DISC DIAGRAMS, WALLSPACES, CUBICAL SMALL-CANCELLATION

The second recitation, on 4 August, will consist of a discussion of a few of the following exercises, according some type of interest-consensus. Please email questions or corrections to `markfhagen@gmail.com`.

Exercise 1. The following exercises illustrate the use of cubical disc diagrams.

(1) A *median graph* is a graph Γ such that for all triples x, y, z of distinct vertices in Γ, there exists a unique vertex $m = m(x, y, z)$ such that

$$d(x, y) = d(m, x) + d(m, y),\ d(y, z) = d(m, y) + d(m, z),\ d(x, z) = d(m, x) + d(m, z).$$

Let $\mathbf{X}$ be a CAT(0) cube complex. Use a disc diagram argument to show that $\mathbf{X}^{(1)}$ is a median graph.

(2) Show that the CAT(0) cube complex $\mathbf{X}$ has the *Helly property*: if $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ is a collection of pairwise-intersecting convex subcomplexes, then $\bigcap_{i=1}^{n} \mathbf{Y}_i \neq \emptyset$.

(3) Let $\overline{\mathbf{Y}} \to \overline{\mathbf{X}}$ be a local isometry of nonpositively-curved cube complexes. Show that this lifts to an isometric embedding $\mathbf{Y} \to \mathbf{X}$ of their universal covers.

Exercise 2. These exercises concern wallspaces, their dual cube complexes, and the attendant group actions.

(1) (Overcubulating!) Consider the tiling of the Euclidean plane shown in Figure 1. What is the cube complex $\mathbf{C}$ dual to the system of "antipodal" walls? The group

$$G \cong \langle a, b \mid a^4, b^2, (ab)^4 \rangle$$

acts properly and cocompactly on this tiling. Does G act properly on $\mathbf{C}$? Find a cocompact action of G on a CAT(0) cube complex by restricting the set of walls.

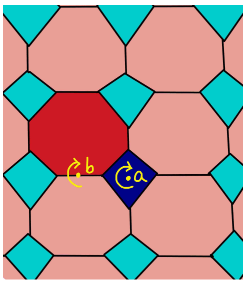

FIGURE 1. The tiling in Exercise 2.(1). G is generated by the order-4 rotation a and the order-2 rotation b. The translation subgroup is $\langle aba, ba^2 \rangle \leq G$.

(2) Consider the usual action of $PGL_2(\mathbb{Z})$ on $\mathbb{H}^2$ by Möbius transformations. Find an equivariant wallspace structure on $\mathbb{H}^2$ and construct the dual cube complex $\mathbf{C}$. Show that the action of $PGL_2(\mathbb{Z})$ on $\mathbf{C}$ is proper. What does $\mathbf{C}$ look like: how many orbits of hyperplanes? Verify that the action of $PGL_2(\mathbb{Z})$ on $\mathbf{C}$ is cocompact. (I suppose it's possible that your example is not proper or cocompact; if so, modify it so it is! It's more fun if $\mathbf{C}$ is not a tree.)

(3) Construct an example of a group G acting properly and with finitely many orbits of hyperplanes on a locally finite CAT(0) cube complex $\mathbf{C}$ such that G is not finitely generated.

Exercise 3. These exercises deal with cubical small-cancellation theory.

(1) (Using grids and splaying.) This exercise asks you to fill in a detail in the sketch of the proof of Theorem 8.7 in the current version of Dani's lecture notes. Let $\overline{\mathbf{X}}$ be a nonpositively-curved cube complex with universal cover $\mathbf{X}$. Let $\{\overline{\mathbf{Y}}_i \to \overline{\mathbf{X}}\}$ be a collection of compact based local isometries, so that we have a cubical presentation

$$\langle \overline{\mathbf{X}} \mid \overline{\mathbf{Y}}_i, \ldots \overline{\mathbf{Y}}_k \rangle.$$

Show that for any path P in a non-contiguous cone-piece or non-contiguous wall-piece, there exists a path Q in a contiguous wall-piece with $|P| \leq |Q|$, i.e. prove the assertion in the notes that "non-contiguous cone-pieces and non-contiguous wall-pieces are dominated by contiguous wall-pieces".

(2) (A $C(6)$ example.) Let

$$G \cong \langle a, b, c \mid (ab)^2, (bc)^2, (ca)^2, (a^3b^3c^3)^2 \rangle.$$

Find a finite-index subgroup $G' \leq G$ and a $C(6)$ cubical presentation for G', with respect to the angling-system discussed in the notes. Is each cone a wallspace?

(3) (Short inner paths) The cubical presentation

$$\langle \overline{\mathbf{X}} \mid \overline{\mathbf{Y}}_i, \ldots \overline{\mathbf{Y}}_k \rangle$$

has *short inner paths* if each cone $\mathbf{Y}_i$ has the following property: let S be a path in $\mathbf{Y}_j$ such that $\Omega_i(S) < \pi$ for all i, where Ω_i is the *total defect* defined below. Then for each local geodesic $S' \to \mathbf{Y}_j$ that is path-homotopic to S, and for each path $Q \to \mathbf{Y}_j$ such that QS' is an essential closed path in $\mathbf{Y}_j$, we have $|S'| < |Q|$. Show that if the cubical presentation is $C'(\frac{1}{24})$, then it has short inner paths. For which $\alpha > \frac{1}{24}$ does this hold?

(4) (Cones embed) Let $\mathbf{X}^*$ be the coned-off complex associated to the cubical presentation:

$$\langle \mathbf{X} \mid \mathbf{Y}_i, \ldots \mathbf{Y}_k \rangle.$$

Denote by $\widetilde{\mathbf{X}}^*$ the cover of $\mathbf{X}$ associated to

$$\langle\langle \{\mathbf{Y}_i\} \rangle\rangle \leq \pi_1 \mathbf{X}.$$

Use the fundamental theorem of small-cancellation theory to show that, under the following hypothesis, each $\mathbf{Y}_i$ embeds in $\widetilde{\mathbf{X}}^*$: there is an angling-system on $\mathbf{X}^*$ such that the ladder theorem holds and with respect to which $\mathbf{X}^*$ has short inner paths.

Definition 1 (Total defect)**.** Let $D \to \mathbf{X}^*$ be a rectified disc diagram of minimal complexity such that the path $S \to \mathbf{Y}_j$ lies on a cone-cell C in D mapping to $\mathbf{Y}_j$. For each corner c in C along S, the *defect* at c is $\pi - \sphericalangle(c)$. The *defect of along S in D* is the sum of the defects at all of the corners c of C along S. The *total defect of S in $\mathbf{Y}_j$*, denoted $\Omega_j(S)$, is the infimum over all such minimal-complexity rectified diagrams D of the total defect of S in D. See Figure 2.

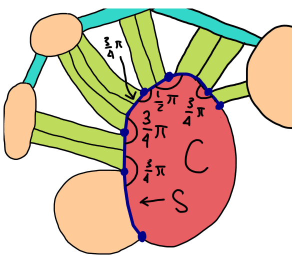

FIGURE 2. The total defect is $\frac{3\pi}{2}$.