{T}}S$ is E'–hyperbolic and admits an HHS structure with constant E' that has index set $(\mathfrak{S} - \mathfrak{T}) \cup \{S\}$ and where the hyperbolic spaces, relations, and projections are all inherited from $(\mathcal{X}, \mathfrak{S})$.*

The domains, hyperbolic spaces, and relative projections for the HHS $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$ are all identical to their counterparts from $\mathfrak{S}$. The projection maps need a little more illumination.

Recall, $\mathcal{C}_{\mathfrak{T}}S$ is the space $\mathcal{X}$ with additional edges attached. If $x \in \mathcal{C}_{\mathfrak{T}}S$ is also a point of $\mathcal{X}$, then for each $W \in (\mathfrak{S} - \mathfrak{T}) \cup \{S\}$, the projection $\pi_W(x)$ is the same as the projection to $\mathcal{C}W$ in $\mathfrak{S}$. If instead x is a point on an edge e that is added to $\mathcal{X}$ to make $\mathcal{C}_{\mathfrak{T}}S$, then $\pi_W(x)$ is the union of the images of two end points of e under π_W.

There are two important consequences of Proposition 4.6 that we will use repeatedly for the remainder of the section. First, Theorem 3.24 applies to the hyperbolic HHS $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$, so we can identify the Gromov boundary of $\mathcal{C}_{\mathfrak{T}}S$ with points in the Gromov boundaries of $\mathcal{C}W$ for $W \in (\mathfrak{S} - \mathfrak{T}) \cup \{S\}$. Second, we can use Lemma 3.25 to relate neighborhoods in $\mathcal{C}W \cup \partial\mathcal{C}W$ to neighborhoods in $\mathcal{C}_{\mathfrak{T}}S \cup \partial\mathcal{C}_{\mathfrak{T}}S$ when $W \in (\mathfrak{S} - \mathfrak{T}) \cup \{S\}$.

In order to prove Theorem 4.1, it remains to show that this second step in the maximization procedure (replacing $\mathfrak{S}$ with $\mathfrak{T}$) does not change the boundary of a G–HHS. The proof of this fact is involved, and we spend the next several subsections developing the necessary machinery and establishing a number of preliminary results. Theorem 4.1 is then proven in Section 4.6.

4.2. **Invariance of hierarchy paths and hierarchical quasiconvexity under maximization.** As in the previous subsection, $(\mathcal{X}, \mathfrak{S})$ is an HHS with constant E and the bounded domain dichotomy, and $(\mathcal{X}, \mathfrak{T})$ is the HHS produced after maximizing $\mathfrak{S}$. We denote the $\sqsubseteq$–maximal domain in both structures by S. The goal of this subsection is establish that hierarchy paths and hierarchical quasiconvexity do not change under the maximization procedure. These results are used to prove that Step 2 of the maximization procedure does not change the boundary, but we expect they will be of broader interest as well.

We start by quoting a result that says hierarchy paths with respect $\mathfrak{S}$ are also hierarchy paths with respect to $\mathfrak{T}$. This was established by the first two authors and Durham during the introduction of the maximization procedure.

Lemma 4.7 ([ABD21, Special case of Lemma 3.6])**.** *For each $\lambda \geqslant 1$ there exists $\lambda' \geqslant \lambda$ for which the following holds: if γ is a λ–hierarchy path in $(\mathcal{X}, \mathfrak{S})$, then $\overline{\pi}_S \circ \gamma$ is an unparametrized (λ', λ')–quasi-geodesic of $\mathcal{C}_{\mathfrak{T}}S$. In particular, every λ–hierarchy path of $(\mathcal{X}, \mathfrak{S})$ is also a λ'–hierarchy path of $(\mathcal{X}, \mathfrak{T})$.*

Next we establish the converse of Lemma 4.7, that hierarchy paths with respect to $\mathfrak{T}$ are also hierarchy paths with respect to $\mathfrak{S}$. This establishes that $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}, \mathfrak{T})$ have the same set of hierarchy paths (with possibly different constants).

Lemma 4.8. *For each $\lambda \geqslant 1$ there exists $\lambda' \geqslant \lambda$ for which the following holds: if γ is a λ–hierarchy path in $(\mathcal{X}, \mathfrak{T})$, then γ is also a λ'–hierarchy path of $(\mathcal{X}, \mathfrak{S})$.*

Proof. Let γ be a λ–hierarchy path in $(\mathcal{X}, \mathfrak{T})$. For each $W \in \mathfrak{T} - \{S\}$, the projection $\pi_W \circ \gamma$ is an unparametrized (λ, λ)–quasi-geodesic because $\pi_W = \overline{\pi}_W$. Now assume $W = S$ or $W \in \mathfrak{S} - \mathfrak{T}$. By Proposition 4.6, the space $\mathcal{C}_{\mathfrak{T}}S$ is a hierarchically hyperbolic space with respect to $\{S\} \cup (\mathfrak{S} - \mathfrak{T})$, where the projection maps are the projection maps in the structure $\mathfrak{S}$. Because $\mathcal{C}_{\mathfrak{T}}S$ is hyperbolic, every quasi-geodesic in $\mathcal{C}_{\mathfrak{T}}S$ is a hierarchy path in every HHS structure by Theorem 3.24. Thus $\pi_W \circ \overline{\pi}_S \circ \gamma$ is an unparametrized (λ', λ')-quasi-geodesic in $\mathcal{C}W$. Because $\overline{\pi}_S$ is the inclusion map, we have that $\pi_W \circ \gamma$ is an unparametrized (λ', λ')–quasi-geodesic in $\mathcal{C}W$. □

Using a result of the third author with Spriano and Tran, the above lemmas imply that the sets of hierarchically quasiconvex subsets of $\mathcal{X}$ with respect to $\mathfrak{S}$ and $\mathfrak{T}$ are the same.

Proposition 4.9. *A subset $\mathcal{Y} \subseteq \mathcal{X}$ is hierarchically quasiconvex with respect to $\mathfrak{S}$ if and only if it is hierarchically quasiconvex with respect to $\mathfrak{T}$. Further, the function of hierarchical quasiconvexity in either $\mathfrak{S}$ or $\mathfrak{T}$ will determine the function in the other.*

Proof. Lemmas 4.7 and 4.8 show that every hierarchy path of $(\mathcal{X}, \mathfrak{S})$ is a hierarchy path of $(\mathcal{X}, \mathfrak{T})$ and vice-versa. The proposition then follows from [RST18, Proposition 5.7], which states that a subset $\mathcal{Y}$ of an HHS is hierarchically quasiconvex if and only if there is a function $F\colon [1,\infty) \to [0,\infty)$ so that for every $\lambda \geqslant 1$, every λ–hierarchy path based on $\mathcal{Y}$ is contained in the $F(\lambda)$–neighborhood of $\mathcal{Y}$. The statement on the function of hierarchical quasiconvexity also follows from [RST18, Proposition 5.7], which additionally shows that the function k of hierarchical quasiconvexity and the function F each determine the other. □

Proposition 4.9 is most relevant for us in the case of the sets $\mathbf{F}_W$ in $(\mathcal{X},\mathfrak{S})$ and $\overline{\mathbf{F}}_W$ in $(\mathcal{X},\mathfrak{T})$. While $\mathbf{F}_W$ might not equal $\overline{\mathbf{F}}_W$ even when $W \in \mathfrak{S} \cap \mathfrak{T}$, they are each hierarchically quasiconvex with respect to their respective structures (Proposition 3.9(1)). By Proposition 4.9, there is thus some k depending only on $(\mathcal{X},\mathfrak{S})$ such that $\mathbf{F}_W$ and $\overline{\mathbf{F}}_W$ are each k–hierarchically quasiconvex with respect to both $\mathfrak{S}$ and $\mathfrak{T}$. In particular, the projection $\overline{\pi}_S(\mathbf{F}_W)$ is a $k(0)$–quasiconvex subspace of the hyperbolic space $\mathcal{C}_{\mathfrak{T}}S$.

Since a subset $\mathcal{Y}$ is hierarchically quasiconvex with respect to $\mathfrak{S}$ if and only if it is hierarchically quasiconvex with respect to $\mathfrak{T}$, such a subset has a gate map with respect to each structure. We denote the gate map in $\mathfrak{S}$ by $\mathfrak{g}_{\mathcal{Y}}$ and the gate map in $\mathfrak{T}$ by $\overline{\mathfrak{g}}_{\mathcal{Y}}$. Our final lemma says these two gate maps are coarsely the same. The key step is relating the the gate map in $\mathfrak{S}$ to the closest point projection onto the image of a hierarchically quasiconvex subset in $\mathcal{C}_{\mathfrak{T}}S$.

Lemma 4.10. *Suppose $\mathcal{Y} \subseteq \mathcal{X}$ is k–hierarchically quasiconvex with respect to $\mathfrak{S}$. There exists $C_1, C_2 \geqslant 0$ depending on k and $\mathfrak{S}$ so that for all $x \in \mathcal{X}$ we have*

$$\mathfrak{p}_{\overline{\pi}_S(\mathcal{Y})}(\overline{\pi}_S(x)) \asymp_{C_1} \overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x)) \text{ and } \mathfrak{g}_{\mathcal{Y}}(x) \asymp_{C_2} \overline{\mathfrak{g}}_{\mathcal{Y}}(x).$$

Proof. Let E be the hierarchy constant for $\mathfrak{S}$ and $\mathfrak{T}$. Fix $x \in \mathcal{X}$, and let y be any point of $\mathcal{Y}$ satisfying $\overline{\pi}_S(y) \in \mathfrak{p}_{\overline{\pi}_S(\mathcal{Y})}(\overline{\pi}_S(x))$. Since $\overline{\pi}_S(y), \overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x))$ and $\mathfrak{g}_{\mathcal{Y}}(x), \overline{\mathfrak{g}}_{\mathcal{Y}}(x)$ all have diameter uniformly bounded in terms of E, the two coarse equalities will follow if we can bound the distances $\overline{d}_S(\overline{\pi}_S(y), \overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x)))$ and $d_{\mathcal{X}}(\mathfrak{g}_{\mathcal{Y}}(x), \overline{\mathfrak{g}}_{\mathcal{Y}}(x))$, respectively.

By Lemma 3.8, there exists $\lambda \geqslant 1$ depending only on k and $\mathfrak{S}$ so that there is a λ–hierarchy path γ in $(\mathcal{X},\mathfrak{S})$ that connects x and y and passes within λ of $\mathfrak{g}_{\mathcal{Y}}(x)$. Let y' be a point on γ with $d_{\mathcal{X}}(y', \mathfrak{g}_{\mathcal{Y}}(x)) \leqslant \lambda$.

By Lemma 4.7, γ is also a hierarchy path in $(\mathcal{X},\mathfrak{T})$, and so $\overline{\pi}_S \circ \gamma$ is an unparametrized (λ',λ')–quasi-geodesic in $\mathcal{C}_{\mathfrak{T}}S$ for some λ' ultimately depending only on $\mathfrak{S}$ and k. By the Morse Lemma (Lemma 2.1), there is $\sigma \geqslant 0$, depending ultimately only on $\mathfrak{S}$ and k, so that $\overline{\pi}_S(\gamma)$ is contained in the σ–neighborhood of any $\mathcal{C}_{\mathfrak{T}}S$–geodesic $[\overline{\pi}_S(x), \overline{\pi}_S(y)]$. Since $y' \in \gamma$ and $\overline{\pi}_S$ is 1–Lipschitz, we know $\overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x))$ is within $\sigma' = \lambda + \sigma$ of $[\overline{\pi}_S(x), \overline{\pi}_S(y)]$. Since $\overline{\pi}_S(y) \in \mathfrak{p}_{\overline{\pi}_S(\mathcal{Y})}(\overline{\pi}_S(x))$, we have $\overline{d}_S(\overline{\pi}_S(y), \overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x))) \leqslant 2\sigma' + 1$, where σ' depends only on $\mathfrak{S}$ and k. This establishes the first coarse equality.

Since $\mathfrak{p}_{\overline{\pi}_S(\mathcal{Y})}(\overline{\pi}_S(x)) \asymp_{\lambda} \overline{\pi}_S(\mathfrak{g}_{\mathcal{Y}}(x))$, the uniform bound on $d_{\mathcal{X}}(\mathfrak{g}_{\mathcal{Y}}(x), \overline{\mathfrak{g}}_{\mathcal{Y}}(x))$ now follows from the uniqueness axiom in $\mathfrak{T}$ (Definition 3.1(7)), because $\pi_U(\mathfrak{g}_{\mathcal{Y}}(x))$ will be uniformly close to $\overline{\pi}_U(\overline{\mathfrak{g}}_{\mathcal{Y}}(x))$ for all $U \in \mathfrak{T} - \{S\}$. □

4.3. A bijection from $\partial_\Delta(\mathcal{X},\mathfrak{S})$ to $\partial_\Delta(\mathcal{X},\mathfrak{T})$.

In this section, we define a simplicial isomorphism

$$\phi\colon \partial_\Delta(\mathcal{X},\mathfrak{S}) \to \partial_\Delta(\mathcal{X},\mathfrak{T}).$$

In Section 4.6, we will prove that this map is a homeomorphism from $\partial(\mathcal{X},\mathfrak{S})$ to $\partial(\mathcal{X},\mathfrak{T})$. By Lemma 4.2, we may assume that the first step of the maximization procedure has already been applied to $(\mathcal{X},\mathfrak{S})$. Thus we have a standing assumption for the remainder of this section that $\mathfrak{S} = \mathfrak{S}_{ess}$.

We first define ϕ for points $p \in \partial_\Delta(\mathcal{X},\mathfrak{S})$ whose support is contained in $\mathfrak{T} - \{S\}$. Recall, if $W \in \mathfrak{T} - \{S\}$, then $\mathcal{C}W = \mathcal{C}_{\mathfrak{T}}W$. Moreover, because $\mathfrak{S} = \mathfrak{S}_{ess}$, we have $W, V \in \mathfrak{S}$ are orthogonal if and only if $W, V \in \mathfrak{T} - \{S\}$ and are orthogonal in $\mathfrak{T}$. Thus, each point $p \in \partial_\Delta(\mathcal{X},\mathfrak{S})$ with $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$ is also a point in $\partial_\Delta(\mathcal{X},\mathfrak{T})$ with the same support. For such points we define $\phi(p) = p$.

Now consider $p \in \partial_\Delta(\mathcal{X}, \mathfrak{S})$ with $\operatorname{supp}(p) \not\subseteq \mathfrak{T} - \{S\}$. As supports are pairwise orthogonal collections of domains and the only non-singleton sets of orthogonal domains of $\mathfrak{S}$ are contained in $\mathfrak{T} - \{S\}$, this implies $\operatorname{supp}(p) = \{P\}$ for some $P \in \{S\} \cup (\mathfrak{S} - \mathfrak{T})$. In this case, we define ϕ using the fact given by Proposition 4.6 that $(\partial \mathcal{C}_{\mathfrak{T}} S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$ is a hyperbolic HHS. By Theorem 3.24, the identity map on $\mathcal{C}_{\mathfrak{T}} S$ extends to a homeomorphism from the Gromov boundary $\partial \mathcal{C}_{\mathfrak{T}} S$ to the HHS boundary $\partial(\mathcal{C}_{\mathfrak{T}} S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$. This homeomorphism gives a bijection from $\{p \in \partial \mathcal{C} W \mid W \in \{S\} \cup (\mathfrak{S} - \mathfrak{T})\}$ to $\partial \mathcal{C}_{\mathfrak{T}} S$. Hence, if $p \in \partial \mathcal{C} W$ for some $W \in \{S\} \cup (\mathfrak{S} - \mathfrak{T})$, then $\phi(p)$ will be the image of p under this identification.

For each $p \in \partial_\Delta(\mathcal{X}, \mathfrak{S})$, we will denote $\phi(p)$ by $\overline{p}$. For each $P \in \mathfrak{S}$, we also define a corresponding domain $\overline{P} \in \mathfrak{T}$ by $\overline{P} = P$ if $P \in \mathfrak{T}$ and $\overline{P} = S$ if $P \in \mathfrak{S} - \mathfrak{T}$. This definition ensures the following basic fact.

Lemma 4.11. *If $p \in \partial_\Delta(\mathcal{X}, \mathfrak{S})$ and $P \in \operatorname{supp}(p)$, then $\overline{P} \in \operatorname{supp}_{\mathfrak{T}}(\overline{p})$. Moreover, $|\operatorname{supp}(p)| = |\operatorname{supp}_{\mathfrak{T}}(\overline{p})|$.*

Proof. As described in the preceding paragraphs, either $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$ or $\operatorname{supp}(p) \subseteq \{S\} \cup (\mathfrak{S} - \mathfrak{T})$. If $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$, then $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$ and $\overline{P} = P$ for each $P \in \operatorname{supp}(p)$. Since $(\mathfrak{S} - \mathfrak{T}) \cup \{S\}$ does not contain any pairwise orthogonal domains, if $\operatorname{supp}(p) \subseteq (\mathfrak{S} - \mathfrak{T}) \cup \{S\}$, then $\operatorname{supp}(p) = \{P\}$ for some $P \in (\mathfrak{S} - \mathfrak{T}) \cup \{S\}$. In this case $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ and $\overline{P} = S$. □

Note that if $p = \sum_{W \in \operatorname{supp}(p)} a_W p_W$, then $\overline{p} = \sum_{W \in \operatorname{supp}(p)} a_{\overline{W}} \overline{p}_{\overline{W}}$ where $a_W = a_{\overline{W}}$ and $\overline{p}_{\overline{W}} = \phi(p_W)$.

4.4. **Defining neighborhoods in $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}, \mathfrak{T})$.** The key step in proving that Step 2 of the maximization procedure does not change the boundary is to understand how basis neighborhoods in $\partial(\mathcal{X}, \mathfrak{T})$ relate to those in $\partial(\mathcal{X}, \mathfrak{S})$.

In addition to assuming that $\mathfrak{S} = \mathfrak{S}_{ess}$, we make the following standing assumption to simplify notation. One consequence of this assumption is that the projection map associated to any W now has codomain $\mathcal{C}W$; this ensures that the preimage $\pi_W^{-1}(X)$ is well defined for a subset $X \subseteq \mathcal{C}W$.

Standing Assumption 4.12. Given an HHS $(\mathcal{X}, \mathfrak{S})$, we will assume that for each $W \in \mathfrak{S}$ and $x \in \mathcal{X}$, $\pi_W(x)$ is a single point instead of a bounded diameter set. This can always be done by replacing the image $\pi_W(x)$ with a choice of a single point in $\pi_W(x)$. This modification gives a hieromorphism where the map on index sets is bijection and the maps between hyperbolic spaces are isometries. Hence, Corollary 3.23 ensures that this assumption does not affect the HHS boundary. Note this assumption may increase the hierarchy constant from E to $3E$.

We also fix the following constant for the remainder of the section.

Notation 4.13. We first fix a constant E larger than twice all the HHS constants for $(\mathcal{X}, \mathfrak{S})$, $(\mathcal{X}, \mathfrak{T})$, and $(\mathcal{C}_{\mathfrak{T}} S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$, as well as the hyperbolicity constant of $\mathcal{C}_{\mathfrak{T}} S$. This includes making E larger than the diameter of the boundary projections in each structure. Next, we define a constant C to be

$$C = 2\kappa + 8E + 1,$$

where E is the constant fixed above and κ is the maximum of the constants for the gate map and the $\mathbf{F}_W$'s from Lemma 3.8 and Proposition 3.9(2). (Essentially, C is chosen large enough to accommodate any coarseness from the HHS properties.)

For convenience, we also define the following subsets of $\mathcal{C}_{\mathfrak{T}} S$.

Definition 4.14. For each $W \in \mathfrak{S} - \mathfrak{T}$, define Y_W to be the subspace of $\mathcal{C}_{\mathfrak{T}} S$ given by

$$Y_W := \overline{\pi}_S(\mathbf{F}_W).$$

Remark 4.15 (Quasiconvexity and boundary of Y_W)**.** Because $\mathbf{F}_W$ is hierarchically quasiconvex in $(\mathcal{X}, \mathfrak{S})$, it is also hierarchically quasiconvex in $(\mathcal{X}, \mathfrak{T})$ by Proposition 4.9. Let k be the function so each $\mathbf{F}_W$ is k-hierarchically quasiconvex with respect to both $\mathfrak{S}$ and $\mathfrak{T}$. By the definition of hierarchical quasiconvexity, the space Y_W is a $k(0)$–quasiconvex subspace of the hyperbolic space $\mathcal{C}_{\mathfrak{T}}S$.

If $q \in \partial(\mathcal{X}, \mathfrak{S})$ with $\operatorname{supp}(q) = \{Q\}$ for some $Q \in \mathfrak{S} - \mathfrak{T}$, then Proposition 3.9(2) ensures that there is a sequence of points (x_n) in $\mathbf{F}_Q$ so that the sequence $(\pi_Q(x_n))$ converges to q in $\mathcal{C}Q \cup \partial\mathcal{C}Q$. Since $Q \in \mathfrak{S} - \mathfrak{T}$, the point $\overline{q}$ is in $\partial\mathcal{C}_{\mathfrak{T}}S$ and Lemma 3.25 ensures that the sequence $(\overline{\pi}_S(x_n))$ will then converge to $\overline{q}$ in $\mathcal{C}_{\mathfrak{T}}S \cup \partial\mathcal{C}_{\mathfrak{T}}S$. Hence $\overline{q} \in \partial Y_Q$.

Fix a basepoint $x_0 \in \mathcal{X}$, and let $\pi_W(x_0)$ be the basepoint with respect to which the boundary $\partial\mathcal{C}W$ is constructed for each $W \in \mathfrak{S}$. Given a point $p = \sum_{i=1}^n a_{U_i} p_{U_i} \in \partial(\mathcal{X}, \mathfrak{S})$ and a basic set $\mathcal{B}_{r,\varepsilon}(p)$ in the topology on $\partial(\mathcal{X}, \mathfrak{S})$, there is an associated collection of neighborhoods $M(r; p_{U_i})$ of p_{U_i} in $\mathcal{C}U_i$. The goal of this subsection is to define an associated collection of sets in the hyperbolic spaces in the structure $\mathfrak{T}$. In Section 4.6, we will discuss how this associated collection of sets is related to a basic set in the topology on $\partial(\mathcal{X}, \mathfrak{T})$. Given a neighborhood $M(r; p_W)$ of $p_W \in \partial\mathcal{C}W$, we will define a corresponding neighborhood $M(R_r; \overline{p}_{\overline{W}})$ in $\mathcal{C}_{\mathfrak{T}}\overline{W} \cup \partial\mathcal{C}_{\mathfrak{T}}\overline{W}$. In what follows, $M^\circ(*; p_W)$ denotes the set $M(*; p_W) \cap \mathcal{C}W$, that is, $M^\circ(*; p_W)$ is the subset of the neighborhood that is *not* in the boundary of $\mathcal{C}W$.

Definition 4.16 (Neighborhoods in $\mathfrak{T}$)**.** Let $M(r; p_W)$ be a neighborhood in $\mathcal{C}W \cup \partial\mathcal{C}W$ of $p_W \in \partial\mathcal{C}W$ for some $W \in \mathfrak{S}$. Let E and C be the constants from Notation 4.13 and assume $r \geqslant r_0$, where r_0 is the constant from Lemma 3.25 for E.

First, we define an intermediate subset $\widetilde{M}(r; \overline{p}_{\overline{W}})$ as

$$\widetilde{M}(r; \overline{p}_{\overline{W}}) := \mathcal{N}_C\left(\overline{\pi}_{\overline{W}}\left(\pi_W^{-1}(\mathcal{N}_C(M^\circ(r; p_W)))\right)\right).$$

We now use $\widetilde{M}(r; \overline{p}_{\overline{W}})$ to define a neighborhood of $\overline{p}_{\overline{W}}$ in $\mathcal{C}_{\mathfrak{T}}\overline{W}$. By a straightforward calculation with Gromov products, $\mathcal{N}_C(M^\circ(r; p_W)) \subseteq M(r - 2C; p_W)$. If $\overline{W} = W$, then $\mathcal{C}W = \mathcal{C}_{\mathfrak{T}}W$ and $p_W = \overline{p}_{\overline{W}}$. Thus we have

$$\widetilde{M}(r; \overline{p}_{\overline{W}}) \subseteq \mathcal{N}_{2C}(M^\circ(r, p_W)) \subseteq M(r - 4C; \overline{p}_{\overline{W}}).$$

If instead $\overline{W} = S$, then $\overline{p}_{\overline{W}}$ is a point in $\partial\mathcal{C}_{\mathfrak{T}}S$. Since $r \geqslant r_0$, we can therefore apply Lemma 3.25 to the E–hyperbolic HHS $(\mathcal{C}_{\mathfrak{T}}S, \{S\} \cup (\mathfrak{S} - \mathfrak{T}))$ to see that

$$\widetilde{M}(r; \overline{p}_{\overline{W}}) \subseteq M(r'; \overline{p}_{\overline{W}})$$

for some r' determined by r and E. Setting $R_r = \max\{r', r - 4C\}$, the desired neighborhood is $M(R_r; \overline{p}_{\overline{W}})$.

The next lemma verifies that R_r is an increasing function of r.

Lemma 4.17. *Given a neighborhood $M(r; p_W)$ in $\mathcal{C}W \cup \partial\mathcal{C}W$ (where $r \geqslant r_0$) and its associated neighborhood $M(R_r; \overline{p}_{\overline{W}})$ in $\mathcal{C}_{\mathfrak{T}}\overline{W} \cup \partial\mathcal{C}_{\mathfrak{T}}\overline{W}$ as in Definition 4.16, the quantity R_r is an increasing linear function of r with the constant of linearity determined by E.*

Proof. From Definition 4.16, $R_r = \max\{r', r - 4C\}$. Since C is determined by the hierarchy constant E, the result follows from Lemma 3.25, as r' is an increasing linear function of r with the constant of linearity determined by E. □

4.5. **How boundary projections behave when switching structures.** We now prove three technical lemmas that let us understand how the boundary projections change when switching from $\mathfrak{S}$ to its maximization $\mathfrak{T}$. These lemmas will be essential in the proof of Theorem 4.1.

The first lemma describes a specific situation when the boundary projection changes by only a uniformly bounded amount.

Lemma 4.18. *Let $q, p \in \partial(G, \mathfrak{S})$, and suppose q is remote to p and $\overline{q}$ is remote to $\overline{p}$. Suppose* $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$ *and* $\operatorname{supp}(q) = \{Q\}$ *for some* $Q \in \{S\} \cup (\mathfrak{S} - \mathfrak{T})$*. If* $W \in \operatorname{supp}(p)$ *or* $W \in \operatorname{supp}(p)^{\perp}$ *with* $W \not\perp Q$*, then we have*

$$\operatorname{diam}_{\mathcal{C}W}\left(\partial\overline{\pi}_W(\overline{q}) \cup \partial\pi_W(q)\right) \leqslant C - 2E.$$

Proof. Let σ be the Morse constant (Lemma 2.1) for a $(1, 20E)$–quasi-geodesic in an E–hyperbolic space. Since $\operatorname{supp}(q) = \{Q\} \subset \{S\} \cup (\mathfrak{S} - \mathfrak{T})$, we have $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) = \{S\}$. Thus for any $W \in \mathfrak{T}$, the boundary projection $\partial\overline{\pi}_W(\overline{q})$ is defined as $\overline{\rho}^S_W(Z)$ where Z is the set of all point of $\mathcal{C}_{\mathfrak{T}}S$ that are at least $E + \sigma$ far from $\overline{\rho}^W_S = \overline{\pi}_S(\mathbf{F}_W)$ and lie on a $(1, 20E)$–quasi-geodesic from a point in $\overline{\rho}^W_S$ to $\overline{q}$.

On the other hand, the boundary projection $\partial\pi_W(q)$ depends on the relation between W and Q. The only way $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$ is if $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$. Since the only orthogonality of $\mathfrak{S}$ or $\mathfrak{T}$ happens in $\mathfrak{T} - \{S\}$, we also have $\operatorname{supp}(p)^{\perp} \subseteq \mathfrak{T} - \{S\}$. This means $Q \not\sqsubseteq W$ as $Q \notin \mathfrak{T} - \{S\}$. Similarly, $Q \not\perp W$ as the only orthogonality occurs among domains of $\mathfrak{T} - \{S\}$. Hence we must have $Q \pitchfork W$ or $W \sqsubsetneq Q$. We consider each case separately, because the definition of the boundary projection $\partial\pi_P(q)$ depends on which relation holds.

If $Q \pitchfork W$, the boundary projection of q to W is defined as $\partial\pi_W(q) = \rho^Q_W$. Since $\operatorname{supp}(q) = \{Q\}$ and $Q \notin \mathfrak{T}$, consider the subspace $Y_Q = \overline{\pi}_S(\mathbf{F}_Q)$ of $\mathcal{C}_{\mathfrak{T}}S$. By Remark 4.15, Y_Q is $k(0)$–quasiconvex subset of $\mathcal{C}_{\mathfrak{T}}S$ and $\overline{q} \in \partial Y_Q$. Hence, Lemma 2.6 provides a constant $A \geqslant 0$ and an $(1, 20E + 2A)$–quasi-geodesic from a point in $\overline{\rho}^W_S$ to $\overline{q}$ that eventually lies in Y_Q. Denote this quasi-geodesic by α. By the Morse Lemma (Lemma 2.1), α is contained in a σ'–neighborhood of any $(1, 20E)$–quasi-geodesic from $\overline{\rho}^W_S$ to $\overline{q}$, where σ' is determined by E. In particular, by going sufficiently far along α, we can find a point $x \in \alpha \cap Y_Q$ and a point $y \in Z$ so that the $\mathcal{C}_{\mathfrak{T}}S$–geodesic from x to y avoids $\mathcal{N}_{2E}(\overline{\rho}^W_S)$. Moreover, we can choose x and y so that they are points in $\mathcal{X}$ in addition to points in $\mathcal{C}_{\mathfrak{T}}S$. The bounded geodesic image axiom (Definition 3.1(8)) in $\mathfrak{T}$ now says $\operatorname{diam}(\overline{\rho}^S_W(x) \cup \overline{\rho}^S_W(y)) \leqslant E$.

Since $x \in \alpha \cap Y_Q$, we have

$$\overline{\rho}^S_W(x) = \pi_W(\overline{\pi}_S^{-1}(x)) \subseteq \pi_W(\mathbf{F}_Q).$$

By Proposition 3.9, $\pi_W(\mathbf{F}_Q) \asymp_\kappa \rho^Q_W$. Putting these together, we have

$$\operatorname{diam}_{\mathcal{C}W}\left(\partial\overline{\pi}_W(\overline{q}) \cup \partial\pi_W(q)\right) = \operatorname{diam}_{\mathcal{C}W}(\overline{\rho}^W_S(Z) \cup \rho^Q_W) \leqslant 2E + \kappa.$$

The definition of C ensures $2E + \kappa \leqslant C - 2E$, finishing the proof in this case.

If $W \sqsubsetneq Q$, the boundary projection $\partial\pi_W(q)$ is defined in terms of projections of quasi-geodesic rays. Since $Q \notin \mathfrak{T} - \{S\}$, $\partial\mathcal{C}Q$ is a subset of $\partial\mathcal{C}_{\mathfrak{T}}S$. Pick $z \in \mathbf{F}_W$, and let α be a $(1, 20E)$–quasi-geodesic in $\mathcal{C}_{\mathfrak{T}}S$ from $\overline{\pi}_S(z) \subseteq \overline{\rho}^W_S$ to $\overline{q} \in \partial\mathcal{C}Q \subset \partial\mathcal{C}_{\mathfrak{T}}S$. By Theorem 3.24, the quasi-geodesic α is a hierarchy path in the hyperbolic HHS $\mathcal{C}_{\mathfrak{T}}S$. Thus $\pi_Q \circ \alpha$ is an unparametrized (λ, λ)–quasi-geodesic in $\mathcal{C}Q$ for some λ determined by E, where here π_Q is the projection map in $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$. Since $\rho^W_Q \asymp_C \pi_Q(\mathbf{F}_W)$ by Proposition 3.9, $\pi_Q \circ \alpha$ gives a quasi-geodesic ray in $\mathcal{C}Q$ that goes from $\pi_Q(z) \in \mathcal{N}_C(\rho^W_Q)$ to q. Hence $\pi_Q(\alpha)$ will be contained in a uniform neighborhood of any $(1, 20E)$–quasi-geodesic from ρ^W_Q to q. There is therefore a point $x \in \alpha$ so that if α_0 is the subray of α starting at x, then $\overline{\rho}^S_W(\alpha_0) \subseteq \partial\overline{\pi}_W(\overline{q})$ and $\rho^Q_W(\pi_Q(\alpha_0)) \subseteq \mathcal{N}_E(\partial\pi_W(q))$ (this second inclusion follows from the bounded geodesic image axiom in $\mathfrak{S}$). In particular, $d_Q(\rho^W_Q, \alpha_0)$ and $\overline{d}_S(\overline{\rho}^W_S, \alpha_0)$ are both strictly larger than E. See Figure 1 for a summary of the situation.

Since $\overline{\pi}_S$ is the inclusion map, we can further select x so that $\overline{\pi}_S^{-1}(x) = x$. Since $d_Q(\rho^W_Q, x) > E$, the consistency axiom in $\mathfrak{S}$ says

$$\operatorname{diam}_{\mathcal{C}W}\left(\rho^Q_W(\pi_Q(\overline{\pi}_S^{-1}(x))) \cup \pi_W(\overline{\pi}_S^{-1}(x))\right) \leqslant E. \tag{1}$$

By our choice of x, we have $\pi_Q(\overline{\pi}_S^{-1}(x)) \subseteq \pi_Q(\alpha_0)$, and hence

$$\rho^Q_W(\pi_Q(\overline{\pi}_S^{-1}(x))) \subseteq \rho^W_Q(\pi_Q(\alpha_0)) \subseteq \mathcal{N}_E(\partial\pi_W(q)).$$

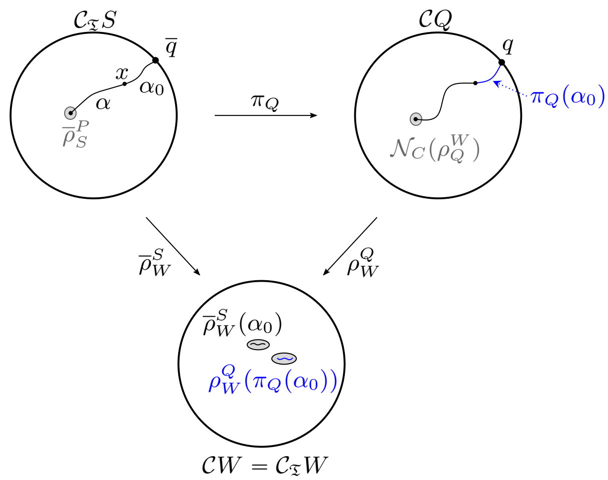

FIGURE 1. Proof of Lemma 4.18 when $W \subsetneq Q$.

Equation (1) then implies

$$\operatorname{diam}_{\mathcal{C}W}\left(\partial\pi_W(q) \cup \pi_W(\overline{\pi}_S^{-1}(x))\right) \leqslant 4E. \tag{2}$$

Similarly, because $\overline{d}_S(\overline{\rho}_S^W, x) > E$, the consistency axiom in $\mathfrak{T}$ says

$$\operatorname{diam}_{\mathcal{C}W}\left(\overline{\rho}_W^S(x) \cup \pi_W(\overline{\pi}_S^{-1}(x))\right) \leqslant E.$$

Since x was chosen so that $\overline{\rho}_W^S(x) \subseteq \partial\overline{\pi}_W(\overline{q})$, this implies

$$\operatorname{diam}_{\mathcal{C}W}\left(\partial\overline{\pi}_W(\overline{q}) \cup \pi_W(\overline{\pi}_S^{-1}(\overline{x}))\right) \leqslant 2E. \tag{3}$$

Applying the triangle inequality to (2) and (3), we obtain

$$\begin{aligned}\operatorname{diam}_{\mathcal{C}W}\left(\partial\pi_W(q) \cup \partial\overline{\pi}_W(\overline{q})\right) &\leqslant \operatorname{diam}_{\mathcal{C}W}\left(\partial\pi_W(q) \cup \pi_W(\overline{\pi}_S^{-1}(\overline{x}))\right) + \operatorname{diam}_{\mathcal{C}W}\left(\pi_W(\overline{\pi}_S^{-1}(\overline{x})) \cup \partial\overline{\pi}_W(\overline{q})\right)\\ &\leqslant 4E + 2E = 6E.\end{aligned}$$

As $6E < C - 2E$, this completes the proof of Lemma 4.18. □

The next two lemmas describe how switching structures affects the interaction of boundary projections with neighborhoods. Roughly, the lemmas state that if we have two points $p, q \in \partial(\mathcal{X}, \mathfrak{S})$ with q remote to p and a domain $P \in \operatorname{supp}(p)$, then if the boundary projection in $(\mathcal{X}, \mathfrak{S})$ of q to P is contained in the neighborhood $M(r; p_P)$ of p_P in $\mathcal{C}P$, then the boundary projection in $(\mathcal{X}, \mathfrak{T})$ of $\overline{q}$ to $\overline{P}$ (or $\overline{q}$ itself) is contained in the associated set $M(R_r; p_{\overline{P}})$ in $\mathcal{C}_{\mathfrak{T}}\overline{P}$. Here, $\overline{P}$ is as defined in Section 4.4; see Definitions 2.3 and 4.16 for the definitions of $M(r; p_P)$ and $M(R_r; \overline{p}_{\overline{P}})$, respectively. The statements are made precise by considering how $\overline{q}$ and $\overline{p}$ are related. Lemma 4.19 handles the case when $\overline{q}$ is remote to $\overline{p}$ and is broken into two subcases depending on whether $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$ or not. Lemma 4.20 handles the case when $\overline{q}$ is not remote to $\overline{p}$ and assumes that $\operatorname{supp}(p) \neq \operatorname{supp}_{\mathfrak{T}}(\overline{p})$, which is the only case we will need for the proof of Theorem 4.1.

Recall, we are still operating under the standing assumptions that $\mathfrak{S} = \mathfrak{S}_{ess}$ (Lemma 4.2) and that $\pi_W(x)$ is a single point for each $x \in \mathcal{X}$ and $W \in \mathfrak{S}$ (Standing Assumption 4.12).

Lemma 4.19. *Let $q, p \in \partial(G, \mathfrak{S})$, and suppose q is remote to p and $\overline{q}$ is remote to $\overline{p}$. Let $r \geqslant r_0$, where r_0 is the lower bound on r required in Definition 4.16.*

(1) If $\operatorname{supp}(p) = \{P\}$ and $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ (including the possibility that $P = S$), then

$$\partial\pi_P(q) \subseteq M(r; p_P) \qquad \Longrightarrow \qquad \partial\overline{\pi}_S(\overline{q}) \subseteq M(R_r; \overline{p}_S).$$

(2) If $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$*, then for any* $P \in \operatorname{supp}(p)$

$$\partial\pi_P(q) \subseteq M(r; p_P) \qquad \Longrightarrow \qquad \partial\overline{\pi}_P(\overline{q}) \subseteq M(R_r; \overline{p}_P).$$

Proof. **Proof of (1).** Suppose $\operatorname{supp}(p) = \{P\}$ and $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$. We first determine how the boundary projections $\partial\overline{\pi}_S(\overline{q})$ and $\partial\pi_P(q)$ are defined. Since $\overline{q}$ is remote to $\overline{p}$, we must have $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) \cap \operatorname{supp}_{\mathfrak{T}}(\overline{q}) = \varnothing$ (see Definition 3.13), and so $S \notin \operatorname{supp}_{\mathfrak{T}}(\overline{q})$. Thus $Q \subsetneq S$ for all $Q \in \operatorname{supp}_{\mathfrak{T}}(\overline{q})$, and the boundary projection (Definition 3.12) of $\overline{q}$ to S is defined as

$$\partial\overline{\pi}_S(\overline{q}) = \bigcup_{Q \in \operatorname{supp}_{\mathfrak{T}}(\overline{q})} \overline{\rho}_S^Q.$$

Moreover, $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) \neq \{S\}$ implies that $\operatorname{supp}(q) = \operatorname{supp}_{\mathfrak{T}}(\overline{q})$, and so $Q \in \mathfrak{T} - \{S\}$ for all $Q \in \operatorname{supp}(q)$. However, since $P \notin \mathfrak{T} - \{S\}$, it is not possible that $P \perp Q$ or $P \subsetneq Q$ for any $Q \in \operatorname{supp}(q)$. Thus for each $Q \in \operatorname{supp}(q)$, either $Q \subsetneq P$ or $Q \pitchfork P$. In either case, the boundary projection of q to P is defined as

$$\partial\pi_P(q) = \bigcup_{Q \in \operatorname{supp}(q)} \rho_P^Q.$$

Fix $r \geqslant r_0$ where r_0 is the lower bound on r require by Definition 4.16. Assume $\partial\pi_P(q) \subseteq M(r; p_P)$ and let $Q \in \operatorname{supp}(q) = \operatorname{supp}_{\mathfrak{T}}(\overline{q})$. Proposition 3.9(2) says $\rho_P^Q \asymp_C \pi_P(\mathbf{F}_Q)$, which implies

$$\mathbf{F}_Q \subseteq \pi_P^{-1}\left(\mathcal{N}_C(M^\circ(r; p_P))\right).$$

However, R_r was chosen so that this implies $\overline{\pi}_S(\mathbf{F}_Q) = \overline{\rho}_S^Q \subseteq M(R_r, \overline{p}_{\overline{P}})$. Thus $\partial\overline{\pi}_S(q) \subseteq M(R_r, \overline{p}_{\overline{P}})$ as desired.

Proof of (2)**.** Suppose $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$. This only occurs when $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$. If $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) \neq \{S\}$, then the result is immediate because $\operatorname{supp}(q) = \operatorname{supp}_{\mathfrak{T}}(\overline{q}) \subseteq \mathfrak{T} - \{S\}$ and we have $\partial\pi_P(q) = \partial\overline{\pi}_P(\overline{q})$ and $\mathcal{N}_{2C}(M^\circ(r; p_P)) \subseteq M(R_r; \overline{p}_{\overline{P}})$; see Definition 4.16. So suppose $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) = \{S\}$. This only occurs when $\operatorname{supp}(q) = \{Q\}$ for some $Q \in \{S\} \cup (\mathfrak{S} - \mathfrak{T})$. Since $Q \notin \mathfrak{T} - \{S\}$ but $\operatorname{supp}(p) \subseteq \mathfrak{T} - \{S\}$, we know $Q \neq P$ and $Q \not\perp P$ for each $P \in \operatorname{supp}(p)$.

Because $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \operatorname{supp}(p) \neq \{S\}$, we have $P \subsetneq S$ for all $P \in \operatorname{supp}(p)$. Let σ be the Morse constant for a $(1, 20E)$–quasi-geodesic in an E–hyperbolic space and fix $P \in \operatorname{supp}(p)$. Let Z be the set of all points in $\mathcal{C}_{\mathfrak{T}}S$ that are at least $E + \sigma$ far from $\overline{\rho}_S^P$ and are contained a $(1, 20E)$–quasi-geodesics from a point in $\overline{\rho}_S^P$ to $\overline{q}_S = \overline{q}$. The boundary projection of $\overline{q}$ to each $P \in \operatorname{supp}(p)$ is then defined as $\partial\overline{\pi}_P(\overline{q}) = \overline{\rho}_P^S(Z)$; see Definition 3.12.

Since P is in both $\mathfrak{S}$ and $\mathfrak{T}$, we have $\mathcal{C}P = \mathcal{C}_{\mathfrak{T}}P$, and so we consider $\partial\overline{\pi}_P(\overline{q})$ as a subset of $\mathcal{C}P$. Since $\partial\pi_P(q)$ has diameter at most E and $\partial\pi_P(q) \subseteq M^\circ(r; p_P)$ by assumption, it follows from Lemma 4.18 (with $W = P$) that $\partial\overline{\pi}_P(\overline{q}) \subseteq \mathcal{N}_C(M^\circ(r; p_P))$. Because $P = \overline{P}$, the set $M(R_r, \overline{p}_P)$ is defined so that $\mathcal{N}_C(M(r; p_P)) \subseteq M(R_r; \overline{p}_P)$, and the result follows. □

Lemma 4.20. *Let* $p, q \in \partial(G, \mathfrak{S})$*, and suppose* q *is remote to* p *but* $\overline{q}$ *is not remote to* $\overline{p}$*. Suppose* $\operatorname{supp}(p) = \{P\}$ *and* $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ *where* $S \neq P$*. If* r_0 *is the lower bound for* r *from Definition 4.16, then for any* $r > r_0$

$$\partial\pi_P(q) \subseteq M(r; p_P) \qquad \Longrightarrow \qquad \overline{q} \in M(R_r; \overline{p}_S).$$

Proof. We have $\operatorname{supp}(p) = \{P\}$ and $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ where $S \neq P$. We first determine the supports of q and $\overline{q}$. Because there are no domains orthogonal to S, no domain in $\operatorname{supp}_{\mathfrak{T}}(\overline{q})$ is orthogonal to S. Since we are assuming that $\overline{q}$ is not remote to $\overline{p}$, we therefore must have $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) \cap \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \varnothing$; see Definition 3.13. It follows that $\operatorname{supp}_{\mathfrak{T}}(\overline{q}) = \{S\}$, and so $\operatorname{supp}(q) = \{Q\}$ for some $Q \in \mathfrak{S}$ with $Q \notin \mathfrak{T} - \{S\}$. Since we are assuming p and q are remote, we also have $P \neq Q$.

Assume $\partial\pi_P(q) \subseteq M(r; p_P)$. We consider two cases, depending on how the boundary projection $\partial\pi_P(q)$ is defined: we first handle the case where either $Q \subsetneq P$ or $Q \pitchfork P$, then we address the case where $Q \supsetneq P$.

Case $Q \subsetneq P$ or $Q \pitchfork P$: When $Q \subsetneq P$ or $Q \pitchfork P$, the boundary projection is defined as $\partial\pi_P(q) = \rho_P^Q$. Since $\partial\pi_P(q) = \rho_P^Q \subseteq M(r;p_P)$ by assumption and $\rho_P^Q \asymp_C \pi_P(\mathbf{F}_Q)$ by Proposition 3.9(2), our choice of R_r yields

$$\overline{\rho}_S^Q = \overline{\pi}_S(\mathbf{F}_Q) \subseteq \overline{\pi}_S(\pi_P^{-1}(\mathcal{N}_C(M(r;p_P))) \subseteq M(R_r;\overline{p}_S)$$

as desired.

Case $Q \supsetneq P$: Let σ be the Morse constant (Lemma 2.1) for a $(1,20E)$–quasi-geodesic in an E–hyperbolic space. Because $Q \supsetneq P$, the boundary projection $\partial\pi_P(q)$ is defined as $\rho_P^Q(Z)$, where Z is the collection of all points on $(1,20E)$–quasi-geodesics in $\mathcal{C}Q$ from a point in ρ_Q^P to q that are at distance at most $E+\sigma$ from ρ_Q^P. Let α be a $(1,20E)$–quasi-geodesic ray α in $\mathcal{C}_{\mathfrak{T}}S$ from a point in $\overline{\pi}_S(\mathbf{F}_P) = Y_P$ to $\overline{q}$. By Theorem 3.24, there is $\lambda \geqslant 1$ determined by E so that α is λ–hierarchy path in the HHS $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S}-\mathfrak{T}) \cup \{S\})$. In particular, $\pi_Q \circ \alpha$ is an unparameterized (λ,λ)–quasi-geodesic ray from a point in $\mathcal{N}_E(\rho_Q^P)$ to q. By the Morse Lemma (Lemma 2.1), $\pi_Q(\alpha)$ is contained in a uniform neighborhood of any $(1,20E)$–quasi-geodesic from a point in ρ_Q^P to q. Hence, by going far enough along α, we can find a subray α_0 so that the consistency and bounded geodesic image axioms in $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S}-\mathfrak{T}) \cup \{S\})$ imply $\rho_P^Q(\pi_Q(\alpha_0)) \subseteq \mathcal{N}_E(\partial\pi_Q(q))$ and $\pi_P(\alpha_0) \asymp_E \rho_P^Q(\pi_Q(\alpha_0))$. As a result,

$$\pi_P(\alpha_0) \subseteq \mathcal{N}_{2E}(M(r;p_P)) \subseteq M(r-4E;p_P) \subseteq M(r-4C;p_P).$$

Lemma 3.25 and our choice of R_r (Definition 4.16) then imply $\alpha_0 \subseteq M(R_r;\overline{p}_{\overline{P}})$. Since α_0 represents $\overline{q}$, this implies $\overline{q} \in M(R_r;\overline{p}_{\overline{p}})$ as desired. ☐

4.6. **Invariance of the boundary under maximization.** We are now ready to prove that the maximization procedure does not change the HHS boundary for proper HHSs with the bounded domain dichotomy. Since every finitely generated group is a proper metric space and every G–HHS structure has the bounded domain dichotomy, Theorem 4.1 is a special case of this result.

Theorem 4.21. *Let $\mathcal{X}$ be a proper geodesic space and $\mathfrak{S}$ an HHS structure for $\mathcal{X}$ with the bounded domain dichotomy. Suppose $\mathfrak{T}$ is the HHS structure produced by maximizing $\mathfrak{S}$. The map $\phi\colon \partial(\mathcal{X},\mathfrak{S}) \to \partial(\mathcal{X},\mathfrak{T})$ defined in Section 4.3 is both a simplicial isomorphism $\partial_\Delta(\mathcal{X},\mathfrak{S}) \to \partial_\Delta(\mathcal{X},\mathfrak{T})$ and a homeomorphism $\partial(\mathcal{X},\mathfrak{S}) \to \partial(\mathcal{X},\mathfrak{T})$. Moreover, the identity map $\mathcal{X} \to \mathcal{X}$ extends continuously to ϕ.*

Proof. By Lemma 4.2, we can assume $\mathfrak{S} = \mathfrak{S}_{ess}$ without loss of generality. We also assume $\pi_W(x)$ is a single point for each $x \in \mathcal{X}$ and $W \in \mathfrak{S}$ by Standing Assumption 4.12.

The fact that ϕ is a simplicial automorphism follows from the fact that we are assuming all the domains of $\mathfrak{S}$ are essential and thus $\mathfrak{S}$ and $\mathfrak{T}$ have identical sets of pairwise orthogonal domains.

Define the map $\Phi\colon \mathcal{X} \cup \partial(\mathcal{X},\mathfrak{S}) \to \mathcal{X} \cup \partial(\mathcal{X},\mathfrak{T})$ to be the identity on $\mathcal{X}$ and ϕ on the boundary. As with ϕ, we will denote the image $\Phi(p)$ by $\overline{p}$.

The map Φ is a bijection, and we will show it is sequentially continuous. Since $\partial(\mathcal{X},\mathfrak{S})$ and $\partial(\mathcal{X},\mathfrak{T})$ are first countable (this is implicit in [DHS17] and explicitly proven in [Hag20, Proposition 1.5]), sequential continuity implies that ϕ is continuous and is a continuous extension of the identity on $\mathcal{X}$. Because $\mathcal{X}$ is proper, $\mathcal{X} \cup \partial(\mathcal{X},\mathfrak{S})$ and $\mathcal{X} \cup \partial(\mathcal{X},\mathfrak{T})$ are compact and Hausdorff [DHS17, Theorem 3.4]. Hence, proving ϕ is a continuous bijection implies it is a homeomorphism.

Let $p \in \partial(\mathcal{X},\mathfrak{S})$ and suppose (p_n) is a sequence in $\mathcal{X} \cup \partial(\mathcal{X},\mathfrak{S})$ that converges to p. To prove Φ is sequentially compact, it suffices to prove $\Phi(p_n) = \overline{p}_n \to \overline{p} = \Phi(p)$.

Continuing the convention from Notation 4.3, we let $\overline{\mathcal{B}}_{r,\varepsilon}(\cdot)$ denote a basis neighborhood in $\mathfrak{T}$ and $\mathcal{B}_{r,\varepsilon}(\cdot)$ denote a basis neighborhood in $\mathfrak{S}$ as described in Definition 3.17. Recall, Definition 4.16 takes any neighborhood $M(r;p_W)$ of $p_W \in \partial CW$ with $r \geqslant r_0$ and produces a neighborhood $M(R_r;\overline{p}_{\overline{W}})$ of $\overline{p}_{\overline{W}} \in \partial\mathcal{C}\overline{W}$ so that Lemmas 4.19 and 4.20 hold. We let the constants E and C be as described in Notation 4.13.

Fix a basis neighborhood $\overline{\mathcal{B}}_{\overline{r},\varepsilon}(\overline{p})$ for some $\overline{r} \geqslant 0$ and $\varepsilon \geqslant 0$. To prove $\overline{p}_n \to \overline{p}$, it suffices to show that $\overline{p}_n \in \overline{\mathcal{B}}_{\overline{r},\varepsilon}(\overline{p})$ for all but finitely many n.

By Lemma 4.17, there exists an r sufficiently large so that the constant R_r is defined and is large enough to ensure $\overline{\mathcal{B}}_{R_r,\varepsilon}(\overline{p}) \subseteq \overline{\mathcal{B}}_{\overline{r},\varepsilon}(\overline{p})$. Fixing this r, it suffices to show that $\overline{p}_n \in \overline{\mathcal{B}}_{R_r,\varepsilon}(\overline{p})$ for all but finitely many n.

Since $p_n \to p$, for each $s \geqslant 1$ and all but finitely many n, we have

$$p_n \in \mathcal{B}_{r+s,\frac{1}{s}}(p).$$

Notice that $M(r+s;p_W) \subseteq M(r;p_W)$ for all $s \geqslant 1$, and so $\mathcal{B}_{r+s,\frac{1}{s}}(p) \subseteq \mathcal{B}_{r,\varepsilon}(p)$ for all sufficiently large s. In fact, it is clear from the definition of the decomposition of the neighborhoods that $\mathcal{B}^*_{r+s,\frac{1}{s}}(p) \subseteq \mathcal{B}^*_{r,\varepsilon}(p)$, where $* \in \{rem, int, non\}$. Moreover, since $r+s \to \infty$ as $s \to \infty$, we have that $d_W(x_0, M(r+s;p_W)) \to \infty$ as $s \to \infty$ for each $W \in \operatorname{supp}(p)$.

We divide the sequence $(\overline{p}_n)$ into three disjoint subsequences, and analyze each in a separate step of the proof. We will show that for each such subsequence, $\overline{p}_n \in \overline{\mathcal{B}}_{R_r,\varepsilon}(\overline{p})$ for all but finitely many n.

Step 1. Consider the subsequence consisting of all n so that $p_n \in \mathcal{B}^{rem}_{r+s,\frac{1}{s}}(p) \subseteq \mathcal{B}^{rem}_{r,\varepsilon}(p)$. If this subsequence is finite, we are done and move on to step two. So suppose it is infinite. There are two further subcases, depending on the support of p and $\overline{p}$. Note that since p_n is remote to p in this case, we can apply Lemmas 4.19 and 4.20 with $q = p_n$.

Step 1(a): Assume $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$. For any n such that $\operatorname{supp}(p_n) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}_n)$, we must have $\overline{p}_n$ is remote to $\overline{p}$, because p_n is remote to p and remoteness is determined by the support set. Hence, we can apply Lemma 4.19(2) to see that condition (R1) from Definition 3.14 for $\overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$ is satisfied. Since $\operatorname{supp}(p)^\perp = \operatorname{supp}_{\mathfrak{T}}(\overline{p})^\perp$, the fact that $p_n \in \mathcal{B}^{rem}_{r,\varepsilon}(p)$ implies (R2) and (R3) are also satisfied for $\overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$ (to see this, note (R2) and (R3) involve ε but not r). This implies $\overline{p}_n \in \overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$, as desired. We can therefore assume that $\operatorname{supp}(p_n) \neq \operatorname{supp}(p_n)$ for all n. This implies that $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$ and $\operatorname{supp}(p_n) = \{Q_n\}$ for some $Q_n \in \mathfrak{S} - \mathfrak{T}$; in particular $Q_n \neq S$.

First assume $\operatorname{supp}(p) = \{S\}$, so $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$. Then $\overline{p}_n$ is not remote to $\overline{p}$, and we will check that $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$. Condition (N1) of Definition 3.15 holds by Lemma 4.20. Condition (N2) holds as $a^{\overline{p}}_S = a^p_S = 1$ and $a^{\overline{p}_n}_S = a^{p_n}_Q = 1$. Finally (N3) vacuously holds, as

$$\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) - (\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) \cap \operatorname{supp}_{\mathfrak{T}}(\overline{p})) = \varnothing.$$

Now assume $\operatorname{supp}(p) \neq \{S\}$. Since $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$, we also have $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$. This means $\overline{p}_n$ is remote to $\overline{p}$ because $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$. We will show that $\overline{p}_n \in \mathcal{B}^{rem}_{R_r,\varepsilon}(\overline{p})$ in this case.

Since $p_n \in \mathcal{B}^{rem}_{r,\varepsilon}(p)$, we have $\partial\pi_P(p_n) \subseteq M(r;p_P)$ for all $P \in \operatorname{supp}(p)$. By Lemma 4.19(2), we therefore have $\partial\overline{\pi}_P(\overline{p}_n) \subseteq M(R_r;\overline{p}_P)$ for all $P \in \operatorname{supp}(p)$, satisfying (R1). The condition (R3) is satisfied because $a^{\overline{p}_n}_W = 0$ for all $W \in \operatorname{supp}_{\mathfrak{T}}(\overline{p}_n)^\perp$, since $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$ and S is not orthogonal to any domain.

For the remaining condition (R2), note that $\operatorname{supp}(p)^\perp = \operatorname{supp}_{\mathfrak{T}}(\overline{p})^\perp$. Because $\operatorname{supp}_{\mathfrak{T}}(p) \neq \{S\}$, we have $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) \subseteq \mathfrak{T}$. Thus, $\mathcal{C}_{\mathfrak{T}}W = \mathcal{C}W$ for each $W \in \operatorname{supp}_{\mathfrak{T}}(\overline{p})$, and we can think of both $\partial\pi_W(p_n)$ and $\partial\overline{\pi}_W(\overline{p}_n)$ as subsets of $\mathcal{C}W$. Recall $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \operatorname{supp}(p_n)$ contains only the domain Q_n. As in Definition 3.13, let $\mathcal{S}_{\overline{p}_n}$ be the union of $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \operatorname{supp}(p)$ with the set of domains in $\operatorname{supp}(p)^\perp$ that are not orthogonal to Q_n (this is the set of domains for which (R2) needs to be verified). By Lemma 4.18, we have

$$d_W(x_0, \partial\pi_W(p_n)) - C \leqslant d_W(x_0, \partial\overline{\pi}_W(\overline{p}_n)) \leqslant d_W(x_0, \partial\pi_W(p_n)) + C \tag{4}$$

for each $W \in \mathcal{S}_{\overline{p}_n}$. The following claim uses (4) to complete the proof that (R2) holds for all but finitely many n.

Claim 4.22. *For any $W \in \mathcal{S}_{\overline{p}_n}$ and $P \in \operatorname{supp}(p)$, we have*

$$\left|\frac{d_W(x_0, \partial\overline{\pi}_W(\overline{p}_n))}{d_P(x_0, \partial\overline{\pi}_P(\overline{p}_n))} - \frac{a_W^{\overline{p}}}{a_P^{\overline{p}}}\right| < \varepsilon \tag{5}$$

for all but finitely many n.

Proof. Recall that for any $s \geqslant 1$, we have that $p_n \in \mathcal{B}_{r+s,\frac{1}{s}}(p)$ for all but finitely many n. When $p_n \in \mathcal{B}^{rem}_{r+s,\frac{1}{s}}(p)$, (R1) and (R2) in $\mathfrak{S}$ imply for any $P \in \operatorname{supp}(p)$ and $W \in \mathcal{S}_{p_n}$ we have

$$\partial\pi_P(p_n)) \subseteq M(r+s; p_P)$$

and

$$\left|\frac{d_W(x_0, \partial\pi_W(p_n))}{d_P(x_0, \partial\pi_P(p_n))} - \frac{a_W^{p}}{a_P^{p}}\right| < \frac{1}{s}. \tag{6}$$

Recall that $d_P(x_0, M(r+s; p_P)) \to \infty$ as $s \to \infty$, which coupled with (4) implies that for all $P \in \operatorname{supp}(p)$, the distance $d_P(x_0, \partial\overline{\pi}_P(\overline{p}_n)) \to \infty$ as $n \to \infty$.

Suppose $W \in \operatorname{supp}(p)^\perp$ and $d_W(x_0, \partial\overline{\pi}_W(\overline{p}_n))$ is uniformly bounded for all n. In this case, $a_W^P = 0$, and since the numerator of the first term of (5) is bounded while the denominator goes to infinity as $n \to \infty$, (5) is satisfied for but finitely many n.

Thus, we may assume without loss of generality that the numerator and denominator of the first term of (5) both go to infinity as $n \to \infty$. Lemma 3.19 now implies

$$\left|\frac{d_W(x_0, \partial\overline{\pi}_W(\overline{p}_n))}{d_P(x_0, \partial\overline{\pi}_P(\overline{p}_n))} - \frac{a_W^{\overline{p}}}{a_P^{\overline{p}}}\right| < \varepsilon$$

because of (4). □

The above shows that $\overline{p}_n \in \overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$ for all but finitely many n when $\operatorname{supp}(p) \neq \{S\}$. Combining this with the earlier proof that $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$ when $\operatorname{supp}(p) = \{S\}$, we conclude that $\overline{p}_n \in \overline{\mathcal{B}}_{R_r,\varepsilon}(\overline{p})$ for all but finitely many n, whenever $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$.

Step 1(b): Assume $\operatorname{supp}(p) \neq \operatorname{supp}_{\mathfrak{T}}(\overline{p})$. Then $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ and $\operatorname{supp}(p) = \{P\}$ for some $P \in \mathfrak{S} - \mathfrak{T}$. We consider two subcases, depending on $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n)$.

- Suppose $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) \neq \{S\}$ for some n. Now $\overline{p}_n$ is remote to $\overline{p}$, because $\operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$. Thus, we show $\overline{p}_n \in \overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$. Lemma 4.19(1) shows that (R1) holds, and conditions (R2) and (R3) vacuously hold as $\operatorname{supp}_{\mathfrak{T}}(\overline{p})^\perp = \varnothing$ and $|\operatorname{supp}_{\mathfrak{T}}(\overline{p})| = 1$. Therefore $\overline{p}_n \in \overline{\mathcal{B}}^{rem}_{R_r,\varepsilon}(\overline{p})$.
- Suppose $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$ and let $\operatorname{supp}(p_n) = \{Q\}$, where we include the possibility that $Q = S$. In this case, $\overline{p}_n$ is not remote to $\overline{p}$, so we show $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$. Condition (N1) holds by Lemma 4.20. Condition (N2) trivially holds, as $\overline{p}_n = (\overline{p}_n)_S$ and $\overline{p} = \overline{p}_S$. Finally, (N3) is vacuously satisfied, as $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \operatorname{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$. Thus $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$.

Step 2. Consider the subsequence consisting of all n so that $p_n \in \mathcal{B}^{non}_{r+s,\frac{1}{s}}(p) \subseteq \mathcal{B}^{non}_{r,\varepsilon}(p)$. If this subsequence is finite, we are done and move on to step three. So suppose it is infinite. There are two further subcases, depending on the support of p and $\overline{p}$.

Step 2(a): Assume $\operatorname{supp}(p) = \operatorname{supp}_{\mathfrak{T}}(\overline{p})$. First suppose that $\operatorname{supp}(p_n) \neq \operatorname{supp}_{\mathfrak{T}}(\overline{p}_n)$ for some n. The only way for this to occur is for $\operatorname{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$ and for $\operatorname{supp}(p_n) = \{Q\}$ for some $Q \in \mathfrak{S} - \mathfrak{T}$. If $\operatorname{supp}(p) \cap \operatorname{supp}(p_n) = \varnothing$, then p_n is remote to p because no domains are orthogonal to Q. However, this contradicts the assumption in this step, so we must have $\operatorname{supp}(p) \cap \operatorname{supp}(p_n) \neq \varnothing$, and this intersection must be $\{Q\}$. Hence $\mathrm{supp}(p) = \{Q\}$, because support sets are collections of pairwise orthogonal domains and there are no domains orthogonal to Q. However, $Q \notin \mathfrak{T}$, and so $\mathrm{supp}_{\mathfrak{T}}(\overline{p}) = \{S\} \neq \mathrm{supp}(p)$, which contradicts the assumptions of this case.

Therefore we must have $\mathrm{supp}(p_n) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}_n)$, which makes $\overline{p}_n$ not remote to $\overline{p}$. We verify that $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$. If $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}) \subseteq \mathfrak{T} - \{S\}$, then $\mathrm{supp}(p_n) = \mathrm{supp}(\overline{p}_n)$ is also contained in $\mathfrak{T} - \{S\}$. This means $p_n = \overline{p}_n$, $p = \overline{p}$, and $\mathrm{supp}(p) \cap \mathrm{supp}(p_n) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}) \cap \mathrm{supp}_{\mathfrak{T}}(\overline{p}_n)$. Hence, the conditions for $\overline{p}_n$ to be in $\overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$ follow from the fact that $p_n \in \mathcal{B}^{non}_{r,\varepsilon}(p)$ and $M(R_r; \overline{p}_W) \subseteq M(r, p_W)$ for each $W \in \mathrm{supp}(p)$ in this case. If instead $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$, then $\mathrm{supp}(p_n) = \mathrm{supp}(\overline{p}_n) = \{S\}$ as well. Thus (N1) is satisfied as $M(R_r; \overline{p}_S) \subseteq M(r, p_S)$ and (N2) and (N3) are trivially true because $a^{\overline{p}}_S = a^{\overline{p}_n}_S = 1$ and $\mathrm{supp}_{\mathfrak{T}}(\overline{p}_n) - (\mathrm{supp}_{\mathfrak{T}}(\overline{p}_n) \cap \mathrm{supp}_{\mathfrak{T}}(\overline{p})) = \varnothing$.

Step 2(b): Assume $\mathrm{supp}(p) \neq \mathrm{supp}_{\mathfrak{T}}(\overline{p})$. In this case, $\mathrm{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$ and $\mathrm{supp}(p) = \{P\}$ for some $P \in \mathfrak{S} - \mathfrak{T}$. Since p_n is not remote to p, either $P \in \mathrm{supp}(p_n)$ or $Q \perp P$ for all $Q \in \mathrm{supp}(p_n)$. However, the latter is impossible, as there are no domains orthogonal to P. So we must have that $P \in \mathrm{supp}(p_n)$, in which case $\mathrm{supp}(p_n) = \{P\}$, as support sets are collections of pairwise orthogonal domains. Thus $\mathrm{supp}_{\mathfrak{T}}(\overline{p}_n) = \{S\}$, making $\overline{p}_n$ not remote to $\overline{p}$.

We show that $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$. Since $p_n \in M(r, p_P) \cap \partial\mathcal{C}P$ and $P \in \mathfrak{S} - \mathfrak{T}$, Lemma 3.25 applied to the hyperbolic HHS $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$ says $\overline{p}_n \in M(R_r, \overline{p}_S)$. Hence Condition (N1) for $\overline{p}_n$ to be in $\overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$ is satisfied. Condition (N3) is clear as $\overline{p}_n = (\overline{p}_n)_S$ and $\overline{p} = \overline{p}_S$, and Condition (N2) is vacuously satisfied as $\mathrm{supp}_{\mathfrak{T}}(\overline{p}_n) = \mathrm{supp}_{\mathfrak{T}}(\overline{p})$. Therefore $\overline{p}_n \in \overline{\mathcal{B}}^{non}_{R_r,\varepsilon}(\overline{p})$.

Step 3. Consider the subsequence consisting of all n so that $p_n \in \mathcal{B}^{int}_{r+s,\frac{1}{s}}(p) \subseteq \mathcal{B}^{int}_{r,\varepsilon}(p)$. If this subsequence is finite, the proof is complete. So suppose it is infinite. There are two further subcases, depending on the support of p and $\overline{p}$. Since Φ restricts to the identity on $\mathcal{X}$, we have $\overline{p}_n = p_n$.

Step 3(a): Assume $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p})$. Since $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p})$, we have $\mathrm{supp}(p)^{\perp} = \mathrm{supp}_{\mathfrak{T}}(\overline{p})^{\perp}$, and so (I2) and (I3) hold automatically because $p_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$. If $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}) \neq \{S\}$, then (I1) follows from the fact that $\mathcal{C}W = \mathcal{C}_{\mathfrak{T}}W$ for $W \in \mathrm{supp}(p)$, $R_r \leqslant r$, and $p_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$. If $\mathrm{supp}(p) = \mathrm{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$, then $\mathcal{C}S \neq \mathcal{C}_{\mathfrak{T}}S$. In this case, (I1) is a consequence of our choice of R_r and Lemma 3.25 applied to the hyperbolic HHS $(\mathcal{C}_{\mathfrak{T}}S, (\mathfrak{S} - \mathfrak{T}) \cup \{S\})$. Thus $p_n \in \overline{\mathcal{B}}^{int}_{R_r,\varepsilon}(\overline{p})$.

Step 3(b): Assume $\mathrm{supp}(p) \neq \mathrm{supp}_{\mathfrak{T}}(\overline{p})$. In this case $\mathrm{supp}_{\mathfrak{T}}(\overline{p}) = \{S\}$, which makes Conditions (I2) and (I3) for p_n to be in $\overline{\mathcal{B}}^{int}_{R_r,\varepsilon}(\overline{p})$ automatically true. For Condition (I1), we have $\pi_P(p_n) \subseteq M(r; p_P)$ for each $P \in \mathrm{supp}(p)$ because $p_n \in \mathcal{B}^{int}_{r,\varepsilon}(p)$. By our choice of R_r (Definition 4.16), this implies $\overline{\pi}_S(p_n) \subseteq M(R_r; \overline{p}_S)$, verifying (I1).

We have show that for all but finitely many n, we have $\overline{p}_n \in \overline{\mathcal{B}}_{R_r,\varepsilon}(\overline{p}) \subseteq \overline{\mathcal{B}}_{r_0,\varepsilon}(\overline{p})$. Therefore $\overline{p}_n \to \overline{p}$ as $n \to \infty$ and Φ is sequentially continuous. This completes the proof of Theorem 4.21. □

5. Applications

We conclude by recording several corollaries of Theorem 4.1, which show that some topological and dynamical properties that are known to hold for maximized G–HHS structures also hold in *every* possible G–HHS structure for the group. We begin by recalling some definitions from [DHS17] about the dynamics of an element of a nearly hierarchically hyperbolic group on the G–HHS structure. The results we cite below were originally formulated in the setting of hierarchically hyperbolic groups, but they continue to hold in the slightly more general context of G–HHS because the definition of the HHS boundary does not interact in any way with the domains whose associated hyperbolic spaces have uniformly bounded diameter. Also, since any HHG is a G–HHS, the results in this section imply all the statements from the introduction about hierarchically hyperbolic groups.

Fix a G–HHS $(G, \mathfrak{S})$ with $\sqsubseteq$–maximal element $S \in \mathfrak{S}$. The *big set in* $\mathfrak{S}$ of an element $g \in G$ is the set of domains $W \in \mathfrak{S}$ so that $\operatorname{diam}(\pi_W(\langle g \rangle)) = \infty$; we denote the big set in $\mathfrak{S}$ by $\operatorname{Big}_{\mathfrak{S}}(g)$. We say g is *irreducible* with respect to $\mathfrak{S}$ if $\operatorname{Big}_{\mathfrak{S}}(g) = \{S\}$. If $\operatorname{Big}_{\mathfrak{S}}(g) \neq \emptyset$ but $\operatorname{Big}_{\mathfrak{S}}(g) \neq \{S\}$, then we say g is *reducible* with respect to $\mathfrak{S}$. Durham, Hagen, and Sisto show the following basic properties of the big sets, which holds more generally assuming that $(G, \mathfrak{S})$ is only a G–HHS.

Lemma 5.1 ([DHS17])**.** *Let $(G, \mathfrak{S})$ be an HHG and $g \in G$.*

(1) $\operatorname{Big}_{\mathfrak{S}}(g) = \emptyset$ if and only if g has finite order. That is, every element of G is either irreducible, reducible, or finite order.
(2) $\operatorname{Big}_{\mathfrak{S}}(g)$ is a pairwise orthogonal subset of domains of $\mathfrak{S}$. In particular, there is $k \in \mathbb{N}$ depending only on $\mathfrak{S}$ so that $g^k W = W$ for each $W \in \operatorname{Big}_{\mathfrak{S}}(g)$.
(3) For each $n \in \mathbb{Z}$ and $W \in \operatorname{Big}_{\mathfrak{S}}(g)$, if $g^n W = W$, then g^n acts loxodromically on $\mathcal{C}W$.

Recall that the action of G on itself is by hieromorphisms where each of the maps between hyperbolic spaces in an isometry; see Section 3.2.3. Hence, the action of G on itself extends continuously to an action by both homeomorphisms and simplicial automorphism on the boundary. The main dynamical property of this action that we will be interested is that of north-south dynamics.

Definition 5.2. Let $(G, \mathfrak{S})$ be a G–HHS that is not virtually cyclic. An element $g \in G$ acts with *north-south dynamics* on $\partial(G, \mathfrak{S})$ if g has exactly two fixed points $\xi^+, \xi^- \in \partial(G, \mathfrak{S})$ and for any disjoint open sets $O^+, O^- \subseteq \partial(G, \mathfrak{S})$ containing ξ^+ and ξ^- respectively, there exists $n \in \mathbb{N}$ so that

$$g^n \cdot (\partial(G, \mathfrak{S}) - O^-) \subseteq O^+.$$

We call ξ^+ the *attracting fixed point* of g and ξ^- the *repelling fixed point.*

Remark 5.3. Combining the work in [DHS17] and [ABD21] yields: $\partial(G, \mathfrak{S}) = \emptyset$ if and only if G is finite; $|\partial(G, \mathfrak{S})| = 2$ if and only if G is virtually $\mathbb{Z}$; and $|\partial(G, \mathfrak{S})| = \infty$ in all other cases. Thus $|\partial(G, \mathfrak{S})| = \infty$ if and only if G is not virtually cyclic.

Durham, Hagen, and Sisto show that the irreducible elements always act with north-south dynamics, and that the set of attracting fixed points of the irreducible elements is dense in the HHS boundary; again this holds equal as well under the assumption that G is a G–HHS.

Theorem 5.4 ([DHS17])**.** *Let $(G, \mathfrak{S})$ be a non-virtually cyclic HHG and $S \in \mathfrak{S}$ be the $\sqsubseteq$–maximal element. If $\mathcal{C}S$ has infinite diameter, then*

(1) if $g \in G$ is irreducible with respect to $\mathfrak{S}$, then g acts with north-south dynamics on $\partial(G, \mathfrak{S})$;
(2) the set of attracting fixed points for the irreducible elements of $(G, \mathfrak{S})$ are dense in $\partial(G, \mathfrak{S})$;
(3) the inclusion of $\partial \mathcal{C}S$ into $\partial(G, \mathfrak{S})$ is a continuous embedding whose image is dense in $\partial(G, \mathfrak{S})$.

A result of the first two authors and Durham established that the hyperbolic space obtained from the maximization procedure is independent of the initial HHG structure (this result is implicit in [ABD21, Theorem 5.1], which proves that the hyperbolic space associated to the nest-maximal element in a maximized structure is the initial object in a particular category and whence unique). Since the proof in [ABD21] only involves the domains in $\mathfrak{S}^\infty$, the proof there for HHGs also establishes the identical result for G–HHSs.

Theorem 5.5 ([ABD21])**.** *Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two HHG structures for the group G. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the maximizations of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ accordingly. If S_1 and S_2 are the $\sqsubseteq$–maximal elements of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively, then $\mathcal{C}S_1$ is quasi-isometric to $\mathcal{C}S_2$. In this case, $\partial \mathcal{C}S_1$ is homeomorphic to $\partial \mathcal{C}S_2$.*

Combining this uniqueness with the density results from Theorem 5.4, some topological properties of the boundary of the maximized hyperbolic space can be expanded to the HHS boundary of any G–HHS. One salient example of such a topological property that can be extended from a dense subset is connectedness.

Corollary 5.6. *Let $(G, \mathfrak{T})$ be a maximized G–HHS. If $T \in \mathfrak{T}$ is the $\sqsubseteq$–maximal element and $\mathcal{C}T$ is one-ended, then for any G–HHS structure $\mathfrak{S}$ for G, the HHS boundary $\partial(G, \mathfrak{S})$ is connected.*

Proof. Let $\mathfrak{S}$ be a G–HHS structure for G and let $\mathfrak{R}$ be the maximization of $\mathfrak{S}$. Let S be the $\sqsubseteq$–maximal element of $\mathfrak{S}$ and $\mathfrak{R}$.

Since G acts coboundedly on $\mathcal{C}T$, one-endedness of $\mathcal{C}T$ is equivalent to connectedness of $\partial\mathcal{C}T$. Now $\mathcal{C}T$ is quasi-isometric to $\mathcal{C}_{\mathfrak{R}}S$ by Theorem 5.5, so $\partial\mathcal{C}T$ being connected implies $\partial\mathcal{C}_{\mathfrak{R}}S$ is connected. Since $\partial\mathcal{C}_{\mathfrak{R}}S$ is dense in $\partial(G, \mathfrak{R})$, this implies $\partial(G, \mathfrak{R})$ is also connected. Thus $\partial(G, \mathfrak{S})$ is connected by Theorem 4.1. □

Remark 5.7. Unfortunately, such an argument with density can not be used to show the HHS boundary is path connected. The topologist's sine curve is an example of a space that has a dense path-connected subset, but is not path-connected.

Our next set of applications involve the Morse elements of a G–HHS and the elements that act by north-south dynamic on the boundary.

A quasi-geodesic γ in a metric space is *Morse* if there exists a function $N\colon [1,\infty)\times[0,\infty) \to [0,\infty)$ so that each (λ, c)–quasi-geodesic with endpoints on γ is contained in the $N(\lambda, c)$–neighborhood of γ. An element g of a finitely generated group G is *Morse* if $\langle g\rangle$ is a Morse quasi-geodesic in the Cayley graph of G with respect to some finite generating set. Since a quasi-geodesic being Morse is preserved by quasi-isometries, whether or not $g \in G$ is Morse is independent of the choice of finite generating set for G.

One of the original applications of the maximization procedure was to characterize Morse elements of an HHG as precisely those that are irreducible with respect to a maximized structure. Since this condition involves only domains in $\mathfrak{S}^{\infty}$, its proof also works identically for G–HHSs.

Theorem 5.8 ([ABD21]). *Let G be a HHG. If $\mathfrak{T}$ is a maximized structure, then $g \in G$ is Morse if and only if g is irreducible with respect to $\mathfrak{T}$.*

Since the boundary is invariant under maximization and the maximized hyperbolic space is unique up to quasi-isometry, we now show that the Morse elements are precisely the set of elements that act with north-south dynamics on the HHS boundary of *any* G–HHS.

Corollary 5.9. *Let $(G, \mathfrak{S})$ be a G–HHS that is not virtually cyclic. An element $g \in G$ acts with north-south dynamics on $\partial(G, \mathfrak{S})$ if and only if g is a Morse element of G. In particular, the set of elements of G that act with north-south dynamics does not depend on the G–HHS structure $\mathfrak{S}$.*

Proof. Let $\mathfrak{T}$ be the maximization of the G–HHS structure $\mathfrak{S}$. Let S be the $\sqsubseteq$–maximal domain of $\mathfrak{S}$ and $\mathfrak{T}$. Since G is not virtually cyclic, both $\partial(G, \mathfrak{T})$ and $\partial(G, \mathfrak{S})$ have an infinite number of points; see Remark 5.3.

Assume that $g \in G$ is a Morse element. By Theorem 5.8, g being Morse is equivalent to g being irreducible with respect to $\mathfrak{T}$. Hence g acts with north-south dynamics on $\partial(G, \mathfrak{T})$ (Theorem 5.4). Since $\partial(G, \mathfrak{T})$ is G–equivariantly homeomorphic to $\partial(G, \mathfrak{S})$ by Theorem 4.1, g must also act on $\partial(G, \mathfrak{S})$ with north-south dynamics.

Now assume $g \in G$ acts by north-south dynamics on $\partial(G, \mathfrak{S})$. Since $\partial(G, \mathfrak{S})$ is Hausdorff and has an infinite number of points, north-south dynamics ensures that g does not have finite order. Hence, $\operatorname{Big}_{\mathfrak{S}}(g) \neq \varnothing$ by Lemma 5.1. Further, [DHS17, Proposition 6.22] says $|\operatorname{Big}_{\mathfrak{S}}(g)| > 1$ implies g stabilizes at least 3 points in $\partial(G, \mathfrak{S})$, which would contradict g having north-south dynamics. Thus, we know $\operatorname{Big}_{\mathfrak{S}}(g)$ contains exactly one domain $W \in \mathfrak{S}$. Since $gW = W$, if

$V \perp W$, then $gV \perp W$ as well. Thus, $\mathfrak{S}_W^\perp \cap \mathfrak{S}^\infty \neq \emptyset$ would imply that the non-empty set of points $\{p \in \partial(G,\mathfrak{S}) : \operatorname{supp}(p) \subseteq \mathfrak{S}_W^\perp \cap \mathfrak{S}^\infty\}$ is stabilized by g. Since this would violate north-south dynamics, we know $\mathfrak{S}_W^\perp \cap \mathfrak{S}^\infty = \emptyset$. Hence, $W \notin \mathfrak{T} - \{S\}$ as maximization removes all non-maximal domains that are not orthogonal to a domain of $\mathfrak{S}^\infty$. This implies $\operatorname{Big}_{\mathfrak{T}}(g) = \{S\}$, which makes g a Morse element by Theorem 5.8. □

Since the set of attracting fixed points for the irreducible elements are dense, we have the same for the attracting fixed points of the Morse elements regardless of the choice of G–HHS structure.

Corollary 5.10. *Let $(G,\mathfrak{S})$ be a G–HHS that is not virtually cyclic. If G contains a Morse element, then the set of attracting fixed points of Morse elements in $\partial(G,\mathfrak{S})$ is dense in $\partial(G,\mathfrak{S})$.*

Proof. Let $\mathfrak{T}$ be the maximization of $\mathfrak{S}$. By Theorem 5.4 and Theorem 5.8, the set of attracting fixed point of the Morse elements is dense in $\partial(G,\mathfrak{T})$. Theorem 4.1 then implies that they are also dense in $\partial(G,\mathfrak{S})$. □

Remark 5.11 (Density of the Morse boundary)**.** The Morse geodesics of a group can be used to make a *Morse boundary* for any finitely generated group; see [Cor19]. The Morse boundary of an HHG has a G–equivariant continuous injection into $\partial(G,\mathfrak{T})$ where $\mathfrak{T}$ is a maximized structure [ABD21, Theorem 6.6] (this result again works for G– HHSs, as well). Every Morse element also has a pair of fixed points in the Morse boundary, and the continuous inclusion sends fixed points to fixed points. Hence, Corollary 5.10 shows that the image of the Morse boundary in the HHS boundary is dense.

Finally, we use the density of Morse elements to show the limit set of a normal subgroup is the entire HHS boundary. The *limit set* of a subgroup H of a G–HHS $(G,\mathfrak{S})$ is the set of all points in $\partial(G,\mathfrak{S})$ that are the limit of a sequence of elements of H.

Corollary 5.12. *Let $(G,\mathfrak{S})$ be a G–HHS that is not virtually cyclic. If G contains a Morse element and N is an infinite normal subgroup of G, then the limit set of N in $\partial(G,\mathfrak{S})$ is all of $\partial(G,\mathfrak{S})$.*

Proof. Let $\mathfrak{T}$ be the maximization of $\mathfrak{S}$, and let S be the $\sqsubseteq$–maximal element of $\mathfrak{S}$ and $\mathfrak{T}$. Since $\partial\mathcal{C}_{\mathfrak{T}}S$ is dense in $\partial(G,\mathfrak{T})$ and by Theorem 4.1 the identity $G \to G$ induces a homeomorphism $G \cup \partial(G,\mathfrak{S}) \to G \cup \partial(G,\mathfrak{T})$, it suffices to prove that the limit set of N contains all of $\partial\mathcal{C}_{\mathfrak{T}}S \subseteq \partial(G,\mathfrak{T})$.

Let $p \in \partial(G,\mathfrak{T})$ with $\operatorname{supp}(p) = \{S\}$. Fix a basis neighborhood $\mathcal{B}_{r,\varepsilon}(p)$ in $\partial(G,\mathfrak{T})$. Since G has at least one Morse element, Corollary 5.10 says there is a Morse element $g \in G$, so that its attracting fixed point ξ is contained in $\mathcal{B}_{r,\varepsilon}(p)$. Proposition 5.14 below shows that there is also a Morse element h in the normal subgroup N with attracting fixed point $\zeta \in \mathcal{C}_{\mathfrak{T}}S \subseteq \partial(G,\mathfrak{S})$. Since the Morse elements of G act with north-south dynamics, there is some $n \in \mathbb{N}$ so that $g^n\zeta \in \mathcal{B}_{r,\varepsilon}(p)$. Now, the sequence $\{g^n h^i g^{-n}\}_{i=1}^\infty$ will converge to $g^n\zeta$ because $\{\pi_S(g^n h^i g^{-n})\}_{i=1}^\infty$ will converge to $g^m\zeta \in \partial\mathcal{C}_{\mathfrak{T}}S$. Since $h \in N$ and N is a normal subgroup, $g^n h^i g^{-n} \in N$ for each $i \in \mathbb{N}$. Thus, p will be a limit point of elements of N. □

Remark 5.13. The conclusion of Corollary 5.12 can fail to hold when G does not contain any Morse elements. For example, if G is the direct product of two infinite G–HHSs $H_1 \times H_2$, then the HHS boundary of G will be the join of the HHS boundaries of H_1 and H_2, and the limit set of each H_i will be exactly one side of this join.

Our last proposition was used in the proof of Corollary 5.12 and is useful in its own right. We note that it can be deduced as a special case of [RST22, Corollary 3.6]; we provide a short proof using the theory of hierarchical hyperbolicity for completeness.

Proposition 5.14. *Let G be a G–HHS containing a Morse element. Then every infinite normal subgroup of G contains an element that is Morse in G.*

Proof. Let $g \in G$ be a Morse element, and let $\mathfrak{S}$ be a maximized structure for G. By Theorem 5.8, $\mathrm{Big}_{\mathfrak{S}}(g) = \{S\}$ where S is the $\sqsubseteq$–maximal element of $\mathfrak{S}$. Let N be a normal subgroup of G. Applying the Rank-Rigidity Theorem ([DHS17, Theorem 9.13]) to the action of N on $(G, \mathfrak{S})$, either N contains a Morse element or it stabilizes a product region $\mathbf{P}_U$ for some $U \in \mathfrak{S}$ ($\mathbf{P}_U$ is the image of $\mathbf{F}_U \times \mathbf{E}_U$ in G). Since g is loxodromic on $\mathcal{C}S$, by taking n large enough we can ensure that ρ_S^U is as far as desired from $\rho_S^{g^nU}$. Since $\rho_S^U \asymp \pi_S(\mathbf{P}_U)$, the orbits $N \cdot \rho_S^N$ and $g^nNg^{-n} \cdot \rho_S^{g^nU}$ both have uniformly bounded diameter for any $n \in \mathbb{Z}$. Since $g^nNg^{-n} = N$, acylindricity of the action of G on $\mathcal{C}S$ implies that N is finite. □

References

[ABD21] Carolyn Abbott, Jason Behrstock, and Matthew G. Durham. Largest acylindrical actions and stability in hierarchically hyperbolic groups. With an appendix by Daniel Berlyne and Jacob Russell. *Trans. Amer. Math. Soc. Ser. B*, 8:66–104, 2021.

[BB97] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BHS17a] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups. *Proc. Lond. Math. Soc. (3)*, 114(5):890–926, 2017.

[BHS17b] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces I: curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.

[BHS19] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces II: combination theorems and the distance formula. *Pacific J. Math.*, 299:257–338, 2019.

[BHS21] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Quasiflats in hierarchically hyperbolic spaces. *Duke Math. J.*, 170:909–996, 2021.

[Bir69] Joan S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[BR20a] Federico Berlai and Bruno Robbio. A refined combination theorem for hierarchically hyperbolic groups. *Groups Geom. Dyn.*, 14(4):1127–1203, 2020.

[BR20b] Daniel Berlyne and Jacob Russell. Hierarchical hyperbolicity of graph products. arXiv:2006.03085, 2020.

[CMM21] Matt Clay, Johanna Mangahas, and Dan Margalit. Right-angled Artin groups as normal subgroups of mapping class groups. *Compos. Math.*, 157(8):1807–1852, 2021.

[Cor19] Matthew Cordes. A survey on Morse boundaries and stability. In *Beyond hyperbolicity*, volume 454 of *London Math. Soc. Lecture Note Ser.*, pages 83–116. Cambridge Univ. Press, Cambridge, 2019.

[CT07] James W. Cannon and William P. Thurston. Group invariant Peano curves. *Geom. Topol.*, 11:1315–1355, 2007.

[Dah18] François Dahmani. The normal closure of big Dehn twists and plate spinning with rotating families. *Geom. Topol.*, 22(7):4113–4144, 2018.

[DHS17] Matthew G. Durham, Mark F. Hagen, and Alessandro Sisto. Boundaries and automorphisms of hierarchically hyperbolic spaces. *Geom. Topol.*, 21(6):3659–3758, 2017.

[DK18] Cornelia Druţu and Michael Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[Hag20] Mark Hagen. Comment on second-countability and metrisability of the HHS boundary. `http://www.wescac.net/boundary_metrisable.pdf`, 2020.

[HMS] Mark Hagen, Alexandre Martin, and Alessandro Sisto. Extra-large type artin groups are hierarchically hyperbolic. arXiv:2109.04387.

[KB02] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

[Mit98] Mahan Mitra. Cannon-Thurston maps for hyperbolic group extensions. *Topology*, 37(3):527–538, 1998.

[Mou18] Sarah C. Mousley. Nonexistence of boundary maps for some hierarchically hyperbolic spaces. *Algebr. Geom. Topol.*, 18(1):409–439, 2018.

[PS20] Harry Petyt and Davide Spriano. Unbounded domains in hierarchically hyperbolic groups. arXiv:2007.12535, 2020.

[RST18] Jacob Russell, Davide Spriano, and Hung C. Tran. Convexity in hierarchically hyperbolic spaces. *Algebr. Geom. Topol.*, arXiv:1809.09303, 2018. To appear.

[RST22] Jacob Russell, Davide Spriano, and Hung Cong Tran. The local-to-global property for Morse quasi-geodesics. *Math. Z.*, 300(2):1557–1602, 2022.

[Spr18] Davide Spriano. Hyperbolic HHS I: Factor systems and quasiconvex subgroups. arXiv:1711.10931, 2018.

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