1 Overview

My primary areas of interest are the worlds of finitely generated groups and low dimensional manifolds. I am especially interested in interactions between group theory, geometry, and topology; recently I have also been pursuing connections between these areas and probabilistic combinatorics.

Geometry and group theory meet via the observation that any finitely generated group can be endowed with a natural metric which is uniquely determined up to *quasi-isometry*, i.e., up to maps which distort distance by a bounded additive and multiplicative amount. With this basic fact as a starting point, Gromov proposed the problem of classifying groups by their (quasi-isometric) geometry. The question of classifying groups in this way is a central question which has received enormous attention, yet about which much work remains to be done. At present, there are a few important families whose geometry has successfully been classified and many more for which a classification theorem appears to be imminent. One beautiful aspect of this program, as I describe below, is the interactions it fosters between group theory and many areas of mathematics including: analysis, combinatorics, computer science, logic, geometry, topology, and others.

For a number of the most fundamental families of groups, I have completely resolved or made significant progress on the quasi-isometric classification question. This includes mapping class groups [6, 29], 3–manifold groups [33, 34], Artin groups [10, 20, 27, 28], relatively hyperbolic groups [15], Coxeter groups [20, 22, 26], and others. I have also used the geometry of groups as a starting point to resolve problems in parts of mathematics outside of group theory, including: Teichmüller theory [6, 15, 23, 26, 31], Kazhdan's Property (T) [16, 17], the Novikov Conjecture [32], random graphs [18, 19, 22], 3–manifold topology [33, 34], geometric measure theory [13, 14], and elsewhere.

In this overview I will just focus on two of the main directions of my recent work. In Section 2 I'll describe recent and ongoing work on **hierarchically hyperbolic group and spaces**, which I introduced with M. Hagen and A. Sisto in a series of papers, which has led to a flurry of activity by ourselves and others. Our work establishes a new framework for studying non-positive curvature phenomena and has already played a key role in resolving a number of long-standing questions and conjectures. The results and techniques in Section 3 primarily involve **probabilistic combinatorics** and are of a quite distinct flavor from much of my other research. This work provides novel insights into the study of **random graphs** and in turn obtain new results about the geometry of **Coxeter groups**. As discussed below, this work has been informed by experiments using computer software written by myself and with collaborators, including some software written with undergraduates working with me in a summer "research experience for undergraduates" program. Finally, in Section 4, I'll briefly mention a few other results of mine to give a further sense of some of the other types of questions I've studied and some of the variety of tools I've developed to do so.

In addition to the research I've carried out, I also actively mentor at all levels and to a diverse collection of students. For undergraduates I have run several REUs, leading to research papers and honors theses for a number of the participants. I currently have several Ph.D. students and have had five graduate so far, each of whom then held a

September 23, 2024

postdoc at a top research university. I have also organized a number of conferences aimed at junior researchers and mentored a number of postdocs, including several with whom I've now written papers [1, 2, 3, 19, 18, 20, 21, 22, 23, 24, 25, 26]. I also wrote an article to introduce beginning graduate students to the field of geometric group theory [7] based on a lecture series I gave in China and which I'm in the process of expanding to a book.

2 Hierarchically hyperbolic groups and spaces

In a series of papers with M. Hagen and A. Sisto, we introduced a class of groups and spaces which we call hierarchically hyperbolic [23, 24, 25, 26]. This class effectively captures the type of negative curvature phenomenon that one sees in many important groups and spaces in low dimensional topology, including mapping class groups, right-angled Artin groups, most 3–manifold groups, Teichmüller space (in any of the well studied metrics), etc.

Using the viewpoint provided by hierarchical hyperbolicity, we have successfully proven a number of exciting new results, including: a general framework for proving quasi-isometric rigidity [26]; distance formulas for cubulated groups and 3–manifold groups [23, 24] which allow one to quasi-isometrically estimate distances in the group in terms of more easily accessed data; acylindricity for cubulated groups [23]; restrictions on quasi-isometric embeddings of nilpotent groups into many spaces [23]; effective bounds on asymptotic dimension [25]; structure theorems for quasiflats [26]; a generalization of Thurston's Dehn filling theorem [21]; etc. We have also been able to give short new proofs of a number of important and difficult theorems, including: the Masur–Minsky distance formula for $\mathcal{MCG}$ [MM2, 24]; acylindricity of the action of $\mathcal{MCG}$ on the curve graph [Bow3, 23] and of a right-angled Artin group on its extension graph [KK, 23], quasi-isometric rigidity of $\mathcal{MCG}$ [29, 26], the Brock–Farb Rank Conjecture for Teichmüller space [31, 23], and others.

The study of hierarchically hyperbolic spaces (HHSs) began with constructing tools to generalize the "hierarchy machinery" for mapping class groups which was developed in [MM1, MM2, 6, 29, 31]. The basic setup allows us to use HHS analogues of tools which played a central role in proving recent major results including quasi-isometric rigidity for mapping class groups [29], the Rank Conjecture for Teichmüller space [31], and the Ending Lamination Conjecture for Kleinian groups [BCM].

Roughly, a hierarchically hyperbolic structure is defined by providing analogues of the following objects and maps from the theory of surfaces: a notion of "subsurfaces" and "complexity" of a subsurface; a "curve graph" associated to each surface (for surfaces this is an infinite δ–hyperbolic graph which encodes the set of all homotopy classes of simple closed curves on a surface and their intersection patterns); relations of "nesting," "disjointness," and "overlapping"; and, projections from the curve graph of one "subsurface" to another with suitable properties. One upshot is that this allows us to prove in new settings, for instance cubical groups, theorems known for mapping class groups; it also provides a new perspective for approaching questions about mapping class groups, which were not accessible from within the theory of mapping class groups (e.g., results in [25, 26] are two examples of the phenomenon where the HHS perspective is crucial, as it allows for more flexible constructions than those purely within the world of mapping class groups, allowing us to resolve long-standing conjectures about mapping class groups via a more general result).

Hierarchical hyperbolicity turns out to include a very broad collections of groups and spaces, including those obtained by a number of combination theorems. Recent papers enlarging the class of spaces with this property includes a number of papers by myself and others: [21, 23, 24, 25, BR2, BR1, DDLS, Hae2, HHP, HP, HRSS, HS, Hug, HV, RS, Rus1, RV1, RV2, Spr2, Vok].

I'll now describe in slightly more detail a few of the results I've been directly involved in to give a taste of the types of work being done using hierarchical hyperbolicity.

Generalizing classical theorems of Morse and Mostow, it was proven that in a higher-rank symmetric space an arbitrary quasiflat must lie near a finite number of standard flats [EF, KL]. A number of long standing questions asked whether generalizations of such a classification of quasiflats holds in other settings; we resolve these with the following.

Theorem 2.1 (Quasiflats Theorem for HHS; [26])**.** *Let $\mathcal{X}$ be an any hierarchically hyperbolic group (or, more generally, any HHS satisfying a minor technical hypothesis). Then any top-dimensional quasiflat in $\mathcal{X}$ is Hausdorff close to a uniformly bounded number of orthants of standard flats.*

This theorem has numerous applications. Applied to the mapping class group this theorem affirms a conjecture of Farb. Using this theorem in the case of the Weil-Petersson metric on Teichmüller space answers a question of Brock [Bro, Question 5.3]. Applied to fundamental groups of non-geometric 3–manifolds allows us to recover an important theorem of Kapovich–Leeb [KL1]. For cubulated groups this results generalizes the main theorems of [BKS] and [Hua], allowing us to, for instance, obtain a quasiflat theorem for all right-angled Coxeter groups. Also, we used this to give a new very short alternate proof of quasi-isometric rigidity for mapping class groups and establish a framework for proving quasi-isometric rigidity in a more general setting.

The following gives a relation between hierarchically hyperbolic groups and acylindrically hyperbolic groups, cf. [Osi2]. For the sake of brevity, we won't define the *automorphism group* of a hierarchically hyperbolic space here, rather, we just note that it includes, in the relevant cases, all elements of $\mathcal{MCG}$ and all isometries of a cube complex with a factor system. The "maximal element" S and $\mathcal{C}S$ referred to below are $\mathcal{X}$ and its contact graph, in case $\mathcal{X}$ is a CAT(0) cube complex, while they are the surface S and its curve graph, when G is the mapping class group of S.

Theorem 2.2 (HHS act acylindrically; [23])**.** *Let $\mathcal{X}$ be hierarchically hyperbolic with respect to the set $\mathfrak{S}$ of hyperbolic spaces and let $G \leq \mathrm{Aut}(\mathfrak{S})$ act properly and cocompactly on $\mathcal{X}$. Let S be the maximal element of $\mathfrak{S}$ and denote by $\mathcal{C}S$ the corresponding hyperbolic space. Then G acts acylindrically on $\mathcal{C}S$.*

The *asymptotic dimension* of a metric space is a well-studied quasi-isometry invariant, introduced by Gromov [Gro], which provides a coarse version of the topological dimension. Early motivation for studying asymptotic dimension was provided by Yu, who showed that groups with finite asymptotic dimension satisfy both the coarse Baum–Connes and the Novikov conjectures [Yu]. It is now known that asymptotic dimension provides coarse analogues for many properties of topological dimension, see [BD]. Using very different techniques a number of groups and spaces have been shown to have finite asymptotic dimension, although good estimates on this dimension have proved difficult in many cases: curve graphs [BF, BB], mapping class groups [BBF], cubulated groups [Wri], graph manifold groups [Smi], and groups hyperbolic relative to ones with finite asymptotic dimension [Osi1].

The following very general theorem, which in addition to covering many new cases, provides a unified proof of finite asymptotic dimension for almost all the cases just mentioned:

Theorem 2.3 (Finite asymptotic dimension for HHS; [25])**.** *Any hierarchically hyperbolic group has finite asymptotic dimension.*

Not only do we prove finiteness, but we obtain very effective bounds on the dimension. For instance, for the mapping class group our bound is quadratic in the complexity of the surface, improving prior bounds which were double exponential [BBF, BB, Web].

Corollary 2.4 (Effective asymptotic dimension bound for $\mathcal{MCG}(S)$; [25])**.** *Let S be a connected oriented surface of finite type of complexity $\xi(S) \geq 2$. Then the asymptotic dimension of $\mathcal{MCG}(S)$ is at most $5\xi(S)^2$.*

In a recent paper with C. Abbott, we found that many elements in a hierarchically hyperbolic groups satisfy the linearly bounded conjugator property, which is useful for solving algorithmic problems in a group. The following is a special case of one of our results concerning Morse elements, which are elements whose geometry is similar to that of the axis of a loxodromic isometry of a hyperbolic space.

Theorem 2.5 (Morse elements have linearly bounded shortest conjugators; [1])**.** *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. There exist constants K, C such that if $a, b \in G$ are Morse elements which are conjugate in G, then there exists $g \in G$ with $ga = bg$ and*

$$|g| \leq K(|a| + |b|) + C.$$

In a paper with C. Abbott and M. Durham, we proved that Morse elements in a hierarchically hyperbolic group admit a number of nice characterizations, see [2, Theorem B].

Arguably, the most basic elements in the mapping class group of a surface are the Dehn twists. In 1974, in Birman's classic monograph [Bir], she notes that for the closed genus two surface the normal closure of the squares of Dehn twists is of index $6!$ in the mapping class group; she then asked whether the index is finite or infinite for arbitrary genus. During the past 50 years there has been a small industry of people showing in various cases that in some cases these quotients were finite and in others infinite. In the following result my collaborators and I not only prove that many of these groups are infinite, but we provide a hierarchically hyperbolic structure on them which gives substantial progress towards geometrically understanding these quotients.

Theorem 2.6 (Quotients of $\mathcal{MCG}$ by powers of Dehn twists; [21, Theorem 2])**.** *For any surface of finite type there exists a constant K_0, so that for any non-zero multiple K of K_0 the quotient of the mapping class group by the normal subgroup generated by K-th powers of Dehn twists is an infinite hierarchically hyperbolic group.*

The study of hierarchically hyperbolic groups/spaces has already had a tremendous impact and is growing quickly. In addition to the numerous problems these techniques have already resolved, they also allow for many new approaches to questions previously out of reach. The power of our approach can be gleamed from both the large number of papers that have already been written about hierarchically hyperbolic groups in just a few years. For instance, some of the papers which use this framework in a crucial way are the following: [1, 2, 3, 4, 21, 23, 24, 25, 26, But, ANS, BR1, BR2, Bow1, Bow2, DDLS, DHS, DMS, DZ, Hae2, Hae1, HHP, HP, HRSS, HS, HMS, Hug, HV, JL, Che, Mou1, Mou2, MR, Pet, PSZ, PS, RS, Rus1, Rus2, RST, RV1, RV2, Sel2, Sel1, She, Sis2, Spr1, Spr2, Vok] and numerous others in preparation.

3 Random graphs and Coxeter groups

I have been interested in proving geometric group theory analogues of some of Erdős–Rényi's foundational work in graph theory. My work has two interlaced aspects: one is making novel contributions in the field of probabilistic combinatorics; the second is applying my graph theoretic results to obtain striking new theorems about groups.

Random graphs

A *random graph* in the Gilbert/Erdős–Rényi model model is obtained by taking a *density* function $p(n)\colon \mathbb{N} \to (0,1)$ and forming a graph on n vertices by independently declaring each pair of vertices to span an edge with probability $p(n)$; we say such a graph is in $\mathcal{G}(n,p)$. Given a function p, we say a random graph satisfies a given property *asymptotically almost surely (a.a.s.)* when, with respect to the density function $p(n)$, the probability that a random graph satisfies the property approaches 1 as $n \to \infty$.

A property of graphs is said to exhibit a *sharp threshold* if there is a *critical density* $p_c = p_c(n)$ such that for any fixed $\epsilon > 0$ if $p < (1-\epsilon)p_c$ then a.a.s. $\mathcal{P}$ does not hold in $\mathcal{G}(n,p)$, while if $p > (1+\epsilon)p_c$ then a.a.s. $\mathcal{P}$ holds in $\mathcal{G}(n,p)$. A quintessential example is a classical theorem of Erdős and Rényi which yields a sharp threshold for connectedness:

Theorem 3.1 (Erdős-Rényi sharp threshold for connectivity; [ER])**.** *There is a sharp threshold for connectivity of a random graph and the critical density is* $\frac{\log(n)}{n}$. *Above this density random graphs are connected and below it they are disconnected.*

An interested family of graphs which my collaborators and I have studied are called $\mathcal{CFS}$ *graphs* ("Constructed From Squares"). These graphs arise naturally in geometric group theory in the context of the large–scale geometry of right–angled Coxeter groups (a special case of these graphs was introduced by Dani–Thomas to study divergence in triangle-free right-angled Coxeter groups [DT]). Roughly, a graph is in $\mathcal{CFS}$ if it can be built inductively by chaining together induced squares in such a way that each square overlaps with one of the previous squares along opposite vertices. The $\mathcal{CFS}$ property provides a strong notion of connectivity, and thus the next result which establishes the critical threshold for the $\mathcal{CFS}$ property, is an analogue of Erdős–Rényi's Theorem 3.1.

Theorem 3.2 (Critical threshold for $\mathcal{CFS}$; [18, 19])**.** *Suppose $p(n)$ is bounded away from 1. There is a sharp threshold for a random graph to have the $\mathcal{CFS}$ property and the critical density is $\sqrt{\sqrt{6}-2}\cdot n^{-\frac{1}{2}}$. Above this density random graphs have $\mathcal{CFS}$ and below it they do not.*

One application of Theorem 3.2 is that it allowed us to establish in [19] a conjecture of Bollobas and Riordan about percolation [BR] and provides a framework for answering other questions, including additional ones asked in [BR].

Applications to geometric group theory

My interest in random graphs arose from studying the geometry of right-angled Coxeter groups. These groups are an extremely interesting source of examples. Indeed, the richness of the class of right-angled Coxeter groups has led to many significant advances in geometric group theory and 3–manifold topology, especially in light of recent work by Agol, Haglund, Wise, and many others [Ago, HW, Wis2, Wis1].

A *right-angled Coxeter group* admits the following concise finite presentation: each generator is of order two and the only defining relators are that certain generators commute. Right-angled Coxeter groups can be described (uniquely) via a finite simplicial graph: the vertices correspond to the (order 2) generators and edges to pairs of generators which commute. Following the usual convention, we denote the graph Γ and the corresponding right-angled Coxeter group by W_Γ.

A powerful invariant for distinguishing metric spaces and groups up to quasi-isometry is that of divergence. The notion of the *divergence* of a metric space originates in work of Gromov and Gersten, and, roughly speaking, measures how far one must travel to connect two points while avoiding a specified ball centered at a third. An illustrative example is symmetric spaces of non-compact type where the order of the divergence of geodesic rays is either exponential (when the rank is one) or linear (when the rank is at least two).

The class of Coxeter groups contains many examples of hyperbolic and relatively hyperbolic groups. There is a criterion for hyperbolicity purely in terms of the presentation graph due to Moussong [Mou] and an algebraic criterion for relative hyperbolicity due to Caprace [Cap]. The class of Coxeter groups includes examples which are non-relatively hyperbolic, for instance, those constructed by Davis–Januszkiewicz [DJ] and, also, ones studied by Dani–Thomas [DT].

In [22], we establish a geometric criterion for relatively hyperbolicity and obtain computable invariants for quasi-isometrically distinguishing many Coxeter groups:

Theorem 3.3 (Canonical relative hyperbolicity structures; [22])**.** *Every Coxeter group either has polynomial divergence or is hyperbolic relative to a canonical (possibly empty) collection of subgroups which each have polynomial divergence. Moreover, for each $n \in \mathbb{N}$ there exist Coxeter groups with divergence which is polynomial of order n and one-ended Coxeter groups which are hyperbolic relative to subgroups which each have divergence which is polynomial of order n.*

Note that relatively hyperbolic groups have exponential divergence (see [Sis1]), while polynomial divergence in the above result is established via establishing a property called *thickness* then applying my results from [12].

Further, the above characterization can be formulated in terms of the combinatorics of the graph [22, Lev2]; this allows us to deduce group theoretic results from graph theoretic ones.

By associating to a random graph, $\Gamma \in \mathcal{G}(n,p)$, its associated W_Γ yields a notion of a *random right-angled Coxeter group*. In the papers [18, 19] we found the exact range of densities for which random right-angled Coxeter group have quadratic divergence. Interestingly, a special case of this theorem is the following which shows that a conjecture of Gromov's about non-existence of polynomial divergence in CAT(0) spaces [Gro, Section $6.B_2$] is false for the generic right-angled Coxeter group!

Theorem 3.4 (The generic RACG as counterexample to Gromov's conjecture; [18, 19])**.** *For any $\epsilon > 0$, if $p\colon \mathbb{N} \to (0,1)$ is bounded away from 1 and satisfies $p(n) \geq \left(\sqrt{\sqrt{6}-2}+\epsilon\right) \cdot n^{\frac{1}{2}}$ for all sufficiently large n, then the random right-angled Coxeter group W_Γ, for $\Gamma \in \mathcal{G}(n,p)$, asymptotically almost surely has quadratic divergence.*

In his Ph.D. thesis, my student Ivan Levcovitz proved that a graph is $\mathcal{CFS}$ if and only if the right-angled Coxeter group with that presentation graph has quadratic divergence [Lev1]. His result together with Theorem 3.2 show that below the density in Theorem 3.4 asymptotically almost surely the random right-angled Coxeter group has divergence which is at least cubic.

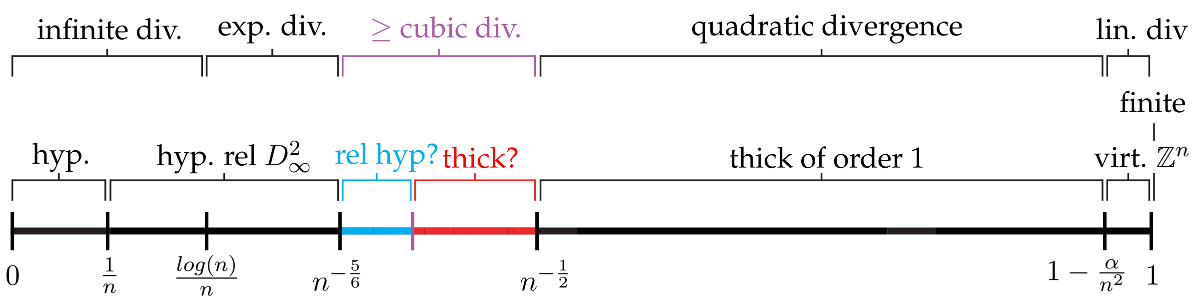


Figure 1: Behavior of random RACG at a spectrum of densities. Each listed property occurs a.a.s. at the given density, as proven in [18, 19, 22, Lev1]. Prevalence of the blue/red properties in the middle are a subject of my current research; indeed my recent preprint [11] makes a huge leap in this direction by showing that the range for relatively hyperbolic range extends all the way up to $(n\log(n))^{-\frac{1}{2}}$, thereby affirming that the statement in the first bullet point of Conjecture 3.5 is the right order of magnitude for the threshold.

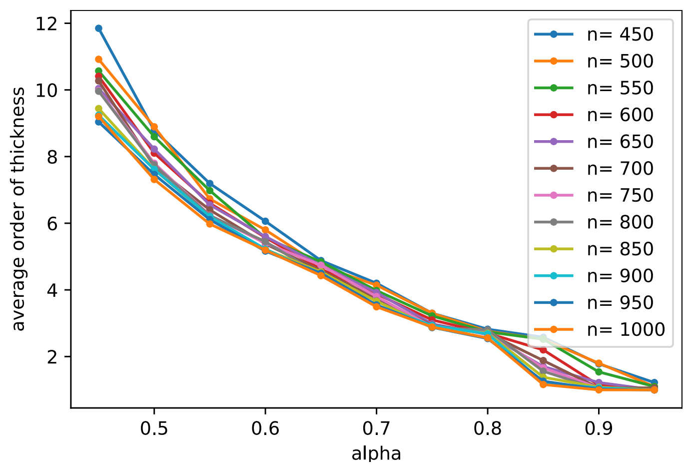
An amazing aspect of Erdős–Rényi's study of connectivity in random graphs is their famous "double jump theorem." Their result shows that the size of the largest connected components of graphs in $\mathcal{G}(n,p)$ has a double jump at $\frac{1}{n}$: for probabilities below this, the size is logarithmic in n; at the threshold, the size is roughly $n^{\frac{2}{3}}$; and, above it, the size of the largest component is linear in n. My viewpoint on the above results about RACG is that they are leading to the following conjecture for RACGs, which would be the first analogue in geometric group theory, by showing that there exists a "double jump" in the qualitative change in behavior at a particular critical threshold:

Conjecture 3.5 (RACG double jump conjecture)**.** *Let $\epsilon > 0$, $\lambda = \sqrt{\sqrt{6}-2}$, and $\Gamma \in \mathcal{G}(n,p)$. One of the following occurs.*

- *If* $0 \leq p(n) \leq \frac{\lambda - \epsilon}{\sqrt{n}}$*, then* W_Γ *has exponential divergence (and is relatively hyperbolic).*
- *If* $p(n) = \frac{\lambda}{\sqrt{n}}$*, then* W_Γ *has polynomial divergence.*
- *If* $\frac{\lambda+\epsilon}{\sqrt{n}} \leq p(n) \leq 1 - \frac{(1+\epsilon)\log n}{n}$*, then* W_Γ *has quadratic divergence.*

Note that my collaborators and I already established the third bullet point in [19, 22].

One of the ways in which we were led to Theorem 3.4 and have found evidence for Conjecture 3.5 is by computer experimentation using software I developed, some of which was done with the help of undergraduates. Recently I generated the data for the figure at right which displays the results of testing random graphs to see if the corresponding RACG is thick and, if so, the order of thickness (which is exactly one less than the order of polynomial divergence); if not thick then they are relatively hyperbolic. This data set looks at a range of vertex sizes, n, and densities of the form $p = \alpha \cdot n^{-\frac{1}{2}}$. For each pair (n, p) of this type, I generated and tested 100 random graphs (note that above $\alpha = 0.55$ they all were thick; below that at least 95% were, except at $\alpha = 0.45$ where for $n < 550$ approximately 70% were thick). The data illustrates the tight correlation between density (as a multiple of $n^{-\frac{1}{2}}$) and order of thickness as a function of the number of vertices. This data provides some computational evidence that $p = \alpha \cdot n^{-\frac{1}{2}}$ is where a threshold will lie for a threshold between thickness and relative hyperbolicity (which should roughly be thought of as an infinite order of thickness).



4 Selected other results

Below is a short summary of some of my other work to give a quick view of a few of my interests beyond those discussed in the two projects described above.

A continuing source of inspiration for me is the example of the mapping class group, $\mathcal{MCG}(S)$ which is the group of isotopy classes of homeomorphisms of a surface, S. This group plays a central role in the topology of surfaces, it is important in Teichmüller theory, algebraic geometry, three-dimensional topology, and is a rich source of interesting phenomena providing a model for study throughout geometric group theory.

The *geometric rank* of a space is the maximal dimension of a euclidean space which can be quasi-isometrically embedded into that space. J. Brock and B. Farb [BF2] formulated

See http://comet.lehman.cuny.edu/behrstock/random.html for an animation with related data.

Rank Conjectures for the mapping class group and for Teichmüller space. Minsky and I resolved these conjectures in [31] by calculating the geometric rank of these spaces. In doing so we showed that the geometric rank of the mapping class group of a surface is equal to the maximal rank of an abelian subgroup, the latter of which was computed by Birman–Lubotzky–McCarthy to be equal to the maximal number of pairwise disjoint homotopy classes of simple closed curves on a given surface [BLM].

One of my first obsessions as a mathematician was a long-standing question: is the mapping class group quasi-isometrically rigid. Loosely speaking, this asks whether the isomorphism class of such groups is determined by their geometry. My collaborators and I resolved this in [29] by proving that a finitely-generated group Γ is quasi-isometric to $\mathcal{MCG}(S)$ if and only if there exists a finite-index subgroup $\Gamma' < \Gamma$ and a homomorphism, with finite kernel and finite-index image, from Γ' to $\mathcal{MCG}(S)$ modulo its center.

I am interested in analytic properties of groups. One such property with roots in representation theory is Kazhdan's Property (T). With C. Druţu and M. Sapir, one of the results we proved is that any group with Property (T) admits only finitely many non-conjugate homomorphisms into any mapping class group [16, 17].

Another property from analysis is the *Rapid Decay property* (a.k.a. the *Haagerup inequality*) which holds for a group G, when the space of rapidly decreasing functions on G, with respect to some length function, is inside the reduced C^*–algebra of G. This is a property which is enjoyed by hyperbolic groups and many, but not all, lattices in semi-simple Lie groups [Haa, dlH]. In work with Minsky, we proved the mapping class group has the Rapid Decay property [32]. This property has many applications, for instance we used it to give a proof of the strong Novikov conjecture for mapping class groups.

My paper [6], introduced a key inequality for studying non-positively curved spaces, which has since been widely used both in my research and by others. This result which people often call the "Behrstock inequality" has played an important role in several Ph.D. theses from the University of Michigan, Utah, Yale, and elsewhere [AK1, Man1, Sun] and numerous publications where people are interested in this inequality or analogues of it, see e.g., [AK2, BBF, BF, BM, CLM, Dur, Run, Man2, SS, ST, Tay2, Tay1].

The fundamental groups of 3–manifolds are an important and rich family of groups. Neumann and I wrote a series of papers which completed resolved the quasi-isometric classification problem for graph manifolds (answering a conjecture of Kapovich–Leeb [KL2] in the process) and established a framework for the general quasi-isometric classification problem for 3–manifold groups [33, 34]. As part of our work we introduced a notion which we call *bisimilarity* — in homage to a related notion in computer science — which is an algorithmically checkable equivalence relation on colored finite graphs. This allows us, for instance, to prove results such as: there are exactly 204535126 quasi-isometry classes of fundamental groups of non-geometric graph manifolds composed of at most 8 Seifert fibered pieces [33], see also [Slo].

Our notion of bisimilarity has now been used to obtain quasi-isometric classifications for other families of groups, see: [28, CRKZ1, CRKZ2, Cas, CM1, CM2, HNT, Mar, NT, Oh, SW].

In the previous section I discussed a bit about the *divergence* function which provides a rich quasi-isometry invariant for metric spaces and finitely generated groups. In symmetric spaces of non-compact type the divergence of geodesic rays is either exponential (when the rank is one) or linear (when the rank is at least two). Gromov discussed this in [Gro, Section $6.B_2$] and explicitly conjectured that all pairs of geodesic rays in the universal cover of a closed Riemannian manifold of non-positive curvature must diverge either linearly or exponentially. In the early 1990's, Gersten provided counterexamples to this conjecture by producing examples of $CAT(0)$ spaces and groups with quadratic divergence [Ger1, Ger2]; he then raised the question of whether one could find $CAT(0)$ spaces whose divergence was polynomial of degree greater than two.

Twenty years after Gromov's question, in joint work with C. Druţu [12], we provide a complete answer to the Gromov/Gersten question [Gro, Ger1, Ger2] by providing an abundant collection of (counter-)examples, including infinite families of pairwise non-quasi-isometric finitely generated $CAT(0)$ groups which each have divergence which is polynomial of degree any fixed integer.

Analogous to the standard divergence function, one can consider "higher dimensional" isoperimetric and divergence functions. The *k–dimensional isoperimetric function* is defined by taking the supremum of filling volumes over all k–dimensional spheres of volume at most a fixed constant times x^k. To make sense of this one needs a way to measure volume. In the case of a group, G, which acts cellularly and properly discontinuously on an n–connected CW–complex, X, such that $X^{(n+1)}/G$ has finitely many cells, for any $1 \leq k \leq n$ we define the *k–dimensional isoperimetric function of G* to be the k–dimensional combinatorial isoperimetric function of X.

Motivated by theorems about groups acting on Hadamard spaces [Gro, Kle], Druţu and I computed the higher divergence and isoperimetric functions of the mapping class group in [13, 14]; one of our results is that these functions exhibit a phase transition at the geometric rank. This result is the mapping class group analogue of a result for Hadamard spaces proven in [Wen, BF1, Leu, Hin].

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